



## Open Mathematics

### Research Article

L.M. Camacho, B.A. Omirov, K.K. Masutova, and I.M. Rikhsiboev\*

# Solvable Leibniz algebras with $NF_n \oplus F_m^1$ nilradical

<https://doi.org/10.1515/math-2017-0115>

Received June 5, 2017; accepted October 4, 2017.

**Abstract:** All finite-dimensional solvable Leibniz algebras  $L$ , having  $N = NF_n \oplus F_m^1$  as the nilradical and the dimension of  $L$  equal to  $n + m + 3$  (the maximal dimension) are described.  $NF_n$  and  $F_m^1$  are the null-filiform and naturally graded filiform Leibniz algebras of dimensions  $n$  and  $m$ , respectively. Moreover, we show that these algebras are rigid.

**Keywords:** Lie algebra, Leibniz algebra, Natural gradation, Null-filiform algebra, Filiform algebra, Solvability, Nilpotence, Nilradical

**MSC:** 17A32, 17A65, 17B30

## 1 Introduction

Leibniz algebras over  $\mathbb{K}$  were first introduced by A. Bloh [1] and called D-algebras. The term Leibniz algebra was introduced in the study of a non-antisymmetric analogue of Lie algebras by Loday [2], being so the class of Leibniz algebras an extension of the one of Lie algebras. In recent years it has been common theme to extend various results from Lie algebras to Leibniz algebras [3,4]. Many results of the theory of Lie algebras have been extended to Leibniz algebras. For instance, the classical results on Cartan subalgebras [5], variations of Engel's theorem for Leibniz algebras have been proven by different authors [6,7] and Barnes has proven Levi's theorem for Leibniz algebras [8].

In an effort to classify Lie algebras, many authors place various restrictions on the nilradical. The first work which was devoted to description of such Lie algebras is the paper [9]. Later, Mubarakjanov proposed the description of solvable Lie algebras with a given structure of nilradical by means of outer derivations [10]. In the papers [11-14], the authors apply the Mubarakjanov's method to classify the solvable Lie algebras with different kinds of nilradicals. Some results of Lie algebra theory generalized to Leibniz algebras in [3] allow us to apply the Mubarakjanov's method for Leibniz algebras. In this sense, we can see the papers [15-18].

It is important to study solvable Leibniz algebras because thanks to the Levi's theorem for Leibniz algebras, a Leibniz algebra is a semidirect sum of the solvable radical and a semisimple Lie algebra. As the semisimple part can be described by simple Lie ideals, the main problem is to understand the solvable radical.

---

**L.M. Camacho:** Dpto. Matemática Aplicada I. Universidad de Sevilla. Avda. Reina Mercedes, s/n. 41012 Sevilla, Spain,  
E-mail: lcamacho@us.es

**B.A. Omirov:** Institute of Mathematics. National University of Uzbekistan, 29, Do'rmon yo'li street., 100125, Tashkent, Uzbekistan,  
E-mail: omirovb@mail.ru

**K.K. Masutova:** Institute of Mathematics. National University of Uzbekistan, 29, Do'rmon yo'li street., 100125, Tashkent, Uzbekistan,  
E-mail: kamilyam81@mail.ru

**\*Corresponding Author: I.M. Rikhsiboev:** Universiti Kuala Lumpur, Malaysian Institute of Industrial Technology, Bandar Seri Alam, 81750 Johor Bahru, Malaysia, E-mail: ikromr@gmail.com

The first aim of the present paper is to classify solvable Leibniz algebras with nilradical  $N = NF_n \oplus F_m^1$  where  $NF_n$  and  $F_m^1$  are the null-filiform and naturally graded filiform Leibniz algebras of dimensions  $n$  and  $m$ , respectively. To obtain this classification, we use the results obtained in [16–18].

The arrangement of this work is as follows. In Section 2 we recall some essential notions and properties of Leibniz algebras. We start Section 3 by establishing the maximal dimension of a solvable Leibniz algebra whose nilradical is  $N = NF_n \oplus F_m^1$ ; thereafter, we present the classification of solvable Leibniz algebras that can be decomposed as a direct sum of their nilradical and a complementary vector space of maximal dimension. Finally, in Section 4 we study the rigidity of the unique solvable Leibniz algebra obtained in the previous section.

Throughout the paper, we consider finite-dimensional vector spaces and algebras over a field of characteristic zero. Moreover, in the multiplication table of an algebra omitted products are assumed to be zero and if it is not noticed we shall consider non-nilpotent solvable algebras.

## 2 Preliminaries

Let us recite some necessary definitions and preliminary results.

A *Leibniz algebra* over a field  $\mathbb{F}$  is a vector space  $L$  equipped with a bilinear map, called bracket,

$$[-, -]: L \times L \rightarrow L,$$

satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

for all  $x, y, z \in L$ .

The set  $\text{Ann}_r(L) = \{x \in L \mid [y, x] = 0, y \in L\}$  is called *the right annihilator of the Leibniz algebra  $L$* . Note that  $\text{Ann}_r(L)$  is an ideal of  $L$  and for any  $x, y \in L$  the elements  $[x, x]$ ,  $[x, y] + [y, x] \in \text{Ann}_r(L)$ .

A linear map  $d: L \rightarrow L$  of a Leibniz algebra  $(L, [-, -])$  is said to be a *derivation* if for all  $x, y \in L$  the following condition holds:

$$d([x, y]) = [d(x), y] + [x, d(y)].$$

For a given element  $x$  of a Leibniz algebra  $L$  the operator of right multiplication  $R_x: L \rightarrow L$ , defined as  $R_x(y) = [y, x]$  for  $y \in L$ , is a derivation. This kind of derivations are called *inner derivations*.

Any Leibniz algebra  $L$  is associated with the algebra of right multiplications  $R(L) = \{R_x \mid x \in L\}$ , which is endowed with a structure of Lie algebra by means of the bracket  $[R_x, R_y] = R_x R_y - R_y R_x$ . Thanks to the Leibniz identity the equality  $[R_x, R_y] = R_{[y, x]}$  holds true. In addition, the algebra  $R(L)$  is antisymmetric isomorphic to the quotient algebra  $L/\text{Ann}_r(L)$ .

**Definition 2.1** ([10]). *Let  $d_1, d_2, \dots, d_n$  be derivations of a Leibniz algebra  $L$ . The derivations  $d_1, d_2, \dots, d_n$  are said to be nil-independent if*

$$\alpha_1 d_1 + \alpha_2 d_2 + \dots + \alpha_n d_n$$

*is not nilpotent for any scalars  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}$ .*

*In other words, if for any  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}$  there exists a natural number  $k$  such that  $(\alpha_1 d_1 + \alpha_2 d_2 + \dots + \alpha_n d_n)^k = 0$ , then  $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ .*

### 2.1 Solvable Leibniz algebras

For a Leibniz algebra  $L$  we consider the following *lower central* and *derived series*:

$$\begin{aligned} L^1 &= L & L^{k+1} &= [L^k, L^1], & k &\geq 1; \\ L^{[1]} &= L, & L^{[s+1]} &= [L^{[s]}, L^{[s]}], & s &\geq 1. \end{aligned}$$

A Leibniz algebra  $L$  is said to be nilpotent (respectively, solvable), if there exists  $n \in \mathbb{N}$  ( $m \in \mathbb{N}$ ) such that  $L^n = 0$  (respectively,  $L^{[m]} = 0$ ).

It should be noted that the sum of any two nilpotent ideals is nilpotent.

The maximal nilpotent ideal of a Leibniz algebra is said to be a nilradical of the algebra.

Obviously, the index of nilpotency of an  $n$ -dimensional nilpotent Leibniz algebra is not greater than  $n + 1$ . The following theorem describes these algebras, algebras with maximal index of nilpotency.

**Theorem 2.2** ([4]). *Any  $n$ -dimensional null-filiform Leibniz algebra admits a basis  $\{e_1, e_2, \dots, e_n\}$  such that the table of multiplication in the algebra has the following form:*

$$NF_n : [e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq n - 1$$

A Leibniz algebra  $L$  is said to be *filiform* if  $\dim L^i = n - i$ , where  $n = \dim L$  and  $2 \leq i \leq n$ .

Due to [4] and [19] it is known that there are three naturally graded filiform Leibniz algebras. In fact, the third type encloses the class of naturally graded filiform Lie algebras.

**Theorem 2.3** ([17]). *Any complex  $n$ -dimensional naturally graded filiform Leibniz algebra is isomorphic to one of the following pairwise non isomorphic algebras:*

$$\begin{aligned} F_n^1 : [e_i, e_1] &= e_{i+1}, & 2 \leq i \leq n - 1, \\ F_n^2 : [e_i, e_1] &= e_{i+1}, & 1 \leq i \leq n - 2, \\ F_n^3(\alpha) : \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \leq i \leq n - 1, \\ [e_i, e_{n+1-i}] = -[e_{n+1-i}, e_i] = \alpha(-1)^{i+1}e_n, & 2 \leq i \leq n - 1, \end{cases} \end{aligned}$$

where  $\alpha \in \{0, 1\}$  for even  $n$  and  $\alpha = 0$  for odd  $n$ .

The following theorems describe solvable Leibniz algebras of maximal dimension with  $NF_n^1$  and  $F_n^1$  nilradical.

**Theorem 2.4** ([17]). *Let  $R$  be a solvable Leibniz algebra whose nilradical is  $NF_n$ . Then there exists a basis  $\{e_1, e_2, \dots, e_n, x\}$  of the algebra  $R$  such that the multiplication table of  $R$  with respect to this basis has the following form:*

$$\begin{aligned} [e_i, e_1] &= e_{i+1}, \quad 1 \leq i \leq n - 1, \\ [e_i, x] &= i e_i, \quad 1 \leq i \leq n, \\ [x, e_1] &= -e_1. \end{aligned}$$

**Theorem 2.5** ([18]). *An arbitrary  $(n + 2)$ -dimensional solvable Leibniz algebra with nilradical  $F_n^1$  is isomorphic to the algebra  $R(F_n^1)$  with the multiplication table:*

$$\begin{aligned} [e_i, e_1] &= e_{i+1}, \quad 2 \leq i \leq n - 1, \quad [e_1, x] = e_1, \\ [e_i, y] &= e_i, \quad 2 \leq i \leq n, \quad [e_i, x] = (i - 1)e_i, \quad 2 \leq i \leq n, \\ [x, e_1] &= -e_1. \end{aligned}$$

Let  $R$  be a solvable Leibniz algebra. Then it can be decomposed in the form  $R = N \oplus Q$ , where  $N$  is the nilradical and  $Q$  is the complementary vector space. Since the square of a solvable Leibniz algebra is contained into the nilradical [3], we get the nilpotency of the ideal  $R^2$  and consequently,  $Q^2 \subseteq N$ .

Let us consider the restrictions to  $N$  of the right multiplication operator on an element  $x \in Q$  (denoted by  $R_{x|N}$ ). From [17], we know that for any  $x \in Q$ , the operator  $R_{x|N}$  is a non-nilpotent derivation of  $N$ . Let  $\{x_1, x_2, \dots, x_m\}$  be a basis of  $Q$ , then for any scalars  $\{\alpha_1, \dots, \alpha_m\} \in \mathbb{C} \setminus \{0\}$ , the matrix  $\alpha_1 R_{x_1|N} + \dots + \alpha_m R_{x_m|N}$  is non nilpotent, which means that the elements  $\{x_1, \dots, x_m\}$  are nil-independent derivations, [10].

**Theorem 2.6** ([17]). *Let  $R$  be a solvable Leibniz algebra and  $N$  be its nilradical. Then the dimension of the complementary vector space to  $N$  is not greater than the maximal number of nil-independent derivations of  $N$ .*

Moreover, similarly as in Lie algebras, for a solvable Leibniz algebra  $R$ , we have  $\dim N \geq \frac{\dim R}{2}$ .

A nilpotent Leibniz algebra is called *characteristically nilpotent* if all its derivations are nilpotent. If the nilradical  $N$  of a Leibniz algebra is characteristically nilpotent then, according to Theorem 2.6, a solvable Leibniz algebra is nilpotent. Therefore, we shall consider solvable Leibniz algebras with non-characteristically nilpotent nilradical. For more details see [17].

