Contractors for flows

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Abstract

We answer a question raised by Lovász and B. Szegedy [Contractors and connectors in graph algebras, *J. Graph Theory* 60:1 (2009)] asking for a contractor for the graph parameter counting the number of B-flows of a graph, where B is a subset of a finite Abelian group closed under inverses. We prove our main result using the duality between flows and tensions in the context of finite Fourier analysis.

Keywords: Graph homomorphism, Fourier transform, contractor, flows, tensions.

1 Introduction

In their paper [1] Lovász and B. Szegedy introduce the notion of connectors and contractors in quantum graph algebras (for reasons of space we quote just one reference in this presently very active area). Connectors and contractors for a graph parameter allow expressions for the parameter of a graph obtained by the connection of two vertices by an edge or the contraction of an edge, in terms of values of the parameter on graphs obtained by operations that preserve simplicity. For example, a connector for the flow polynomial

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is any path of length two or more. Contractors for graph parameters can be thought of as a generalization of the deletion-contraction identity for the chromatic polynomial, flow polynomial, and other specializations of the Tutte polynomial.

In [1, Section 2.3] Lovász and B. Szegedy comment that "there does not seem to be a simple explicit construction for a contractor" for the number of B-flows, where B is a subset of a finite Abelian group closed under inverses. In Section 4 we give an explicit construction.

We prove our main result (Theorem 3.4, and its special case for B-flows, Theorem 4.2) using the Fourier transform on finite Abelian groups. The Fourier transform expresses any weighted function on the set of tensions of a graph (effectively a homomorphism function to an edge-weighted graph) as a related weighted function on the set of flows of the graph. This in particular allows the number of B-flows to be written as the number of homomorphisms to an edge-weighted graph (Lemma 4.1).

For graphs G and H, let $\hom(G, H)$ denote the number of homomorphisms from G to H. The definition can be extended to edge-weighted graphs H, as explained in Section 2.1. In Section 3 we consider the problem of finding a contractor for the homomorphism function $\hom(\cdot, H)$. The minimum polynomial of the adjacency matrix of H can be used to construct a contractor for $\hom(\cdot, H)$ if the adjacency matrix of H does not have eigenvalue 0 (Theorem 3.2). Theorem 3.3 gives a contractor for $\hom(\cdot, H)$ when the adjacency matrix of H has no diagonal entry equal to an off-diagonal entry. The relevant edge-weighted graph H that gives the number of B-flows turns out to have eigenvalue 0, and when B is a strict subset of non-zero group values there are off-diagonal zero entries as well as zeroes on the diagonal of the adjacency matrix of H. However, the special structure of H as an edge-weighted Cayley graph allows us to obtain a contractor by using the minimum polynomial of the graph whose edge weights are obtained by taking the Fourier transform of the edge weights of H (Theorem 3.4).

2 Preliminaries

2.1 Homomorphism functions

Given two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, a *homomorphism* from G to H, written as $\psi : G \to H$, is a mapping $\psi : V(G) \to V(H)$ such that $\psi(u)\psi(v) \in E(H)$ whenever $uv \in E(G)$. The graph parameter $\hom(G, H)$ counting the number of homomorphisms from G to H can be generalized to edge-weighted graphs H as follows.

Let H be a graph with a real weight $\beta_H(ij)$ associated with each edge $ij \in E(H)$, G an unweighted graph, and fix $U \subseteq V(G)$. Define for every function $\phi : U \to V(H)$ the weight

$$\hom_\phi(G, H) = \sum_{\substack{\psi:V(G)\to V(H)\\ \psi_{|U}=\phi}} \prod_{uv\in E(G)} \beta_H(\psi(u), \psi(v)).$$

Then $\hom(G, H) = \sum_{\phi:U\to V(H)} \hom_\phi(G, H)$.

2.2 Flows and tensions

Let $G = (V, E)$ be a graph with an arbitrary fixed orientation of its edges, and let D be the $V \times E$ incidence matrix of G, with (v, e)-entry given by: $+1$ if v is the head of e; -1 if v is the tail of e; and 0 if v is not an endpoint of e, or e is a loop on v.

Suppose Γ is a finite commutative ring with unity, and let $\Gamma^V = \{f : V \to \Gamma\}$ and $\Gamma^E = \{g : E \to \Gamma\}$. The linear transformation $D : \Gamma^E \to \Gamma^V$ of Γ-modules has kernel equal to the set of Γ-*flows* of G. This is to say a function $g : E \to \Gamma$ is a Γ-flow if and only if, for each vertex $v \in V$,

$$\sum_{\substack{e=uv\\ v \text{ head of } e}} g(e) - \sum_{\substack{e=uv\\ v \text{ tail of } e}} g(e) = 0.$$

If $B = -B$ is a subset of Γ then a B-*flow* of G is a Γ-flow taking values in B.

