Graph homomorphisms, the Tutte polynomial and "q-state Potts uniqueness"

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Abstract

We establish for which weighted graphs H homomorphism functions from multigraphs G to H are specializations of the Tutte polynomial of G, answering a question of Freedman, Lovász and Schrijver.

We introduce a new property of graphs called "q-state Potts uniqueness" and relate it to chromatic and Tutte uniqueness, and also to "chromatic–flow uniqueness", recently studied by Duan, Wu and Yu.

Keywords: Tutte polynomial, chromatic polynomial, flow polynomial, q-state Potts partition function, graph homomorphism, homomorphism profile

1 Introduction

In [7] it was shown that evaluations of the q-state Potts partition function of a graph G are the only evaluations of the Tutte polynomial of G that are also homomorphism counting functions from G to a multigraph H. We extend this result to homomorphisms from G to an edge-weighted graph H (Theorem 2.3). This answers a question of Freedman, Lovász and Schrijver [6, Example 3.3].

The search for chromatically unique graphs has been an active area of research [10,11] ever since Read introduced the concept of chromatically equivalent graphs in 1968. Interest has spread to polynomial invariants related to the chromatic polynomial, such as the Tutte polynomial [12] and flow polynomial [5]. We initiate here the study of "q-state Potts equivalent" graphs and "q-state Potts uniqueness", focussing on the case $q = 2$. We remark however that there are examples of graphs that are 2-state Potts equivalent but not q-state Potts equivalent for $q \geq 3$. The 2-state Potts partition function is not only a specialization of the Tutte polynomial but also of the "Ising polynomial" of Andrén and Markström [2]: a pair of Tutte equivalent or "isomagnetic" graphs are also 2-state Potts equivalent.

2 Homomorphisms and the Tutte polynomial

A *homomorphism* from a multigraph G to a multigraph H is a function from $V(G)$ to $V(H)$ which takes edges of G to edges of H (preserving parallel classes). The function $\hom(G, H)$ counting the number of homomorphisms from a multigraph G to H is extended to edge-weighted graphs H with adjacency matrix $A(H) = (h_{u,v})$ by setting

$$\hom(G, H) = \sum_{f:V(G)\to V(H)} \prod_{ij\in E(G)} h_{f(i),f(j)}.$$

The vector $(\hom(G, H) : H \in \mathcal{H})$ is called the *right $\mathcal{H}$-profile* of G, and the vector $(\hom(G, H) : G \in \mathcal{G})$ the *left $\mathcal{G}$-profile* of H.

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As usual C_k, P_k and K_k denote the cycle, path and complete graph on k vertices.

Example 2.1 If $\mathcal{G} = \{P_1\} \cup \{C_k : k \geq 1\}$ then the left $\mathcal{G}$-profiles of H and H' are the same if and only if H and H' are *cospectral*. (See [8].)

If $\mathcal{H} = \{K_q : q \geq 1\}$ then G and G' have the same right $\mathcal{H}$-profile if and only if G and G' are *chromatically equivalent*.

We denote by $T(G; x, y)$ the *Tutte polynomial* of G and by $P(G; q, y)$ the *q-state Potts partition function* of G. These polynomials are related by the equation $P(G; q, y) = q^{k(G)}(y-1)^{r(G)}T(G; \frac{y-1+q}{y-1}, y)$; taking $y = 0$ gives the chromatic polynomial.

Say a function h on multigraphs is *$\mathcal{G}$-local* if it takes non-zero values and has the property that for each $G \in \mathcal{G}$ the quotients $h(G)/h(G/e)$ and $h(G)/h(G\backslash e)$ each depend only on whether e is a bridge, loop or ordinary edge.

Let $K_q^{a,b}$ denote the edge-$\mathbb{C}$-weighted complete graph on q vertices with loops attached at each vertex, having weight a on loops and weight b on non-loops. It is an easy consequence of definitions that $P(G; q, y) = \hom(G, K_q^{y,1})$ for $y \in \mathbb{C}$. A multigraph can be regarded as an edge-$\mathbb{N}$-weighted graph with edge weights indicating multiplicities. The proof of [7, Theorem 2.7] yields the following:

Theorem 2.2 *Let H be a connected multigraph and $\mathcal{G} = \{K_1^{k,0}, K_2^{0,k}, C_k, P_k : k \geq 1\}$. The following statements are equivalent:*

(i) *There exist $x, y \in \mathbb{Q}$ and a $\mathcal{G}$-local function h such that* $\hom(G, H) = h(G)T(G; x, y)$ *for every graph $G \in \mathcal{G}$.*

(ii) *There exist $a, b, q \in \mathbb{N}$, $q \geq 1$, such that $H \cong K_q^{a,b}$.*

Theorem 2.2 implies that $K_q^{a,b}$ for $a, b \in \mathbb{N}$ is amongst connected multigraphs determined by its left $\{K_1^{k,0}, K_2^{0,k}, C_k, P_k : k \geq 1\}$-profile. The star on $k+1$ vertices is denoted by $K_{1,k}$; $K_{1,0} = P_1$ is an isolated vertex.