### 2.2 The second cohomology group of a Leibniz algebra

For acquaintance with the definition of cohomology group of Leibniz algebras and its applications to the description of the variety of Leibniz algebras (similar to the Lie algebras case) we refer the reader to the papers [2, 20-24]. Here we just recall that the second cohomology group of a Leibniz algebra  $L$  with coefficients in a corepresentation  $M$  is the quotient space

$$HL^2(L, M) = ZL^2(L, M)/BL^2(L, M),$$

where the 2-cocycles  $\varphi \in ZL^2(L, M)$  and the 2-coboundaries  $f \in BL^2(L, M)$  are defined as follows

$$(d^2\varphi)(a, b, c) = [a, \varphi(b, c)] - [\varphi(a, b), c] + [\varphi(a, c), b] + \varphi(a, [b, c]) - \varphi([a, b], c) + \varphi([a, c], b) = 0 \quad (1)$$

and

$$f(a, b) = [d(a), b] + [a, d(b)] - d([a, b]) \text{ for some linear map } d.$$

The linear reductive group  $GL_n(\mathbb{F})$  acts on the variety of  $n$ -dimensional Leibniz algebras,  $Leib_n$ , as follows:

$$(g * \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y))), \quad g \in GL_n(\mathbb{F}), \lambda \in Leib_n.$$

The orbits ( $Orb(-)$ ) under this action are the isomorphism classes of algebras. Note that, Leibniz algebras with open orbits are called rigid.

Due the work [20], the nullity of the second cohomology group with coefficients itself gives a sufficient condition for the rigidity of the algebras.

### 3 Main results

Let  $NF_n$  be an  $n$ -dimensional null-filiform Leibniz algebra with a basis  $\{e_1, e_2, \dots, e_n\}$  and  $F_m^1$  an  $m$ -dimensional filiform Leibniz algebra from the first class with a basis  $\{f_1, f_2, \dots, f_m\}$ , then we have the following multiplication:

$$NF_n : [e_i, e_1] = e_{i+1}, 1 \leq i \leq n - 1; F_m^1 : \begin{cases} [f_1, f_1] = f_3, \\ [f_i, f_1] = f_{i+1}, 2 \leq i \leq m - 1. \end{cases}$$

Let us consider the direct sum of these algebras  $N = NF_n \oplus F_m^1$ . The following proposition describes derivations of the algebra  $N$ .

**Proposition 3.1.** Any derivation of the algebra  $N = NF_n \oplus F_m^1$  has the following matrix form:

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_n & 0 & 0 & 0 & \dots & 0 & \beta_m \\ 0 & 2\alpha_1 & \alpha_2 & \dots & \alpha_{n-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 3\alpha_1 & \dots & \alpha_{n-2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & n\alpha_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & \gamma_n & \delta_1 & \delta_2 & \delta_3 & \dots & \delta_{m-1} & \delta_m \\ 0 & 0 & 0 & \dots & \tau_n & 0 & \delta_1 + \delta_2 & \delta_3 & \dots & \delta_{m-1} & \sigma_m \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 2\delta_1 + \delta_2 & \dots & \delta_{m-2} & \delta_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & (m-1)\delta_1 + \delta_2 \end{pmatrix}.$$

*Proof.* The proof is going by straightforward calculation of derivation property. □

From this proposition it is easy to see that the number of nil-independent outer derivations of the algebra  $N$  is equal to 3.

Now we consider solvable Leibniz algebra  $R = N + Q$ , where  $N = NF_n \oplus F_m^1$  and the dimension of  $Q$  is no more than three. Thus, we study the case  $\dim Q = 3$ , i.e.  $\dim R = n + m + 3$ . Several papers described solvable Leibniz algebras with a given nilradical [15-17]. The most interesting cases are when the complementary space of nilradical has the maximum possible. Namely, they have the second group of cohomology trivial. For this reason, we consider the case  $\dim Q = 3$ .

From the work [17], it follows that any solvable Leibniz algebra whose nilradical is  $NF_n$  has dimension  $n + 1$ . It is also known that any solvable Leibniz algebra whose nilradical is  $F_m^1$  has dimension either  $m + 1$  or  $m + 2$ , [16]. In work [18], it was found a unique  $(m + 2)$ -dimensional solvable Leibniz algebra with nilradical  $F_m^1$ . Then in the case of the solvable Leibniz algebras  $R$  with nilradical  $N = NF_n \oplus F_m^1$  and  $\dim Q = 3$ , there is only one possible case.

Taking into account Theorems 2.2 and 2.5, we have the following multiplication of the algebra  $R$ :

$$\begin{aligned}
 [e_i, e_1] &= e_{i+1}, \quad 1 \leq i \leq n - 1, & [f_i, f_1] &= f_{i+1}, \quad 2 \leq i \leq m - 1, \\
 [e_i, x] &= ie_i, \quad 1 \leq i \leq n, & [f_1, y] &= f_1, \\
 [x, e_1] &= -e_1, & [f_i, y] &= (i - 1)f_i, \quad 2 \leq i \leq m, \\
 & & [f_i, z] &= f_i, \quad 2 \leq i \leq m, \\
 & & [y, f_1] &= -f_1,
 \end{aligned} \tag{2}$$

where  $\{x, y, z\}$  be a basis of the space  $Q$ .

In the following theorem, solvable Leibniz algebras with nilradical  $N = NF_n \oplus F_m^1$  and  $\dim Q = 3$  are described.

**Theorem 3.2.** Any  $(n+m+3)$ -dimensional solvable Leibniz algebra with nilradical  $N = NF_n \oplus F_m^1$  is isomorphic to the following algebra:

$$L : \begin{cases} [e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq n - 1, & [f_i, f_1] = f_{i+1}, \quad 2 \leq i \leq m - 1, \\ [e_i, x] = ie_i, \quad 1 \leq i \leq n, & [f_1, y] = f_1, \\ [x, e_1] = -e_1, & [f_i, y] = (i - 1)f_i, \quad 2 \leq i \leq m, \\ & [f_i, z] = f_i, \quad 2 \leq i \leq m, \\ & [y, f_1] = -f_1. \end{cases}$$

*Proof.* From the above argumentations we have the multiplication (2) and we introduce the following denotations for the algebra  $R$  (according to the Mubarakzjanov’s method [10]):

$$\begin{aligned}
 [x, f_1] &= \sum_{i=1}^n \alpha_i^1 e_i + \sum_{i=1}^m \beta_i^1 f_i, & [x, y] &= \sum_{i=1}^n \lambda_i^1 e_i + \sum_{i=1}^m \mu_i^1 f_i, \\
 [x, f_2] &= \sum_{i=1}^n \alpha_i^2 e_i + \sum_{i=1}^m \beta_i^2 f_i, & [y, x] &= \sum_{i=1}^n \lambda_i^2 e_i + \sum_{i=1}^m \mu_i^2 f_i, \\
 [y, e_1] &= \sum_{i=1}^n \gamma_i^1 e_i + \sum_{i=1}^m \delta_i^1 f_i, & [x, z] &= \sum_{i=1}^n \lambda_i^3 e_i + \sum_{i=1}^m \mu_i^3 f_i, \\
 [z, e_1] &= \sum_{i=1}^n \gamma_i^2 e_i + \sum_{i=1}^m \delta_i^2 f_i, & [z, x] &= \sum_{i=1}^n \lambda_i^4 e_i + \sum_{i=1}^m \mu_i^4 f_i, \\
 [f_i, x] &= \sum_{j=1}^n a_{ij} e_j + \sum_{j=1}^m b_{ij} f_j, \quad 1 \leq i \leq m, \\
 [e_i, y] &= \sum_{j=1}^n c_{ij}^1 e_j + \sum_{j=1}^m d_{ij}^1 f_j, \quad 1 \leq i \leq n, \\
 [e_i, z] &= \sum_{j=1}^n c_{ij}^2 e_j + \sum_{j=1}^m d_{ij}^2 f_j, \quad 1 \leq i \leq n,
 \end{aligned}$$

From Leibniz identity it follows that  $[y, e_2] = [y, [e_1, e_1]] = 0$  and by induction we can easily find that  $[y, e_i] = 0$ , with  $2 \leq i \leq n$ . Analogously, we have  $[z, e_i] = 0$ , with  $2 \leq i \leq n$ .

We consider  $[x, f_3] = [x, [f_2, f_1]] = [[x, f_2], f_1] = \sum_{i=3}^m \beta_{i-1}^2 f_i$ . Similarly, and using the induction method, it is possible to show that  $[x, f_i] = \sum_{j=i}^m \beta_{j-i+2}^2 f_j$  for  $3 \leq i \leq m$ .

However, from the equality  $[x, [f_3, y]] = [[x, f_3], y]$ , it follows  $\beta_i^2 = 0$ , with  $2 \leq i \leq m - 1$ , i.e.

$$\begin{aligned} [x, f_i] &= 0, & 3 \leq i \leq m, \\ [x, f_2] &= \sum_{i=1}^n \alpha_i^2 e_i + \beta_1^2 f_1 + \beta_2^2 f_2 + \beta_m^2 f_m. \end{aligned}$$

Taking the following change  $x' = x - \mu_1^1 f_1 - \sum_{i=2}^m \frac{1}{i-1} \mu_i^1 f_i$ , we have  $[x', y] = \sum_{i=1}^n \lambda_i^1 e_i$ .

Next, making  $y' = y - \sum_{i=1}^n \frac{1}{i} \lambda_i^2 e_i$ , we obtain  $[y', x] = \sum_{i=1}^m \mu_i^2 f_i$ .

Finally, the change of basis  $z' = z - \sum_{i=1}^n \frac{1}{i} \lambda_i^4 e_i$  allows to obtain  $[z', x] = \sum_{i=1}^m \mu_i^4 f_i$ .

Let us apply the Leibniz identity on the following triples of elements:

$$\begin{aligned} \{x, x, y\} &\implies \lambda_i^1 = 0, 2 \leq i \leq n \implies [x, y] = \lambda_1^1 e_1; \\ \{y, y, x\} &\implies \mu_i^2 = 0, 2 \leq i \leq m \implies [y, x] = \mu_1^2 f_1; \\ \{z, y, x\} &\implies \mu_i^4 = 0, 1 \leq i \leq m \implies [z, x] = 0. \end{aligned}$$

We observe that  $[x, y] + [y, x] \in \text{Ann}_r(R)$ , thus  $[e_1, \lambda_1^1 e_1 + \mu_1^2 f_1] = 0$ , i.e.  $\lambda_1^1 = 0$ . From the identity  $0 = [f_2, [x, y] + [y, x]] = [f_2, \mu_1^2 f_1]$  we have  $\mu_1^2 = 0$ .

Analogously,  $[e_1, y] + [y, e_1]$  and  $[e_1, z] + [z, e_1] \in \text{Ann}_r(R)$ , then,

$$[e_1, [e_1, y] + [y, e_1]] = 0 \text{ and } [f_2, [e_1, y] + [y, e_1]] = 0,$$

so we have  $c_{11}^1 = -\gamma_1^1$ ,  $d_{11}^1 = -\delta_1^1$  and  $c_{11}^2 = -\gamma_1^2$ ,  $d_{11}^2 = -\delta_1^2$ .

Similarly, from  $[e_1, [x, z] + [z, x]] = 0$  and  $[f_2, [x, z] + [z, x]] = 0$  we deduce  $\lambda_1^3 = \mu_1^3 = 0$ .

Considering the following equalities:

$$\begin{aligned} [x, f_2] = [x, [f_2, y]] = -[x, [y, f_2]] = 0 &\implies \beta_1^2 = \beta_2^2 = \beta_m^2 = 0, \alpha_i^2 = 0, 1 \leq i \leq n, \\ [x, [e_1, f_1]] = -[[x, f_1], e_1] = 0 &\implies \alpha_i^1 = 0, 1 \leq i \leq n - 1, \\ [y, [e_1, f_1]] = [z, [e_1, f_1]] = 0 &\implies \delta_i^1 = \delta_i^2 = 0, 2 \leq i \leq m - 1, \end{aligned}$$

we immediately get

$$\begin{aligned} [x, f_1] &= \alpha_n^1 e_n + \sum_{i=1}^m \beta_i^1 f_i, \\ [y, e_1] &= \sum_{i=1}^n \gamma_i^1 e_i + \delta_1^1 f_1 + \delta_m^1 f_m, \\ [z, e_1] &= \sum_{i=1}^n \gamma_i^2 e_i + \delta_1^2 f_1 + \delta_m^2 f_m. \end{aligned}$$

As a result of  $[x, f_1] + [f_1, x] \in \text{Ann}_r(R)$ , we observe that  $a_{11} = 0$  and  $b_{11} = -\beta_1^1$ .