The transpose linear transformation $D^\top : \Gamma^V \to \Gamma^E$ has image the set of Γ-*tensions* of G. A Γ-tension $g : E \to \Gamma$ arises from a vertex Γ-colouring $f : V \to \Gamma$ of G, by setting $g(e) = f(v) - f(u)$ for each edge e with tail u and head v. A B-*tension* of G is a Γ-tension of G taking values in B.

2.3 Contractors

For fixed positive integer k, a *k-labelled graph* (G, λ) is a finite graph G (without loops but possibly multiple edges) together with a function $\lambda : [k] \to V(G)$. A k-labelled *quantum graph* is a formal linear combination of k-labelled graphs with coefficients in $\mathbb{R}$. Let $\mathcal{G}_k$ denote the set of k-labelled quantum graphs and $\mathcal{G}_k^0$ the subset of those whose labelled vertices are independent.

For two k-labelled graphs X and Y, the *product* XY is defined [1] by taking the disjoint union of X and Y and then identifying vertices which share the same label.

A 2-labelled quantum graph $Z \in \mathcal{G}_2$ is a *contractor* for a graph parameter h if for all $X \in \mathcal{G}_2^0$ we have $h(XZ) = h(XK_1)$ where K_1 is one vertex carrying both labels 1 and 2.

Informally, attaching a contractor Z at two non-adjacent vertices acts like identifying these vertices as far as the value of the graph parameter h is concerned.

3 Contractors for homomorphism functions

Let H be an edge-weighted graph with adjacency matrix $A = (\beta(ij))$. To every $X \in \mathcal{G}_2$ assign a $V(H) \times V(H)$-matrix $M(X) = M_H(X)$ with (i, j)-entry equal to $\hom_\phi(X, H)$ where $\phi(1) = i$ and $\phi(2) = j$.

We let K_n denote the complete graph on n vertices and $\overline{K}_n$ its complement, consisting of n isolated vertices. The 2-labelled path on $\ell + 1$ vertices, $\ell \geq 1$, with one endpoint labelled 1 and the other endpoint labelled 2 is written as P_ℓ. In this notation, $P_1 = K_2$.

A fundamental observation with regard to contractors for homomorphism functions is the following:

Lemma 3.1 [1] *A 2-labelled quantum graph Z is a contractor for* $\hom(\,\cdot\,, H)$ *if and only if $M(Z) = I$.*

Using this lemma we can write down a a quantum path contractor for $\hom(\,\cdot\,, H)$ when the adjacency matrix of H does not have 0 as an eigenvalue.

Theorem 3.2 *Suppose the adjacency matrix A of H has minimum polynomial $p_A(t) = p_0 + p_1 t + \cdots + p_\ell t^\ell$. If $p_0 \neq 0$ (i.e, if 0 is not an eigenvalue of A), then a contractor for* $\hom(\,\cdot\,, H)$ *is given by* $Z = -\frac{1}{p_0}\left[p_1 P_1 + p_2 P_2 + \cdots + p_\ell P_\ell\right]$.

The graph P_1^k (k-fold product of P_1 with itself) is a *thick edge*, consisting of two vertices joined by k parallel edges, with one vertex labelled 1 and the other vertex labelled 2. By straightforward polynomial interpolation we have a quantum thick edge contractor for $\hom(\,\cdot\,, H)$ when the adjacency matrix of H satisfies a suitable condition:

Theorem 3.3 *Suppose the adjacency matrix $A = (\beta(ij))$ of H has the property that $\{\beta(ij) : i, j \in V(H), i \neq j\} \cap \{\beta(ii) : i \in V(H)\} = \emptyset$, and that $|\{\beta(ij) : i, j \in V(H)\}| = \ell$. Then there exist constants $r_0, r_1, \ldots, r_\ell$ (computable by Lagrange interpolation) such that $Z = r_0 \overline{K}_2 + r_1 P_1 + r_2 P_1^2 + \cdots + r_\ell P_1^\ell$, is a contractor for* $\hom(\,\cdot\,, H)$.

We are now left with the problem of constructing a contractor for $\hom(\,\cdot\,,H)$ when the adjacency matrix of H has 0 as an eigenvalue, or when at least one of its diagonal entries also appears amongst its off-diagonal entries. When H is a weighted Cayley graph we are able to find such a contractor as follows.

Suppose that A has rows and columns indexed by an additive Abelian group Γ of order n and has (i,j)-entry equal to $\beta(i-j)$ for all $i,j\in\Gamma$, where $\beta:\Gamma\to\mathbb{C}$ satisfies $\beta(-i)=\beta(i)$. Thus A is the adjacency matrix of a weighted Cayley graph on Γ. Define $\widehat{A}$ to be the matrix with (i,j)-entry the Fourier transform $\widehat{\beta}(i-j)$. Let us call a matrix Γ-*circulant* if, like the matrix A, it takes the form $(\alpha(i-j))$ for some function $\alpha\in\mathbb{C}^\Gamma$ where $\mathbb{C}^\Gamma$ denotes the vector space over $\mathbb{C}$ of all functions from Γ to $\mathbb{C}$.