Theorem 2.3 *For any $a, b \in \mathbb{C}$ the graph $K_q^{a,b}$ is determined up to isomorphism amongst all edge-$\mathbb{C}$-weighted graphs by its left $\{C_k, K_{1,k} : 0 \leq k \leq q\}$-profile.*

Proof (sketch) The adjacency matrix of a graph with the same $\{K_{1,k} : 0 \leq k \leq q\}$-profile as $K_q^{a,b}$ must be a $q \times q$ matrix with constant row and column sums $a + (q-1)b$. A symmetric matrix cospectral with the adjacency matrix $(a-b)I + bJ$ of $K_q^{a,b}$ is also similar to it by the spectral theorem. These two facts suffice to determine that the adjacency matrix of a graph with the same

$\{C_k, K_{1,k} : 0 \leq k \leq q\}$-profile as $K_q^{a,b}$ must equal $(a-b)I + bJ$. □

It can also be shown [8] that $K_q^{a,b}$ is amongst edge-$\mathbb{C}$-weighted graphs determined by its left $\{K_1^{k,0}, K_2^{0,k} : 0 \leq k \leq q\}$-profile. We also note that [6, Example 3.3] includes the result that there is an edge-$\mathbb{R}$-weighted graph H such that $\hom(G, H) = (1-x)^{k(G)}(1-y)^{|V|}T(G; x, y)$ if and only if $(x-1)(y-1) = q$ for integers $q \geq 1$. This provides an alternative proof of Theorem 2.2 in the case where $\mathcal{G}$ is the set of all multigraphs.

3 q-state Potts uniqueness

A multigraph G is *Tutte unique* if $T(G; x, y) = T(G'; x, y)$ implies $G \cong G'$, for every other graph G'. The following is motivated by Theorem 2.2:

Definition 3.1 [7] A multigraph is *colouring unique* if it is determined by its right $\{K_q^{y,1} : q, y \in \mathbb{N}\}$-profile.

Our main result here is the following:

Theorem 3.2 *A multigraph G is Tutte unique if and only if it is colouring unique.*

Proof (sketch) $P(G; q, y)$ for $q, y \in \mathbb{N}$ includes all evaluations of $T(G; x, y)$ at (x, y) for integers $x, y \geq 2$. Use [1, Lemma 2.1] to prove $T(G; x, y)$ is determined by interpolation of its values on a sufficiently large rectangle of integer points (x, y), $x, y \geq 2$. □

Having proved Theorem 3.2 it is natural to consider either fixing y or fixing q in Definition 3.1. The former includes chromatic uniqueness ($y = 0$) and flow uniqueness ($y = 1 - q$).

Definition 3.3 A multigraph is *q-state Potts unique* if it is determined by its right $\{K_q^{y,1} : y \in \mathbb{N}\}$-profile.

We focus on $q = 2$, the case of the Ising model. On the one hand we can list a number of graph invariants determined by the 2-state Potts partition function of G, such as for each $0 \leq i \leq |E(G)|$ the number of Eulerian subgraphs of G of size i. On the other hand we have examples that show that some invariants are not (such as connectedness). There remain many invariants of G which we neither know to be determined by $P(G; 2, y)$ nor the contrary.

Graphs such as the wheel W_5 shown to be "chromatic–flow unique" by Duan, Wu and Yu [5] are also 2-state Potts unique. We prove the following:

Theorem 3.4 *The ladders L_k for $k \geq 6$, Möbius ladders M_k for $k \geq 4$ and squares of cycles C_k^2 for $k \geq 10$ are all 2-state Potts unique.*

Proof (sketch) Most graph invariants obtained from the chromatic and flow polynomial used in the proofs in [5] are also determined by the 2-state Potts partition function; if not, then a minor change of argument suffices. □

Let $\theta(a_1, \ldots, a_s)$ denote the *s-bridge graph* consisting of internally disjoint paths of lengths $a_1, \ldots, a_s$ joining two terminal vertices. The flow polynomial $F(G; q)$ cannot distinguish any pair of s-bridge graphs: it is always equal to $q^{-1}[(q-1)^s + (-1)^s(q-1)]$. Chen et al. [4] (see also [11]) establish the equivalence classes to which s-bridge graphs belong under chromatic equivalence.

Proposition 3.5 *Non-isomorphic multibridge graphs $\theta(a_1, \ldots, a_s)$ have different 2-state Potts partition functions.*

The sizes of the Eulerian subgraphs of $\theta(2, 2, 3, 4)$ are $0, 4, 5, 5, 6, 6, 7, 11$ and those of $\theta(2, 3, 3)$ edge-glued with C_4 are $0, 4, 5, 5, 6, 7, 8, 9$. Hence these graphs have different 2-state Potts partition functions, whereas they have the same chromatic polynomial [11].

4 Conclusion

The problem of determining a graph by its left or right profile has been studied in various contexts, leading to interesting notions of left- and right-convergence (see [3] for a survey) and homomorphism dualities (see for example [9]). Here we have seen how Tutte uniqueness corresponds to being determined by a right profile by weighted complete graphs (Theorem 3.2). Moreover, graphs in this family are determined by their left profile by cycles and stars (Theorem 2.3). The theory of graph homomorphisms provides a fresh perspective on old problems about polynomial graph invariants as well as raising interesting new questions, some of which are explored in a forthcoming paper [8].

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