Since,  $[f_1, [x, f_1]] = 0$  and  $[f_1, [x, e_1]] = 0$  it follows that  $a_{1,i} = b_{1,j} = 0$ , for  $1 \leq i \leq n - 1, 2 \leq j \leq m - 1$ , i.e.

$$[f_1, x] = a_{1,n} e_n - \beta_1^1 f_1 + b_{1,m} f_m,$$

and from the equalities  $[f_i, [x, e_1]] = 0$ , for  $2 \leq i \leq m$ , it follows that  $a_{i,j} = 0$ , for  $2 \leq i \leq m$  and  $1 \leq j \leq n - 1$ , that is

$$[f_i, x] = a_{i,n} e_n + \sum_{j=1}^m b_{ij} f_j, 2 \leq i \leq m.$$

We know that  $[x, f_i] + [f_i, x] \in Ann_r(R)$  for  $2 \leq i \leq m$ , then  $0 = [f_2, [x, f_i] + [f_i, x]] = b_{i1} f_3$ , that is  $b_{i1} = 0$  for  $2 \leq i \leq m$ .

Now, we proceed by looking at the product of certain elements of  $Ann_r(R)$ . Summarizing the following identities

$$[e_1, [y, e_i] + [e_i, y]] = [f_2, [y, e_i] + [e_i, y]] = [e_1, [z, e_i] + [e_i, z]] = [f_2, [z, e_i] + [e_i, z]] = 0,$$

we have the following Leibniz brackets of the basic elements:

$$(3) \left\{ \begin{array}{ll} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-1, & [f_i, f_1] = f_{i+1}, & 2 \leq i \leq m-1, \\ [e_i, x] = i e_i, & 1 \leq i \leq n, & [f_1, y] = f_1, & \\ [x, e_1] = -e_1, & & [f_i, y] = (i-1) f_i, & 2 \leq i \leq m, \\ & & [f_i, z] = f_i, & 2 \leq i \leq m, \\ & & [y, f_1] = -f_1, & \\ [x, f_1] = \alpha_n^1 e_n + \sum_{i=1}^m \beta_i^1 f_i, & & [f_1, x] = a_{1,n} e_n - \beta_1^1 f_1 + b_{1,m} f_m, & \\ [y, e_1] = \sum_{i=1}^n \gamma_i^1 e_i + \delta_1^1 f_1 + \delta_m^1 f_m, & & [f_i, x] = a_{i,n} e_n + \sum_{j=2}^m b_{ij} f_j, & 2 \leq i \leq m, \\ [z, e_1] = \sum_{i=1}^n \gamma_i^2 e_i + \delta_1^2 f_1 + \delta_m^2 f_m, & & [e_1, y] = -\gamma_1^1 e_1 + \sum_{j=2}^n c_{1j}^1 e_j - \delta_1^1 f_1 + \sum_{j=2}^m d_{1j}^1 f_j, & \\ [x, z] = \sum_{i=2}^n \lambda_i^3 e_i + \sum_{i=2}^m \mu_i^3 f_i, & & [e_i, y] = \sum_{j=2}^n c_{ij}^1 e_j + \sum_{j=2}^m d_{ij}^1 f_j, & 2 \leq i \leq n, \\ [e_1, z] = -\gamma_1^2 e_1 + \sum_{j=2}^n c_{1j}^2 e_j - \delta_1^2 f_1 + \sum_{j=2}^m d_{1j}^2 f_j, & & [e_i, z] = \sum_{j=2}^n c_{ij}^2 e_j + \sum_{j=2}^m d_{ij}^2 f_j, & 2 \leq i \leq n. \end{array} \right.$$

The Leibniz identity on the following triples imposes further constraints on (3).

Leibniz identity	Constraint
$\{f_m, x, f_1\}$	$\Rightarrow b_{m,j} = 0, 2 \leq j \leq m-1$
$\{f_{m-1}, x, f_1\}$	$\Rightarrow a_{m,n} = 0, b_{m-1,m-1} = \beta_1^1 + b_{m,m}, b_{m-1,i} = 0, 2 \leq i \leq m-2$

As a result of the above constraints we observe that

$$[f_m, x] = b_{mm} f_m, \\ [f_{m-1}, x] = a_{m-1,n} e_n + (\beta_1^1 + b_{m,m}) f_{m-1} + b_{m-1,m} f_m.$$

By the induction on decrease  $i$  ( $2 \leq i \leq m$ ) and using the Leibniz identity for the elements  $\{f_i, x, f_1\}$  we get

$$[f_i, x] = ((m-i)\beta_1^1 + b_{mm}) f_i + \sum_{j=i+1}^m b_{i-j+m,m} f_j, \quad 3 \leq i \leq m, \\ [f_2, x] = a_{2,n} e_n + ((m-2)\beta_1^1 + b_{mm}) f_2 + \sum_{i=3}^m b_{m-i+2,m} f_i.$$

Considering the Leibniz identity on the triple  $\{e_1, y, e_1\}$  we get  $[e_2, y] = -2\gamma_1^1 e_2 + \sum_{i=3}^n c_{1,i-1}^1 e_i$ .

Also, using the equalities  $[e_i, y] = [[e_{i-1}, e_1], y] = [[e_{i-1}, y], e_1] - [e_{i-1}, [y, e_1]]$ , for  $3 \leq i \leq n$  and by induction method on  $i$ , we obtain that

$$[e_i, y] = -i\gamma_1^1 e_i + \sum_{j=i+1}^n c_{1,j-i+1}^1 e_j, \quad 2 \leq i \leq n.$$

Analogously, we can get  $[e_i, z] = -i\gamma_1^2 e_i + \sum_{j=i+1}^n c_{1,j-i+1}^2 e_j$ , for  $2 \leq i \leq n$ .

The Leibniz identity on the triples  $\{f_2, x, y\}$ ,  $\{f_1, x, y\}$  and  $\{x, f_1, y\}$ , (in this order) gives

$$\begin{aligned} [f_1, x] &= a_{1,n}e_n - \beta_1^1 f_1, \\ [f_2, x] &= a_{2,n}e_n + ((m-2)\beta_1^1 + b_{mm})f_2, \\ [f_i, x] &= ((m-i)\beta_1^1 + b_{mm})f_i, \quad 3 \leq i \leq m, \\ [x, f_1] &= \alpha_n^1 e_n + \beta_1^1 f_1 + \beta_2^1 f_2. \end{aligned}$$

Now, applying the Leibniz identity on the triples  $\{x, f_1, z\}$ ,  $\{y, x, f_1\}$ ,  $\{f_i, x, z\}$ ,  $\{f_i, x, y\}$ , with  $1 \leq i \leq m$ ,  $\{e_i, z, x\}$ ,  $\{e_i, y, x\}$ , with  $1 \leq i \leq n$ , (in this order) it follows

$$\begin{aligned} \alpha_n^1 \gamma_1^2 &= \beta_2^1 = \mu_i^3 = 0, \quad 2 \leq i \leq m-1, \\ a_{1n} &= 0, \\ c_{1i}^1 &= c_{1i}^2 = 0, \quad 2 \leq i \leq n-1, \\ a_{2n}(n\gamma_1^2 + 1) &= 0. \end{aligned}$$

From the above results we can simplify the family (3) as follows:

$$(4) : \left\{ \begin{array}{ll} [e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq n-1, & [f_i, f_1] = f_{i+1}, \quad 2 \leq i \leq m-1, \\ [e_i, x] = ie_i, \quad 1 \leq i \leq n, & [f_1, y] = f_1, \\ [x, e_1] = -e_1, & [f_i, y] = (i-1)f_i, \quad 2 \leq i \leq m, \\ & [f_i, z] = f_i, \quad 2 \leq i \leq m, \\ & [y, f_1] = -f_1, \\ [x, f_1] = \alpha_n^1 e_n + \beta_1^1 f_1, & [f_1, x] = -\beta_1^1 f_1, \\ [f_2, x] = a_{2n}e_n + ((m-2)\beta_1^1 + b_{mm})f_2, & \\ [y, e_1] = \sum_{i=1}^n \gamma_i^1 e_i + \delta_1^1 f_1 + \delta_m^1 f_m, & [f_i, x] = ((m-i)\beta_1^1 + b_{mm})f_i, \quad 3 \leq i \leq m, \\ [z, e_1] = \sum_{i=1}^n \gamma_i^2 e_i + \delta_1^2 f_1 + \delta_m^2 f_m, & [e_1, y] = -\gamma_1^1 e_1 + c_{1n}^1 e_n - \delta_1^1 f_1 + \sum_{j=2}^m d_{1j}^1 f_j, \\ [x, z] = \sum_{i=2}^n \lambda_i^3 e_i + \mu_m^3 f_m, & [e_i, y] = -i\gamma_1^1 e_i, \quad 2 \leq i \leq n \\ [e_1, z] = -\gamma_1^2 e_1 + c_{1n}^2 e_n - \delta_1^2 f_1 + \sum_{j=2}^m d_{1j}^2 f_j, & [e_i, z] = -i\gamma_1^2 e_i, \quad 2 \leq i \leq n \end{array} \right.$$

Finally, considering the Leibniz identity on the following triples we obtain:

Leibniz identity	Constraint
$\{x, z, e_1\}$	$\implies \delta_1^2(\beta_1^1 + 1) = 0, \lambda_j^3 = 0, 2 \leq j \leq n-2, \lambda_{n-1}^3 = \delta_1^2 \alpha_n^1 - c_{1n}^2, d_{1i}^2 = 0, 2 \leq i \leq m,$
$\{x, y, e_1\}$	$\implies c_{1n}^1 = \delta_1^1 \alpha_n^1, \delta_1^1(\beta_1^1 + 1) = 0, d_{1i}^1 = 0, 2 \leq i \leq m,$
$\{x, z, y\}$	$\implies \mu_m^3 = 0,$
$\{x, z, x\}$	$\implies \lambda_n^3 = 0, c_{1n}^2 = \delta_1^2 \alpha_n^1,$
$\{e_1, y, x\}$	$\implies c_{1n}^1 = 0,$
$\{e_1, z, x\}$	$\implies c_{1n}^2 = 0,$
$\{y, e_1, x\}$	$\implies \gamma_i^1 = 0, 2 \leq i \leq n, \delta_m^1(1 - b_{mm}) = 0,$
$\{z, e_1, x\}$	$\implies \gamma_i^2 = 0, 2 \leq i \leq n, \delta_m^2(1 - b_{mm}) = 0,$



Leibniz identity	Constraint
$\{y, e_1, y\}$	$\implies \delta_m^1(\gamma_1^1 + m - 1) = 0,$
$\{z, e_1, z\}$	$\implies \delta_m^2(\gamma_1^2 + 1) = 0,$
$\{f_2, e_1, y\}$	$\implies \delta_1^1 = 0,$
$\{f_2, e_1, z\}$	$\implies \delta_1^2 = 0,$
$\{y, e_1, z\}$	$\implies \delta_m^1(\gamma_1^2 + 1) = 0,$
$\{z, e_1, y\}$	$\implies \delta_m^2(\gamma_1^1 + m - 1) = 0,$
$\{x, f_1, x\}$	$\implies \alpha_n^1(\beta_1^1 + n) = 0,$
$\{x, f_1, y\}$	$\implies \alpha_n^1(n\gamma_1^1 + 1) = 0,$
$\{x, f_1, z\}$	$\implies \alpha_n^1\gamma_1^2 = 0.$