Theorem 3.4 *Suppose H is a connected graph with adjacency matrix a Γ-circulant matrix A and that $\widehat{H}$ has adjacency matrix $\widehat{A}$. Suppose further that A has eigenvector $\mathbf{1}$ with eigenvalue λ_1, has minimum polynomial $p_A(t)$, and $\frac{p_A(t)}{t-\lambda_1}=q(t)=q_0+q_1t+\cdots+q_{\ell-1}t^{\ell-1}$. Then a contractor for $\hom(\,\cdot\,,\widehat{H})$ is given by*

$$Z=\frac{1}{q(\lambda_1)}\sum_{0\le k\le \ell-1}q_kP_1^k.$$

4 A contractor for the number of B-flows

In this section, we apply Theorem 3.4 to the special case where $H=\mathrm{Cayley}(\Gamma,B)$ for additive Abelian group Γ, and where $B\subseteq\Gamma\setminus 0$ satisfies $-B=B$. We first establish the connection between flows and tensions in the context of Fourier analysis. Thus, Lemma 4.1 below expresses the number of B-flows in terms of homomorphisms to a weighted Cayley graph. To reach this result we use standard concepts from finite Fourier analysis, which can be found in [2].

Let $\{\delta_x : x\in\Gamma\}$ be the set of indicator functions defined by $\delta_x(y)=1$ if $x=y$, and $\delta_x(y)=0$ if $x\neq y$. This notation is extended to subsets B of Γ, defining $\delta_B=\sum_{x\in B}\delta_x$. Let $\widehat{\delta_B}$ be the Fourier transform of δ_B.

Lemma 4.1 *Let $B=-B$ be a subset of an additive Abelian group Γ. Let $H=\mathrm{Cayley}(\Gamma,B)$ be the graph on vertex set Γ with an edge joining vertices i and j if and only if $j-i\in B$, and let $\widehat{H}$ be the edge-weighted Cayley graph on vertex set Γ with edge ij having weight $\widehat{\delta}_B(j-i)$.*

Then the number of B-tensions of a graph $G=(V,E)$ with $c(G)$ connected components is equal to $|\Gamma|^{-c(G)}\hom(G,H)$ and the number of B-flows of G is equal to $|\Gamma|^{-|V|}\hom(G,\widehat{H})$.

The adjacency matrix A of H has (i,j)-entry $\delta_B(i-j)$ and eigenvalues $\widehat{\delta_B}(i)$ for $i \in \Gamma$. The minimum polynomial $p_A(t)$ of A has degree ℓ equal to the number of distinct values of $\widehat{\delta_B}(c)$ for $c \in \Gamma$. The largest eigenvalue of A is $\widehat{\delta_B}(0) = |B|$, belonging to the eigenvector $\mathbf{1}$, and as in Theorem 3.4 we set $\frac{p_A(t)}{t-|B|} = q(t) = q_0 + q_1 t + \cdots + q_{\ell-1}t^{\ell-1}$.

Theorem 4.2 *Let Γ be an additive Abelian group of order n and suppose $H = \text{Cayley}(\Gamma, B)$ is a connected graph with adjacency matrix A. Let $\widehat{H}$ be the edge-weighted graph on vertex set Γ with adjacency matrix $\widehat{A}$. Then,*

(i) a contractor for the number of B-flows is given by

$$Z = \frac{n}{q(|B|)} \sum_{0 \le k \le \ell-1} q_k P_1^k;$$

(ii) a contractor for $n^{c(G)}$ times the number of B-tensions of G is given by

$$Z = \frac{1}{q_0}\left[\frac{q(|B|)}{n}\overline{K}_2 - \sum_{1 \le k \le \ell-1} q_k P_k\right].$$

If B does not contain a set of generators for Γ then the Γ-circulant graph $H = \text{Cayley}(\Gamma, B)$ is not connected. The connected components of $\text{Cayley}(\Gamma, B)$ correspond to the cosets of the subgroup generated by B. Theorem 4.2 is easily extended to this disconnected case as follows:

Theorem 4.3 *Let Γ be an additive Abelian group of order n and A the adjacency matrix of $H = \text{Cayley}(\Gamma, B)$. Suppose H has r isomorphic connected components, i.e., its adjacency matrix A is permutation-equivalent to a matrix of the form $I \otimes A_1$ for $r \times r$ identity matrix I and some $n/r \times n/r$ matrix A_1. Then, a contractor for the number of B-flows is given by*

$$Z = \frac{n}{rq(|B|)} \sum_{0 \le k \le \ell-1} q_k P_1^k.$$

References

[1] Lovász, L., and B. Szegedy, *Contractors and connectors in graph algebras*, J. Graph Theory **60**:1 (2009), 11–30.

[2] Terras, A., "Fourier analysis on finite groups and applications", London Math. Soc. Stud. Texts **43**, Cambridge Univ. Press, Cambridge, 1999.