Summarizing the above results and renaming the following parameters  $a_{2n} = a, \alpha_n^1 = \alpha, \beta_1^1 = \beta, b_{mm} = b, \gamma_1^i = \gamma_i, \delta_m^i = \delta_i$  we can simplify the family (4) as follows:

$$(5) : \left\{ \begin{array}{ll} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-1, [f_i, f_1] = f_{i+1}, \quad 2 \leq i \leq m-1, \\ [x, e_1] = -e_1, & [x, f_1] = \alpha e_n + \beta f_1, \\ [y, e_1] = \gamma_1 e_1 + \delta_1 f_m, & [y, f_1] = -f_1, \\ [z, e_1] = \gamma_2 e_1 + \delta_2 f_m, & \\ [e_i, x] = i e_i, & 1 \leq i \leq n, [e_i, y] = -i \gamma_1 e_i, \quad 1 \leq i \leq n, \\ [f_1, x] = -\beta f_1, & [f_1, y] = f_1, \\ [f_2, x] = a e_n + ((m-2)\beta + b) f_2, & [f_i, y] = (i-1) f_i, \quad 2 \leq i \leq m, \\ [f_i, x] = ((m-i)\beta + b) f_i, & 3 \leq i \leq m, \\ [e_i, z] = -i \gamma_2 e_i, & 1 \leq i \leq n, \\ [f_i, z] = f_i, & 2 \leq i \leq m \end{array} \right.$$

with the relations:

$$\begin{aligned} \delta_k(b-1) &= \delta_k(\gamma_1 + m - 1) = \delta_k(\gamma_2 + 1) = 0, \quad k = 1, 2, \\ a(n\gamma_1 + 1) &= a(n\gamma_2 + 1) = 0, \\ \alpha(n\gamma_1 + 1) &= \alpha(\beta + n) = \alpha\gamma_2 = 0. \end{aligned}$$

Now, by performing a change of basis in the family (5)  $x' = x + \beta y - ((m-1)\beta + b)z$  we have:

$$\begin{aligned} [e_i, e_1] &= e_{i+1}, & 1 \leq i \leq n-1, [f_i, f_1] &= f_{i+1}, \quad 2 \leq i \leq m-1, \\ [x, e_1] &= -\theta e_1 + \delta f_m, & [x, f_1] &= \alpha e_n, \\ [y, e_1] &= \gamma_1 e_1 + \delta_1 f_m, & [y, f_1] &= -f_1, \\ [z, e_1] &= \gamma_2 e_1 + \delta_2 f_m, & & \\ [e_i, x] &= i \theta e_i & 1 \leq i \leq n, [e_i, y] &= -i \gamma_1 e_i, \quad 1 \leq i \leq n, \\ [f_2, x] &= a e_n, & [f_1, y] &= f_1, \\ & & [f_i, y] &= (i-1) f_i, \quad 2 \leq i \leq m, \\ [e_i, z] &= -i \gamma_2 e_i, \quad 1 \leq i \leq n, \\ [f_i, z] &= f_i, \quad 2 \leq i \leq m \end{aligned}$$

Since  $R_x$  is not nilpotent,  $\theta \neq 0$ . Thus we can assume  $\theta' = 1$  (making  $x' = \frac{1}{\theta}x$ ).

The Leibniz identity on the following triples imposes further constraints on the parameters.

Leibniz identity	Constraint
$\{x, f_1, x\}$	$\implies \alpha = 0,$
$\{y, e_1, x\}$	$\implies \delta_1 = 0,$
$\{z, e_1, x\}$	$\implies \delta_2 = 0,$
$\{x, e_1, x\}$	$\implies \delta = 0,$
$\{f_2, x, z\}$	$\implies a(n\gamma_2 + 1) = 0,$
$\{f_2, x, y\}$	$\implies a(n\gamma_1 + 1) = 0.$

We can distinguish the following cases:

- Case  $a \neq 0$ . The restrictions imply that  $\gamma_1 = \gamma_2 = -\frac{1}{n}$ . By performing a change of basis

$$f'_1 = f_1, f'_2 = nf_2 - ae_n, f'_i = nf_i, y' = y - \frac{1}{n}x, z' = z - \frac{1}{n}x$$

we obtain the algebra  $L$ .

- Case  $a = 0$ . Making the change  $y' = y + \gamma_1x, z' = z + \gamma_2x$ , we obtain the algebra  $L$ . □

## 4 Rigidity of the algebra $L$

In order to describe the second group of cohomology of the algebra  $L$  we need the description of its derivations. The general form of the derivations of  $L$  is given in the following proposition.

**Proposition 4.1.** *A derivation  $d$  of the algebra  $L$  has the following form:*

$$\begin{aligned} d_1(e_i) &= ie_i, \quad 1 \leq i \leq n, & d_3(f_1) &= f_1, \\ d_2(e_i) &= e_{i+1}, \quad 1 \leq i \leq n-1, & d_3(f_i) &= (i-2)f_i, \quad 3 \leq i \leq m, \\ d_2(x) &= -e_1, & d_4(f_i) &= f_i, \quad 2 \leq i \leq m, \\ & & d_5(f_i) &= f_{i+1}, \quad 2 \leq i \leq m-1, \\ & & d_5(y) &= -f_1. \end{aligned}$$

*Proof.* The proof is carried out by straightforward calculations of derivation properties. □

From Proposition 4.1 we conclude that  $\dim BL^2(L, L) = (m+n+3)^2 - 5$ . The general form of an element of the space  $ZL^2(R(L, L))$  is presented below.

**Proposition 4.2.**  $\dim ZL^2(L, L) = (m+n+3)^2 - 5$ .

*Proof.* Let  $\varphi \in ZL^2(L, L)$ . We set  $e_{n+m+1} := x, e_{n+m+2} := y, e_{n+m+3} := z, e_{n+i} := f_i, 1 \leq i \leq m$  and

$$\varphi(e_i, e_j) = \sum_{k=1}^{n+m+3} a_{i,j}^k e_k, \quad 1 \leq i, j \leq n+m+3.$$

For  $\varphi \in ZL^2(L, L)$  we shall verify equation (1). We consider  $b = c \in L$ , then we get  $[a, \varphi(b, b)] + \varphi(a, b^2) = 0$  for all  $a \in L$ .

If  $b = e_1$ , then we have

$$\varphi(e_i, e_2) = -ia_{1,1}^{n+m+1} e_i - a_{1,1}^1 e_{i+1}, \quad 1 \leq i \leq n,$$

$$\begin{aligned} \varphi(f_1, e_2) &= -a_{1,1}^{n+m+2} f_1, \quad \varphi(x, e_2) = a_{1,1}^1 e_1, \quad \varphi(y, e_2) = a_{1,1}^{n+1} f_1, \quad \varphi(z, e_2) = 0, \\ \varphi(f_i, e_2) &= -((i-1)a_{1,1}^{n+m+2} + a_{1,1}^{n+m+3}) f_i - a_{1,1}^{n+1} f_{i+1}, \quad 2 \leq i \leq m. \end{aligned}$$

From the multiplication table of the algebra  $L$  it is easy to see that  $\varphi(b, b) \in I_1 \oplus I_2$  for all  $b \in L$ ,  $b \neq e_1$ , where  $I_1 = \text{span}\{e_2, \dots, e_n\}$  and  $I_2 = \text{span}\{f_2, \dots, f_m\}$ .

If  $b, c \in I_1 \oplus I_2$ , then we obtain  $[a, \varphi(b, c)] = 0$  for all  $a \in L$ , and, consequently,  $\varphi(b, c) \in I_1 \oplus I_2$ , i.e.  $\varphi(I_i, I_j) \subseteq I_1 \oplus I_2$ ,  $1 \leq i, j \leq 2$ .

If  $a, b, c \in Q$ , (remember that  $L = N \oplus Q$  where  $Q$  is the complementary vector space of the nilradical  $N = NF_n \oplus F_m^1$ ), then we derive

$$\begin{aligned} a_{n+m+1, n+m+2}^1 &= -a_{n+m+2, n+m+1}^1, \\ a_{n+m+1, n+m+3}^1 &= -a_{n+m+3, n+m+1}^1, \\ a_{n+m+2, n+m+3}^{n+1} &= -a_{n+m+3, n+m+2}^{n+1} \end{aligned}$$

and

$$\begin{aligned} \varphi(x, x) &\in I_1, & \varphi(x, y) &\in F_m^1 \oplus \langle e_1 \rangle, & \varphi(x, z) &\in I_2 \oplus \langle e_1 \rangle, \\ \varphi(y, x) &\in NF_n \oplus \langle f_1 \rangle, & \varphi(y, y) &\in I_2, & \varphi(y, z) &\in F_m^1, \\ \varphi(z, x) &\in NF_n, & \varphi(z, y) &\in F_m^1, & \varphi(z, z) &\in I_2. \end{aligned}$$

From now on, we consider the equation (1) with combinations of the triples  $\{a, b, c\}$ , where  $a, b, c$  are the elements of  $L$ .

Triples	Relations
$\{x, x, e_i\}, \{x, y, e_i\}, \{x, e_i, x\}, i \geq 3$	$\Rightarrow \varphi(x, e_i) \in \langle e_1 \rangle, i \geq 3$
$\{x, x, f_1\}, \{x, y, f_1\}, \{x, f_1, x\}$	$\Rightarrow \varphi(x, f_1) = -a_{n+1,1}^2 e_1 + a_{n+m+1, n+1}^{n+1} f_1 + \sum_{i=3}^m a_{n+m+1, n+1}^{n+i} f_i,$
$\{x, y, f_i\}, \{x, f_i, y\}, 2 \leq j \leq m$	$\Rightarrow \varphi(x, f_i) \in \langle e_1 \rangle, 2 \leq i \leq m,$
$\{y, y, e_i\}, \{x, y, e_i\}, \{y, x, e_i\}, \{y, e_i, x\}, \{y, e_i, e_1\}$	$\Rightarrow \varphi(y, e_i) = a_{i-1,1}^{n+1} f_1, 3 \leq i \leq n,$
$\{z, z, f_i\}, \{y, z, f_i\}, \{z, x, f_i\}, \{z, f_i, y\}$	$\Rightarrow \varphi(z, f_i) = 0, 3 \leq i \leq m,$
$\{y, y, e_1\}, \{y, x, e_1\}$	$\Rightarrow \varphi(y, e_1) \in NF_n \oplus \langle f_1 \rangle,$
$\{y, y, f_i\}, \{y, x, f_i\}, \{y, f_i, y\}$	$\Rightarrow \varphi(y, f_i) \in \langle f_1 \rangle, 2 \leq i \leq m,$
$\{z, z, f_1\}, \{z, x, f_1\}, \{z, y, f_1\}$	$\Rightarrow \varphi(z, f_1) = a_{n+m+3, n+1}^{n+1} f_1 + \sum_{i=3}^m a_{n+m+3, n+1}^{n+i} f_i,$
$\{z, z, e_i\}, \{z, x, e_i\}, \{z, e_i, x\}$	$\Rightarrow \varphi(z, e_i) = 0, 3 \leq i \leq n,$
$\{z, x, e_1\}$	$\Rightarrow \varphi(z, e_1) = \sum_{i=1}^n a_{n+m+3,1}^i e_i,$
$\{f_i, x, z\}$	$\Rightarrow \varphi(f_i, x) \in NF_n \oplus \langle f_1 \rangle; \varphi(f_i, x) \in NF_n \oplus I_2, 2 \leq i \leq m,$
$\{e_i, y, e_j\}, 1 \leq i, j \leq n$	$\Rightarrow \varphi(I_1, I_1) \subseteq I_1; \varphi(e_i, y) \in I_1 \oplus F_m^1, 2 \leq i \leq n,$
$\{e_i, e_j, e_1\}, 1 \leq i, j \leq n,$	$\Rightarrow \varphi(e_i, e_j) = -i a_{j-1,1}^{n+m+1} e_i + (a_{j-2,1}^{n+m+1} - a_{j-1,1}^1) e_{i+1}$ $3 \leq j \leq n, 1 \leq i \leq n,$
$\{x, e_i, e_1\}, 2 \leq i \leq n-1$	$\Rightarrow \varphi(x, e_i) = (a_{i-1,1}^1 - a_{i-2,1}^{n+m+1}) e_1, 3 \leq i \leq n;$
$\{f_1, f_j, f_1\}$	$\Rightarrow \varphi(f_1, f_i) = -a_{n+i-1, n+1}^{m+n+2} f_1, 3 \leq i \leq m;$
$\{y, f_1, f_2\}, \{f_1, z, f_2\}$	$\Rightarrow \varphi(f_1, f_2) = a_{n+1, n+2}^{n+1} f_1;$
$\{y, e_1, f_1\}, \{y, f_1, e_1\}$	$\Rightarrow \varphi(f_1, e_1) = \sum_{s=1}^n a_{n+1,1}^s e_s + a_{n+1,1}^{n+1} f_1;$
$\{y, e_i, f_1\}, 2 \leq i \leq n$	$\Rightarrow \varphi(f_1, e_i) = -a_{i, n+1}^{n+1} f_1, 2 \leq i \leq n;$
$\{f_1, x, f_1\}, \{f_1, z, f_1\}$	$\Rightarrow \varphi(f_1, f_1) = \sum_{i=2}^m a_{n+1, n+1}^{n+i} f_i;$

Given the restrictions above and the cocycle property  $(d^2\varphi)(x, e_1, f_i) = 0$  for  $1 \leq i \leq m$ ,  $(d^2\varphi)(e_1, y, f_1) = 0$  and  $(d^2\varphi)(f_i, x, e_j) = 0, 1 \leq i \leq m, 1 \leq j \leq n$ , we derive

$$\begin{aligned} \varphi(e_1, f_1) &\in F_m^1 \oplus \langle e_1, e_2 \rangle, \quad \varphi(e_1, f_2) \in NF_n, \quad \varphi(e_1, f_i) \in \langle e_1, e_2 \rangle, \quad 3 \leq i \leq m, \\ \varphi(f_i, e_1) &\in NF_n \oplus I_2, \quad \varphi(f_i, e_j) \in I_2, \quad 2 \leq i \leq m, 2 \leq j \leq n. \end{aligned}$$

Analogously, we get

$$\{f_i, x, f_j\}, 2 \leq i, j \leq m \Rightarrow \varphi(I_2, I_2) \subseteq I_2;$$

$$\begin{aligned}
\{e_i, f_j, x\}, \{e_i, e_1, f_j\}, 2 \leq i \leq n, 2 \leq j \leq m &\Rightarrow \varphi(I_1, I_2) \subseteq I_1; \\
\{f_i, e_j, x\}, 2 \leq i \leq m, 2 \leq j \leq n &\Rightarrow \varphi(I_2, I_1) \subseteq I_2; \\
\{e_i, x, f_1\}, 2 \leq i \leq n &\Rightarrow \varphi(e_i, f_1) \in I_1 \bigoplus (F_m^1 \setminus \langle f_2 \rangle), 2 \leq i \leq n; \\
\{e_1, x, f_2\} &\Rightarrow \varphi(e_1, f_2) \in \langle e_1, e_2 \rangle; \\
\{f_i, y, e_1\}, 2 \leq i \leq m &\Rightarrow \varphi(f_i, y) \in L \setminus \langle x \rangle, 2 \leq i \leq m; \\
\{e_i, z, e_1\}, 2 \leq i \leq n &\Rightarrow \varphi(e_i, z) \in NF_n \bigoplus I_2, 2 \leq i \leq n.
\end{aligned}$$

Summarizing the above discussion and from  $(d^2\varphi)(f_i, f_1, f_1) = 0, 2 \leq i \leq m, (d^2\varphi)(f_i, f_j, z) = 0, 1 \leq i, j \leq m, (d^2\varphi)(f_i, f_j, y) = 0, 1 \leq i, j \leq m$  and  $(d^2\varphi)(y, y, f_1) = 0$  we conclude:

$$\begin{aligned}
\varphi(f_m, e_1) &= \sum_{s=1}^n a_{n+m,1}^s e_s + a_{n+m,1}^{n+m} f_m, \\
\varphi(f_i, f_1) &= -\sum_{s=1}^{n-1} a_{n+i+1,1}^{s+1} e_s + a_{n+i,n+1}^n e_n + \sum_{s=1}^m a_{n+i,n+1}^{n+s} f_s + a_{n+i+1,1}^1 x + a_{n+i,n+1}^{n+m+2} y + a_{n+i,n+1}^{n+m+3} z, 2 \leq i \leq m-1, \\
\varphi(f_m, f_1) &= -a_{n+m,n+1}^n e_n + \sum_{i=1}^m a_{n+m,n+1}^{n+s} f_s + a_{n+m,n+1}^{n+m+2} y + a_{n+m,n+1}^{n+m+3} z, \\
\varphi(f_1, f_1) &= \sum_{i=3}^m a_{n+1,n+1}^{n+i} f_i, \\
\varphi(f_3, z) &= \sum_{s=1}^n a_{n+2,n+1}^s e_s + (a_{n+2,n+1}^{n+1} - a_{n+2,n+m+3}^{n+m+2}) f_1 + (a_{n+1,n+m+3}^{n+m+2} + a_{n+1,n+m+3}^{n+m+3}) f_2 + \\
&\quad + (a_{n+2,n+m+3}^{n+2} + a_{n+1,n+m+3}^{n+1}) f_3 + \sum_{s=4}^m a_{n+2,n+m+3}^{n+s-1} f_s + a_{n+2,n+1}^{n+m+1} x + a_{n+2,n+1}^{n+m+2} y + a_{n+2,n+1}^{n+m+3} z, \\
\varphi(f_i, z) &= \sum_{s=1}^n a_{n+i-1,n+1}^s e_s + (a_{n+i-1,n+1}^{n+1} - a_{n+i-2,n+1}^{n+m+2}) f_1 + (i-2) \left( \frac{i-1}{2} a_{n+1,n+m+3}^{n+m+2} + a_{n+1,n+m+3}^{n+m+3} \right) f_{i-1} + \\
&\quad + \left( a_{n+2,n+m+3}^{n+2} + (i-2) a_{n+1,n+m+3}^{n+1} \right) f_i + \sum_{s=i+1}^m a_{n+2,n+m+3}^{n+s-i+2} f_s + a_{n+i-1,n+1}^{n+m+1} x + a_{n+i-1,n+1}^{n+m+2} y + \\
&\quad + a_{n+i-1,n+1}^{n+m+3} z, 4 \leq i \leq m, \\
\varphi(f_i, f_2) &= (a_{n+2,n+2}^{n+2} + (i-2) a_{n+1,n+2}^{n+1}) f_i + a_{n+2,n+2}^{n+3} f_{i+1}, 2 \leq i \leq m, \\
\varphi(f_i, f_3) &= -(i-1) a_{n+2,n+1}^{n+m+2} + a_{n+2,n+1}^{n+m+3} f_i - (a_{n+1,n+2}^{n+1} + a_{n+2,n+1}^{n+1}) f_{i+1}, 2 \leq i \leq m, \\
\varphi(f_i, f_j) &= -(i-1) a_{n+j-1,n+1}^{n+m+2} + a_{n+j-1,n+1}^{n+m+3} f_i + (a_{n+j-2,n+1}^{n+m+2} - a_{n+j-1,n+1}^{n+m+1}) f_{i+1} + \\
&\quad + a_{n+i,n+j}^{n+i+j-1} f_{i+j-1}, 4 \leq j \leq m, 2 \leq i \leq m, \\
\varphi(f_1, y) &= -\sum_{i=1}^n a_{n+m+2,n+1}^s e_s - a_{n+m+2,n+1}^{n+1} f_1 + \sum_{s=3}^{m-1} (s-2) a_{n+1,n+1}^{n+s+1} f_s + a_{n+1,n+m+2}^{n+m} f_m - \\
&\quad - a_{n+m+2,n+1}^{n+m+1} x - a_{n+m+2,n+1}^{n+m+2} y - a_{n+m+2,n+1}^{n+m+3} z, \\
\varphi(f_2, y) &= \sum_{s=1}^n a_{n+2,n+m+2}^s e_s + \sum_{s=2}^m a_{n+2,n+m+2}^{n+s} f_s + a_{n+2,n+m+2}^{n+m+1} x - a_{n+1,n+2}^{n+1} y + (a_{n+1,n+2}^{n+1} - a_{n+2,n+2}^{n+2}) z, \\
\varphi(f_3, y) &= -2 \sum_{s=1}^{n-1} a_{n+3,1}^{s+1} e_s + 2a_{n+2,n+1}^n e_n + (a_{n+2,n+1}^{n+1} + a_{n+1,n+2}^{n+1}) f_1 + \\
&\quad + (a_{n+2,n+1}^{n+2} + a_{n+1,n+m+2}^{n+m+2} + a_{n+1,n+m+2}^{n+m+3}) f_2 + (a_{n+2,n+m+2}^{n+2} - a_{n+m+2,n+1}^{n+1}) f_3 + \\
&\quad + \sum_{s=4}^m (a_{n+2,n+m+2}^{n+s-1} - (s-3) a_{n+2,n+1}^{n+s}) f_s + 2a_{n+3,1}^1 x + 2a_{n+2,n+1}^{n+m+2} y + 2a_{n+2,n+1}^{n+m+3} z, \\
\varphi(f_i, y) &= -(i-1) \sum_{s=1}^{n-1} a_{n+i,1}^{s+1} e_s + (i-1) a_{n+i-1,n+1}^n e_n + (i-2) (a_{n+i-1,n+1}^{n+1} - a_{n+i-2,n+1}^{n+m+2}) f_1 + \\
&\quad + \sum_{s=2}^{i-2} (i-s) \left( \sum_{t=1}^{s-1} a_{n+i+t-s,n+1}^{n+t+1} \right) f_s + \left( \sum_{t=2}^{i-1} a_{n+t,n+1}^{n+t} - \frac{(i-2)(i-1)}{2} a_{n+m+2,n+1}^{n+m+2} - (i-2) a_{n+m+2,n+1}^{n+m+3} \right) f_{i-1} + \\
&\quad + (a_{n+2,n+m+2}^{n+2} - (i-2) a_{n+m+2,n+1}^{n+1}) f_i + \sum_{s=i+1}^m (a_{n+2,n+m+2}^{n+s-i+2} - (s-i) \sum_{t=1}^{i-2} a_{n+i-t,n+1}^{n+s-t+1}) f_s + \\
&\quad + (i-1) a_{n+i,1}^1 x + (i-1) a_{n+i-1,n+1}^{n+m+2} y + (i-1) a_{n+i-1,n+1}^{n+m+3} z, 4 \leq i \leq m, \\
\varphi(y, y) &= \sum_{i=2}^{m-2} (i-1) (a_{n+m+2,n+1}^{n+i+1} - a_{n+1,n+1}^{n+i+2}) f_i + (m-2) a_{n+m+2,n+1}^{n+m} - a_{n+1,n+m+2}^{n+m} f_{m-1} + a_{n+m+2,n+m+2}^{n+m} f_m.
\end{aligned}$$

and the following restrictions

$$\begin{aligned}
a_{n+1,n+m+3}^{n+i} &= a_{n+1,n+1}^{n+i+1}, & 2 \leq i \leq m-1, \\
a_{n+2,n+m+3}^{n+m+2} &= -a_{n+1,n+2}^{n+1}, & a_{n+2,n+m+3}^{n+1} = -a_{n+2,n+2}^{n+3}, & a_{n+2,n+m+3}^{n+m+3} = a_{n+1,n+2}^{n+1} - a_{n+2,n+2}^{n+2}, \\
a_{n+i,n+2}^{n+i+1} &= a_{n+2,n+2}^{n+3}, & 3 \leq i \leq m-1, & a_{n+i,n+3}^{n+i+2} = 0, & 2 \leq i \leq m-2.
\end{aligned}$$

Analogously, considering the equation (1) for the bellow listed triples we obtain the corresponding relations

Triples	Relations
$\{z, z, f_2\}, \{f_2, z, f_2\}, \{z, f_2, a\},$ where $a \in \{x, z, e_1, f_1\},$	$\Rightarrow \varphi(z, f_2) = 0,$
$\{z, z, f_1\}, \{z, z, y\}$	$\Rightarrow \varphi(z, z) = \sum_{i=2}^{m-1} a_{n+m+3, n+1}^{n+i+1} f_i + \frac{1}{m-1} a_{n+m+3, n+m+2}^{n+m} f_m,$ $\varphi(z, y) = a_{n+m+3, n+m+2}^{n+1} f_1 + \sum_{i=2}^{m-1} (i-1) a_{n+m+3, n+1}^{n+i+1} f_i + a_{n+m+3, n+m+2}^{n+m} f_m;$
$\{e_1, e_n, e_1\}$	$\Rightarrow a_{n,1}^{n+m+1} = 0, a_{n,1}^1 = a_{n-1,1}^{n+m+1};$
$\{f_1, e_1, y\}$	$\Rightarrow a_{1, n+m+2}^{n+m+2} = a_{1, n+m+2}^{n+m+3} = 0, a_{n+m+2, n+1}^{n+m+1} = -a_{n+1,1}^1,$ $a_{n+m+2, n+1}^s = a_{n+1,1}^{s+1}, 1 \leq s \leq n-1;$
$\{f_i, e_1, y\}, 2 \leq i \leq m-1$	$\Rightarrow a_{n+2, n+m+2}^{n+m+1} = a_{n+2,1}^1, a_{n+2, n+m+2}^s = -a_{n+2,1}^{s+1}, 1 \leq s \leq n-1,$ $a_{1, n+m+2}^{n+1} = a_{n+2,1}^{n+3}, a_{n+i,1}^{n+s} = 0, s \in [2; i-1] \cup [i+2; m],$ $a_{n+i,1}^{n+i+1} = a_{n+2,1}^{n+3}, 2 \leq i \leq m-1;$
$\{x, e_1, y\}$	$\Rightarrow a_{1, n+m+2}^{n+m+1} = 0, a_{1, n+m+2}^2 = a_{n+m+2, n+m+1}^1, a_{1, n+m+2}^s = 0, 3 \leq s \leq n,$ $a_{1, n+m+2}^{n+s} = (s-1) a_{n+m+1,1}^{n+s}, 2 \leq s \leq m, a_{n+m+1,1}^{n+1} = a_{n+2,1}^{n+3}.$

Now, considering the equations  $(d^2\varphi)(x, x, e_1) = (d^2\varphi)(x, e_1, f_1) = (d^2\varphi)(f_1, x, e_1) = (d^2\varphi)(e_1, f_1, y) = (d^2\varphi)(y, f_1, x) = (d^2\varphi)(y, x, f_1) = (d^2\varphi)(f_2, x, e_1) = (d^2\varphi)(f_i, x, y) = 0,$  for  $2 \leq i \leq m$  and we can derive

$$\begin{aligned} \varphi(x, x) &= \sum_{i=2}^{n-1} (i a_{n+m+1,1}^{i+1} - a_{1, n+m+1}^{i+1}) e_i + a_{n+m+1, n+m+1}^n e_n, \\ \varphi(e_1, x) &= -a_{n+m+1,1}^1 e_1 + a_{n+m+1,1}^2 e_2 + \sum_{i=3}^n a_{1, n+m+1}^i e_i - a_{n+2,1}^{n+3} f_1 - \sum_{i=2}^m a_{n+m+1,1}^{n+i} f_i - \\ &\quad - a_{n+m+1,1}^{n+m+1} x - a_{n+m+1,1}^{n+m+2} y - a_{n+m+1,1}^{n+m+3} z, \\ \varphi(e_1, f_1) &= -a_{n+1,1}^1 e_1 + a_{n+1,1}^2 e_2 - a_{n+1,1}^{n+1} f_1 + \sum_{i=3}^m a_{n+m+1,1}^{n+i-1} f_i, \\ \varphi(f_1, x) &= \sum_{i=1}^{n-1} i a_{n+1,1}^{i+1} e_i + n a_{n+m+2, n+1}^n e_n - a_{n+m+1, n+1}^{n+1} f_1, \\ \varphi(x, e_1) &= \sum_{i=1}^n a_{n+m+1,1}^i e_i + a_{n+2,1}^{n+3} f_1 + \sum_{i=2}^m a_{n+m+1,1}^{n+i} f_i + a_{n+m+1,1}^{n+m+1} x + a_{n+1,1}^{n+1} y + (a_{n+2,1}^{n+2} - a_{n+1,1}^{n+1}) z, \\ \varphi(f_2, x) &= \sum_{i=1}^{n-1} i a_{n+2,1}^{i+1} e_i + a_{n+2, n+m+1}^n e_n + a_{n+2, n+m+1}^{n+2} f_2 + a_{n+2, n+m+1}^{n+3} f_3, \\ \varphi(f_i, x) &= \sum_{s=1}^{n-1} s a_{n+i,1}^{s+1} e_s - n a_{n+i-1, n+1}^n e_n + a_{n+i, n+m+1}^{n+i} f_i + a_{n+2, n+m+1}^{n+3} f_{i+1}, 3 \leq i \leq m. \end{aligned}$$

and the following restrictions:

$$\begin{aligned} a_{n+m+1,1}^{n+m+2} &= a_{n+1,1}^{n+1} \\ a_{n+2, n+m+2}^n &= -\frac{1}{n} a_{n+2, n+m+1}^n, \\ a_{n+m+1, n+m+2}^{n+1} &= a_{n+2, n+m+1}^{n+3}. \end{aligned}$$

Analogously, applying the same arguments:

$$\begin{aligned}
 \{x, y, f_1\}, \{y, x, y\} &\Rightarrow \varphi(x, y) = -a_{n+m+2, n+m+1}^1 e_1 + a_{n+2, n+m+1}^{n+3} f_1 + \sum_{i=2}^{m-1} (i-1) a_{n+m+1, n+1}^{n+i+1} f_i + \\
 &\quad + a_{n+m+1, n+m+2}^{n+m} f_m, \\
 \{y, e_1, y\} &\Rightarrow a_{n+m+2, n+m+1}^{n+1} = -a_{n+2, n+m+1}^{n+3}; \\
 &\Rightarrow a_{n+m+2, 1}^{n+1} = -a_{n+2, 1}^{n+3}; \\
 \{f_1, e_1, z\}, \{y, z, f_1\}, \{f_1, z, y\} &\Rightarrow \varphi(f_1, z) = -a_{n+m+3, n+1}^{n+1} f_1 + \sum_{s=2}^{m-1} a_{n+1, n+1}^{n+s+1} f_s + \frac{1}{m-2} a_{n+1, n+m+2}^{n+m} f_m, \\
 &\quad a_{n+m+2, n+m+3}^{n+i} = a_{n+m+2, n+1}^{n+i+1} - a_{n+1, n+1}^{n+i+2}, \quad 2 \leq i \leq m-2, \\
 &\quad a_{n+m+2, n+m+3}^{n+m} = a_{n+m+2, n+1}^{n+m} - a_{n+1, n+m+3}^{n+m}, \quad a_{1, n+m+3}^{n+m} = 0, \\
 &\quad a_{n+m+2, n+1}^{n+2} = a_{n+1, n+1}^{n+3}; \\
 \{e_1, f_1, z\} &\Rightarrow a_{1, n+m+3}^{n+i} = a_{n+m+1, 1}^{n+i}, \quad 2 \leq i \leq m-1; \\
 \{x, z, f_1\} &\Rightarrow a_{n+m+1, n+m+3}^{n+i} = a_{n+m+1, n+1}^{n+i+1}, \quad 2 \leq i \leq m-1; \\
 \{x, z, y\} &\Rightarrow a_{n+m+1, n+m+3}^{n+m} = \frac{1}{m-1} a_{n+m+1, n+m+2}^{n+m}; \\
 \{f_1, e_i, e_1\}, 2 \leq i \leq n &\Rightarrow \varphi(f_1, e_i) = -a_{i-1, 1}^{n+m+2} f_1, \quad 3 \leq i \leq n, \\
 &\quad a_{n, 1}^{n+m+2} = 0; \\
 \{f_2, e_2, y\} &\Rightarrow a_{2, n+m+2}^{n+1} = -a_{1, 1}^{n+1}, \\
 \{f_2, e_1, z\}, \{f_2, z, x\}, \{f_2, z, y\} &\Rightarrow a_{1, n+m+3}^{n+1} = a_{1, n+m+3}^{n+m+3} = 0, \\
 &\quad \varphi(f_2, z) = -\sum_{s=1}^{n-1} a_{n+2, 1}^{s+1} e_s - \frac{1}{n} a_{n+2, n+m+1}^{n+1} e_n - a_{n+2, n+2}^{n+3} f_1 + a_{n+2, n+m+3}^{n+2} f_2 + \\
 &\quad + a_{n+m+3, n+m+2}^{n+1} f_3 + a_{n+1, 1}^1 x - a_{n+1, n+2}^{n+1} y + (a_{n+1, n+2}^{n+1} - a_{n+2, n+2}^{n+2}) z.
 \end{aligned}$$

To complete the proof, we use the following equations in order to obtain some expressions for 2-cocycle  $\varphi$  and some restrictions.

$$\begin{aligned}
 (d^2\varphi)(f_j, e_1, f_1) = (d^2\varphi)(f_j, e_i, e_1) = 0, \quad 2 \leq j \leq m, \quad 2 \leq i \leq n, &\Rightarrow \varphi(f_j, e_i), \quad 2 \leq j \leq m, \quad 1 \leq i \leq n, \\
 \left. \begin{aligned}
 (d^2\varphi)(e_i, f_j, f_1) = (d^2\varphi)(e_i, f_2, x) = 0 \quad 2 \leq i \leq n, \quad 2 \leq j \leq m-1, \\
 (d^2\varphi)(e_i, f_2, y) = 0 \quad 1 \leq i \leq n, \\
 (d^2\varphi)(e_1, f_i, x) = (d^2\varphi)(x, f_i, e_1) = 0 \quad 3 \leq i \leq m,
 \end{aligned} \right\} &\Rightarrow \varphi(e_i, f_j), \quad 1 \leq i \leq n, \quad 2 \leq j \leq m, \\
 (d^2\varphi)(x, f_i, x) = (d^2\varphi)(f_i, y, f_j) = 0, \quad 2 \leq i, j \leq m &\Rightarrow \varphi(x, f_i), \quad \varphi(y, f_i), \quad 2 \leq i \leq m, \\
 &\quad a_{n+i, n+j}^{n+i+j-1} = 0, \quad 2 \leq i \leq m, \quad 4 \leq j \leq m, \\
 (d^2\varphi)(e_i, f_1, e_1) = 0, \quad 1 \leq i \leq n, &\Rightarrow \varphi(e_i, f_1), \quad 2 \leq i \leq n, \\
 &\quad a_{n, 1}^{n+s} = 0, \quad 2 \leq s \leq m-1.
 \end{aligned}$$

Similarly, we get

$$\begin{aligned}
 \{y, x, e_1\} &\Rightarrow a_{n+m+2, n+m+1}^i = i a_{n+m+2, 1}^{i+1}, \quad 1 \leq i \leq n-1; \\
 \{z, x, e_1\} &\Rightarrow a_{n+m+3, n+m+1}^i = i a_{n+m+3, 1}^{i+1}, \quad 1 \leq i \leq n-1; \\
 \{y, z, y\} &\Rightarrow a_{n+m+2, n+m+3}^{n+m} = \frac{1}{m-1} a_{n+m+2, n+m+2}^{n+m}; \\
 \{e_i, z, x\}, 1 \leq i \leq n &\Rightarrow a_{1, n+m+3}^{n+m} = a_{n+m+1, 1}^{n+m}, \quad a_{1, n+m+3}^2 = a_{n+m+3, 1}^2, \quad a_{1, n+m+3}^i = 0, \quad 3 \leq i \leq n; \\
 &\quad a_{i, n+m+1}^{n+s} = i a_{i-1, 1}^{n+s}, \quad 2 \leq s \leq m, \quad 2 \leq i \leq n; \\
 \{x, y, e_1\} &\Rightarrow a_{1, n+m+2}^1 = -a_{n+m+2, 1}^1; \\
 \{x, z, e_1\} &\Rightarrow a_{1, n+m+3}^1 = -a_{n+m+3, 1}^1.
 \end{aligned}$$

Given the restrictions above, by applying the multiplication of the algebra  $L$  and checking the general condition of cocycle for the other basis elements, we get a general form of the 2-cocycle  $\varphi$ .

$$\begin{aligned}
 \varphi(e_i, e_1) &= \sum_{s=1}^n a_{i,1}^s e_s + \sum_{s=1}^m a_{i,1}^{n+s} f_s + a_{i,1}^{n+m+1} x + a_{i,1}^{n+m+2} y + a_{i,1}^{n+m+3} z, \quad 1 \leq i \leq n-1, \\
 \varphi(e_n, e_1) &= a_{n-1,1}^{n+m+1} e_1 + \sum_{i=2}^{n-1} \left( a_{n-i,1}^{n+m+1} - \sum_{s=1}^{i-1} a_{n-i+s,1}^s \right) e_i + a_{n,1}^{n+m+1} e_n, \\
 \varphi(f_1, e_1) &= \sum_{s=1}^n a_{n+1,1}^s e_s + a_{n+1,1}^{n+1} f_1, \\
 \varphi(f_i, e_1) &= \sum_{s=1}^n a_{n+i,1}^s e_s + (a_{n+2,1}^{n+2} + (i-2)a_{n+1,1}^{n+1}) f_i + a_{n+2,1}^{n+3} f_{i+1}, \quad 2 \leq i \leq m, \\
 \varphi(x, e_1) &= \sum_{i=1}^n a_{n+m+1,1}^i e_i + a_{n+2,1}^{n+3} f_1 + \sum_{i=2}^m a_{n+m+1,1}^{n+i} f_i + \left( \frac{2}{(n+2)(n-1)} \sum_{i=1}^n a_{i,1}^i \right) x + a_{n+1,1}^{n+1} y + \\
 &\quad + (a_{n+2,1}^{n+2} - a_{n+1,1}^{n+1}) z, \\
 \varphi(y, e_1) &= \sum_{i=1}^n a_{n+m+2,1}^i e_i - a_{n+2,1}^{n+3} f_1, \\
 \varphi(z, e_1) &= \sum_{i=1}^n a_{n+m+3,1}^i e_i, \\
 \varphi(e_i, e_2) &= -i a_{1,1}^{n+m+1} e_i - a_{1,1}^1 e_{i+1}, \quad 1 \leq i \leq n, \\
 \varphi(e_i, e_j) &= -i a_{j-1,1}^{n+m+1} e_i + (a_{j-2,1}^{n+m+1} - a_{j-1,1}^1) e_{i+1}, \quad 3 \leq j \leq n, \quad 1 \leq i \leq n, \\
 \varphi(f_1, e_i) &= -a_{i-1,1}^{n+m+2} f_1, \quad 2 \leq i \leq n, \\
 \varphi(f_i, e_j) &= -((i-1)a_{j-1,1}^{n+m+2} + a_{j-1,1}^{n+m+3}) f_i - a_{j-1,1}^{n+1} f_{i+1}, \quad 2 \leq j \leq n, \quad 2 \leq i \leq m, \\
 \varphi(x, e_2) &= a_{1,1}^1 e_1, \\
 \varphi(x, e_i) &= (a_{i-1,1}^1 - a_{i-2,1}^{n+m+1}) e_1, \quad 3 \leq i \leq n, \\
 \varphi(y, e_i) &= a_{i-1,1}^{n+1} f_1, \quad 2 \leq i \leq n, \\
 \varphi(e_1, f_1) &= -a_{n+1,1}^1 e_1 + a_{n+1,1}^2 e_2 - a_{n+1,1}^{n+1} f_1 + \sum_{i=3}^m a_{n+m+1,1}^{n+i-1} f_i, \\
 \varphi(e_i, f_1) &= -i a_{n+1,1}^1 e_i + a_{n+1,1}^2 e_{i+1} + a_{i-1,1}^{n+m+2} f_1 - \sum_{s=3}^m a_{i-1,1}^{n+s-1} f_s, \quad 2 \leq i \leq n, \\
 (6) : \quad \varphi(f_1, f_1) &= \sum_{i=3}^m a_{n+1, n+1}^{n+i} f_i, \\
 \varphi(f_i, f_1) &= -\sum_{s=1}^{n-1} a_{n+i+1,1}^{s+1} e_s + a_{n+i, n+1}^n e_n + \sum_{s=1}^m a_{n+i, n+1}^{n+s} f_s + a_{n+i+1,1}^1 x + a_{n+i, n+1}^{n+m+2} y + \\
 &\quad + a_{n+i, n+1}^{n+m+3} z, \quad 2 \leq i \leq m-1, \\
 \varphi(f_m, f_1) &= a_{n+m-1, n+1}^{n+m+2} f_1 - \sum_{i=3}^{m-1} \left( \sum_{s=1}^{i-2} a_{n+m-s, n+1}^{n+i-s} \right) f_i + a_{n+m, n+1}^{n+m} f_m, \\
 \varphi(x, f_1) &= -a_{n+1,1}^2 e_1 + a_{n+m+1, n+1}^{n+1} f_1 + \sum_{i=3}^m a_{n+m+1, n+1}^{n+i} f_i, \\
 \varphi(y, f_1) &= \sum_{i=1}^{n-1} a_{n+1,1}^{i+1} e_i + a_{n+m+2, n+1}^n e_n + a_{n+m+2, n+1}^{n+1} f_1 + a_{n+1, n+1}^{n+3} f_2 + \sum_{i=3}^m a_{n+m+2, n+1}^{n+i} f_i - \\
 &\quad a_{n+1,1}^1 x + a_{n+m+2, n+1}^{n+m+2} y + \frac{1}{m-1} \left( \sum_{i=2}^m a_{n+i, n+1}^{n+i} - \frac{m(m-1)}{2} a_{n+m+2, n+1}^{n+m+2} \right) z, \\
 \varphi(z, f_1) &= a_{n+m+3, n+1}^{n+1} f_1 + \sum_{i=3}^m a_{n+m+3, n+1}^{n+i} f_i, \\
 \varphi(e_i, f_j) &= -i a_{n+j,1}^1 e_i + a_{n+j,1}^2 e_{i+1}, \quad 2 \leq j \leq m, \quad 1 \leq i \leq n, \\
 \varphi(f_1, f_2) &= a_{n+1, n+2}^{n+1} f_1, \\
 \varphi(f_i, f_2) &= ((i-2)a_{n+1, n+2}^{n+1} + a_{n+2, n+2}^{n+2}) f_i + a_{n+2, n+2}^{n+3} f_{i+1}, \quad 2 \leq i \leq m, \\
 \varphi(f_1, f_i) &= -a_{n+i-1, n+1}^{m+n+2} f_1, \quad 3 \leq i \leq m, \\
 \varphi(f_i, f_3) &= -((i-1)a_{n+2, n+1}^{n+m+2} + a_{n+2, n+1}^{n+m+3}) f_i - (a_{n+1, n+2}^{n+1} + a_{n+2, n+1}^{n+1}) f_{i+1}, \quad 2 \leq i \leq m, \\
 \varphi(f_i, f_j) &= -((i-1)a_{n+j-1, n+1}^{n+m+2} + a_{n+j-1, n+1}^{n+m+3}) f_i + (a_{n+j-2, n+1}^{n+m+2} - a_{n+j-1, n+1}^{n+1}) f_{i+1}, \quad 4 \leq j \leq m, \quad 2 \leq i \leq m, \\
 \varphi(x, f_i) &= -a_{n+i,1}^2 e_1, \quad 2 \leq i \leq m, \\
 \varphi(y, f_2) &= -a_{n+2, n+2}^{n+3} f_1, \\
 \varphi(y, f_3) &= (a_{n+1, n+2}^{n+1} + a_{n+2, n+1}^{n+1}) f_1, \\
 \varphi(y, f_i) &= (a_{n+i-1, n+1}^{n+1} - a_{n+i-2, n+1}^{n+m+2}) f_1, \quad 4 \leq i \leq m, \\
 \varphi(e_1, x) &= -a_{n+m+1,1}^1 e_1 + a_{n+m+1,1}^2 e_2 + \sum_{i=3}^n a_{1, n+m+1}^i e_i - a_{n+2,1}^{n+3} f_1 - \sum_{i=2}^m a_{n+m+1,1}^{n+i} f_i -
 \end{aligned}$$

$$\begin{aligned}
 & -\left(\frac{2}{(n+2)(n-1)} \sum_{i=1}^n a_{i,1}^i\right)x - a_{n+1,1}^{n+1}y + (a_{n+1,1}^{n+1} - a_{n+2,1}^{n+2})z, \\
 \varphi(e_2, x) &= a_{1,1}^1 e_1 - 2a_{n+m+1,1}^1 e_2 + (a_{n+m+1,1}^2 - a_{1,1}^3) e_3 + \sum_{s=4}^n (a_{1,n+m+1}^{s-1} - (s-2)a_{1,1}^s) e_s + 2 \sum_{s=1}^m a_{1,1}^{n+s} f_s + \\
 & \quad + 2a_{1,1}^{n+m+1} x + 2a_{1,1}^{n+m+2} y + 2a_{1,1}^{n+m+3} z, \\
 \varphi(e_i, x) &= (i-1)(a_{i-1,1}^1 - a_{i-2,1}^{n+m+1}) e_1 + \sum_{s=2}^{i-2} (i-s) \left( \sum_{t=1}^s a_{i-s+t-1,1}^t - a_{i-s-1,1}^{n+m+1} \right) e_s + \\
 & \quad + \left( \frac{(n+i)(n-i+1)}{(n+2)(n-1)} \sum_{t=1}^{i-1} a_{t,1}^t - \frac{(i+1)(i-2)}{(n+2)(n-1)} \sum_{t=i}^n a_{t,1}^t \right) e_{i-1} - ia_{n+m+1,1}^1 e_i + \\
 & \quad + (a_{n+m+1,1}^2 - \sum_{t=1}^{i-1} a_{t,1}^{t+2}) e_{i+1} + \sum_{s=i+2}^n (a_{1,n+m+1}^{s-i+1} - (s-i) \sum_{t=1}^{i-1} a_{t,1}^{s-i+t+1}) e_s + i \sum_{s=1}^m a_{i-1,1}^{n+s} f_s + \\
 & \quad + ia_{i-1,1}^{n+m+1} x + ia_{i-1,1}^{n+m+2} y + ia_{i-1,1}^{n+m+3} z, \quad 3 \leq i \leq n, \\
 \varphi(f_1, x) &= \sum_{i=1}^{n-1} ia_{n+1,1}^{i+1} e_i + na_{n+m+2,n+1}^n e_n - a_{n+m+1,n+1}^{n+1} f_1, \\
 \varphi(f_2, x) &= \sum_{i=1}^{n-1} ia_{n+2,1}^{i+1} e_i + a_{n+2,n+m+1}^n e_n + a_{n+2,n+m+1}^{n+2} f_2 + a_{n+2,n+m+1}^{n+3} f_3, \\
 \varphi(f_i, x) &= \sum_{s=1}^{n-1} sa_{n+i,1}^{s+1} e_s - na_{n+i-1,n+1}^n e_n + (a_{n+2,n+m+1}^{n+2} - (i-2)a_{n+m+1,n+1}^{n+1}) f_i + a_{n+2,n+m+1}^{n+3} f_{i+1}, \\
 & \quad 3 \leq i \leq m, \\
 \varphi(x, x) &= \sum_{i=2}^{n-1} (ia_{n+m+1,1}^{i+1} - a_{1,n+m+1}^{i+1}) e_i + a_{n+m+1,n+m+1}^n e_n, \\
 \varphi(y, x) &= \sum_{i=1}^{n-1} ia_{n+m+2,1}^{i+1} e_i + a_{n+m+2,n+m+1}^n e_n - a_{n+2,n+m+1}^{n+3} f_1, \\
 \varphi(z, x) &= \sum_{i=1}^{n-1} ia_{n+m+3,1}^{i+1} e_i + a_{n+m+3,n+m+1}^n e_n, \\
 \varphi(e_1, y) &= -a_{n+m+2,1}^1 e_1 + a_{n+m+2,1}^2 e_2 + a_{n+2,1}^{n+3} f_1 + \sum_{i=2}^m (i-1)a_{n+m+1,1}^{n+i} f_i, \\
 \varphi(e_i, y) &= -ia_{n+m+2,1}^1 e_i + a_{n+m+2,1}^2 e_{i+1} - a_{i-1,1}^{n+1} f_1 - \sum_{s=2}^m (s-1)a_{i-1,1}^{n+s} f_s, \quad 2 \leq i \leq n, \\
 \varphi(f_1, y) &= -\sum_{i=1}^{n-1} a_{n+1,1}^{s+1} e_s - a_{n+m+2,n+1}^n e_n - a_{n+m+2,n+1}^{n+1} f_1 + \sum_{s=3}^{m-1} (s-2)a_{n+1,n+1}^{n+s+1} f_s + a_{n+1,n+m+2}^{n+m} f_m + \\
 & \quad + a_{n+1,1}^1 x - a_{n+m+2,n+1}^{n+2} y - \frac{1}{m-1} \left( \sum_{t=2}^m a_{n+t,n+1}^{n+t} - \frac{m(m-1)}{2} a_{n+m+2,n+1}^{n+m+2} \right) z, \\
 \varphi(f_2, y) &= -\sum_{s=1}^{n-1} a_{n+2,1}^{s+1} e_s - \frac{1}{n} a_{n+2,n+m+1}^n e_n + \sum_{s=2}^m a_{n+2,n+m+2}^{n+s} f_s + a_{n+2,1}^1 x - a_{n+1,n+2}^{n+1} y + \\
 & \quad + (a_{n+1,n+2}^{n+1} - a_{n+2,n+2}^{n+2}) z, \\
 \varphi(f_3, y) &= -2 \sum_{s=1}^{n-1} a_{n+3,1}^{s+1} e_s + 2a_{n+2,n+1}^n e_n + (a_{n+2,n+1}^{n+1} + a_{n+1,n+2}^{n+1}) f_1 + \\
 & \quad + \frac{1}{m-1} \left( (m-2)a_{n+2,n+1}^{n+2} - \sum_{t=3}^m a_{n+t,n+1}^{n+t} + \frac{(m-1)(m-2)}{2} a_{n+m+2,n+1}^{n+m+2} \right) f_2 + \\
 & \quad + (a_{n+2,n+m+2}^{n+2} - a_{n+m+2,n+1}^{n+1}) f_3 + \sum_{s=4}^m (a_{n+2,n+m+2}^{n+s-1} - (s-3)a_{n+2,n+1}^{n+s}) f_s + \\
 & \quad + 2a_{n+3,1}^1 x + 2a_{n+2,n+1}^{n+m+2} y + 2a_{n+2,n+1}^{n+m+3} z, \\
 \varphi(f_i, y) &= -(i-1) \sum_{s=1}^{n-1} a_{n+i,1}^{s+1} e_s + (i-1)a_{n+i-1,n+1}^n e_n + (i-2)(a_{n+i-1,n+1}^{n+1} - a_{n+i-2,n+1}^{n+2}) f_1 + \\
 & \quad + \sum_{s=2}^{i-2} (i-s) \left( \sum_{t=1}^{s-1} a_{n+i+t-s,n+1}^{n+t+1} \right) f_s + \\
 & \quad + \left( \frac{m-i+1}{m-1} \sum_{t=2}^{i-1} a_{n+t,n+1}^{n+t} - \frac{i-2}{m-1} \sum_{t=i}^m a_{n+t,n+1}^{n+t} + \frac{(i-2)(m-i+1)}{2} a_{n+m+2,n+1}^{n+m+2} \right) f_{i-1} + \\
 & \quad + (a_{n+2,n+m+2}^{n+2} - (i-2)a_{n+m+2,n+1}^{n+1}) f_i + \sum_{s=i+1}^m (a_{n+2,n+m+2}^{n+s-i+2} - (s-i) \sum_{t=1}^{i-2} a_{n+i-t,n+1}^{n+s-t+1}) f_s + \\
 & \quad + (i-1)a_{n+i,1}^1 x + (i-1)a_{n+i-1,n+1}^{n+m+2} y + (i-1)a_{n+i-1,n+1}^{n+m+3} z, \quad 4 \leq i \leq m, \\
 \varphi(x, y) &= -a_{n+m+2,1}^2 e_1 + a_{n+2,n+m+1}^{n+3} f_1 + \sum_{i=2}^{m-1} (i-1)a_{n+m+1,n+1}^{n+i+1} f_i + a_{n+m+1,n+m+2}^{n+m} f_m, \\
 \varphi(y, y) &= \sum_{i=2}^{m-2} (i-1)(a_{n+m+2,n+1}^{n+i+1} - a_{n+1,n+1}^{n+i+2}) f_i + ((m-2)a_{n+m+2,n+1}^{n+m} - a_{n+1,n+m+2}^{n+m}) f_{m-1} + \\
 & \quad + a_{n+m+2,n+m+2}^{n+m} f_m,
 \end{aligned}$$



$$(8) : \left\{ \begin{aligned} \varphi(z, y) &= a_{n+m+3, n+m+2}^{n+1} f_1 + \sum_{i=2}^{m-1} (i-1) a_{n+m+3, n+1}^{n+i+1} f_i + a_{n+m+3, n+m+2}^{n+m} f_m, \\ \varphi(e_1, z) &= -a_{n+m+3, 1}^1 e_1 + a_{n+m+3, 1}^2 e_2 + \sum_{i=2}^m a_{n+m+1, 1}^{n+i} f_i, \\ \varphi(e_i, z) &= -i a_{n+m+3, 1}^1 e_i + a_{n+m+3, 1}^2 e_{i+1} - \sum_{s=2}^m a_{i-1, 1}^{n+s} f_s, \quad 2 \leq i \leq n, \\ \varphi(f_1, z) &= -a_{n+m+3, n+1}^{n+1} f_1 + \sum_{s=2}^{m-1} a_{n+1, n+1}^{n+s+1} f_s + \frac{1}{m-2} a_{n+1, n+m+2}^{n+m} f_m, \\ \varphi(f_2, z) &= -\sum_{s=1}^{n-1} a_{n+2, 1}^{s+1} e_s - \frac{1}{n} a_{n+2, n+m+1}^{n+3} e_n - a_{n+2, n+2}^{n+3} f_1 + a_{n+2, n+m+3}^{n+2} f_2 + \\ &\quad + a_{n+m+3, n+m+2}^{n+1} f_3 + a_{n+1, 1}^1 x - a_{n+1, n+2}^{n+1} y + (a_{n+1, n+2}^{n+1} - a_{n+2, n+2}^{n+2}) z, \\ \varphi(f_3, z) &= -\sum_{s=1}^{n-1} a_{n+3, 1}^{s+1} e_s + a_{n+2, n+1}^{n+1} e_n + (a_{n+2, n+1}^{n+1} + a_{n+1, n+2}^{n+1}) f_1 + \\ &\quad + (a_{n+2, n+m+3}^{n+2} - a_{n+m+3, n+1}^{n+1}) f_3 + a_{n+m+3, n+m+2}^{n+1} f_4 + \\ &\quad + a_{n+3, 1}^1 x + a_{n+2, n+1}^{n+m+2} y + a_{n+2, n+1}^{n+m+3} z, \\ \varphi(f_i, z) &= -\sum_{s=1}^{n-1} a_{n+i, 1}^{s+1} e_s + a_{n+i-1, n+1}^{n+1} e_n + (a_{n+i-1, n+1}^{n+1} - a_{n+i-2, n+1}^{n+m+2}) f_1 + \\ &\quad + (a_{n+2, n+m+3}^{n+2} - (i-2) a_{n+m+3, n+1}^{n+1}) f_i + a_{n+m+3, n+m+2}^{n+1} f_{i+1} + \\ &\quad + a_{n+i, 1}^1 x + a_{n+i-1, n+1}^{n+m+2} y + a_{n+i-1, n+1}^{n+m+3} z, \quad 4 \leq i \leq m, \\ \varphi(x, z) &= -a_{n+m+3, 1}^2 e_1 + \sum_{i=2}^{m-1} a_{n+m+1, n+1}^{n+i+1} f_i + \frac{1}{m-1} a_{n+m+1, n+m+2}^{n+m} f_m, \\ \varphi(y, z) &= -a_{n+m+3, n+m+2}^{n+1} f_1 + \sum_{i=2}^{m-2} (a_{n+m+2, n+1}^{n+i+1} - a_{n+1, n+1}^{n+i+2}) f_i + \\ &\quad + (a_{n+m+2, n+1}^{n+m} - \frac{1}{m-2} a_{n+1, n+m+2}^{n+m}) f_{m-1} + \frac{1}{m-1} a_{n+m+2, n+m+2}^{n+m} f_m, \\ \varphi(z, z) &= \sum_{i=2}^{m-1} a_{n+m+3, n+1}^{n+i+1} f_i + \frac{1}{m-1} a_{n+m+3, n+m+2}^{n+m} f_m. \end{aligned} \right.$$

Taking into account the expressions (6), (7) and (8), we derive that the dimension of  $ZL^2(L, L)$  is equal to  $(m + n + 3)^2 - 5$ . □

Based on Proposition 4.2, we have the following corollary.

**Corollary 4.3.**  $\dim HL^2(L, L) = 0$ .

Thus, according to the results of the paper [20], we derive the following theorem.

**Theorem 4.4.** *The algebra  $L$  is rigid.*

**Acknowledgement:** The authors were supported by Ministerio de Economía y Competitividad (Spain), grant MTM2016-79661-P (European FEDER support included) and Ministry of Education and Science of the Republic of Kazakhstan, grant No. 0828/GF4. The last author was supported by Fundamental Research Grant FRGS/1/2016/STG06/UNIKL/02/1 from Malaysia.

## References

- [1] Bloh A., On a generalization of the concept of Lie algebras, Dokl. Akad. Nauk SSSR, 1965, 165, 471–473.
- [2] Loday J.-L., Une version non commutative des algèbres de Lie: les algèbres de Leibniz, Enseign. Math., 1993, (2) 39 (3-4), 269–293.
- [3] Ayupov S.A., Omirov B.A., On Leibniz algebras. Algebra and operators theory. Proceedings of the Colloquium in Tashkent 1997. Kluwer Academic Publishers, 1998, 1-12.
- [4] Ayupov S.A., Omirov B.A., On some classes of nilpotent Leibniz algebras, Siberian Math. J., 2001, 42(1), 15–24.
- [5] Omirov B.A., Conjugacy of Cartan subalgebras of complex finite dimensional Leibniz algebras, J. Algebra, 2006, 302, 887–896.
- [6] Barnes D.W., Some theorems on Leibniz algebras, Comm. Algebra, 2011, 39(7), 463–2472.

- [7] Bosko L., Hedges A., Hird J.T., Schwartz N., Stagg K., Jacobson's refinement of Engel's theorem for Leibniz algebras, *Involve*, 2011, 4(3), 293–296.
- [8] Barnes D.W., On Levi's theorem for Leibniz algebras, *Bull. Aust. Math. Soc.*, 2012, 86 (2), 184–185.
- [9] Malcev A.I., Solvable Lie algebras, *Amer. Math. Soc. Translation*, 1950, 36 (27).
- [10] Mubarakzjanov G.M., On solvable Lie algebras (Russian), *Izv. Vysš. Učehn. Zaved. Matematika*, 1963, 32 (1), 114–123.
- [11] Campoamor-Stursberg R., Solvable Lie algebras with a  $\mathbb{N}$ -graded nilradical of maximum nilpotency degree and their invariants, *J. Phys. A.*, 2010, 43(18), 195–202.
- [12] Ndogmo J.C., Winternitz P., Solvable Lie algebras with abelian nilradicals, *J. Phys. A*, 1994, 27 (2), 405–423.
- [13] Tremblay S., Winternitz P., Solvable Lie algebras with triangular nilradicals, *J. Phys. A*, 1998, 31 (2), 789–806.
- [14] Wang Y., Lin J., Deng S., Solvable Lie algebras with quasifiliform nilradicals, *Comm. Algebra*, 2008, 36 (11), 4052–4067.
- [15] Camacho L.M., Omirov B.A., Masutova K.K., Solvable Leibniz algebras with Filiform Nilradical, *Bull. Malays. Math. Sci. Soc.*, 2016, 39, 283–303.
- [16] Casas J.M., Ladra M., Omirov B.A., Karimjanov I.A., Classification of solvable Leibniz algebras with naturally graded filiform nilradical, *Linear Algebra Appl.*, 2013, 438 (7), 2973–3000.
- [17] Casas J.M., Ladra M., Omirov B.A., Karimjanov I.A., Classification of solvable Leibniz algebras with null-filiform nilradical, *Linear Multilinear Algebra*, 2013, 61 (6), 758–774.
- [18] Ladra M., Masutova K.K., Omirov B.A., Corrigendum to "Classification of solvable Leibniz algebras with naturally graded filiform nilradical", *Linear Algebra Appl.*, 2013, 438(7), 2973–3000, *Linear Algebra Appl.*, 2016, 507, 513–517.
- [19] Vergne M., Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes, *Bull. Soc. Math. France*, 1970, 98, 81–116.
- [20] Balavoine D., Déformations et rigidité géométrique des algèbres de Leibniz, *Comm. Algebra*, 1996, 24 (3), 1017–1034.
- [21] Gerstenhaber M., On the deformation of rings and algebras, I, II, III, *Ann. of Math.*, 1964, 79, 59–103; 1966, 84, 1–19; 1968, 88, 1–34.
- [22] Goze M., Khakimjanov Y., Nilpotent Lie algebras, vol. 361 of *Mathematics and its Applications*, Kluwer Academic Publishers Group, Dordrecht, 1996.
- [23] Loday J.-L., Pirashvili T., Universal enveloping algebras of Leibniz algebras and (co)homology, *Math. Ann.*, 1993, 296 (1), 139–158.
- [24] Nijenhuis A., Richardson R.W., Jr., Cohomology and deformations in graded Lie algebras, *Bull. Amer. Math. Soc.*, 1966, 72, 1–29.