Gröbner bases and logarithmic $\mathcal{D}$-modules

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Abstract

Let $\mathbf{C}[x] = \mathbf{C}[x_1, \ldots, x_n]$ be the ring of polynomials with complex coefficients and A_n the Weyl algebra of order n over $\mathbf{C}$. Elements in A_n are linear differential operators with polynomial coefficients. For each polynomial f, the ring $M = \mathbf{C}[x]_f$ of rational functions with poles along f has a natural structure of a

left A_n-module which is finitely generated by a classical result of I.N. Bernstein. A central problem in this context is how to find a finite presentation of M starting from the input f. In this paper we use Gröbner base theory in the non-commutative frame of the ring A_n to compare M to some other A_n-modules arising in Singularity Theory as the so-called logarithmic A_n-modules. We also show how the analytic case can be treated with computations in the Weyl algebra if the input data f is a polynomial.

Keywords: Gröbner bases; Weyl algebra; $\mathcal{D}$-modules; Free divisors; Spencer divisors

1. Introduction

Let us denote by $R_n = \mathbf{C}[x_1, \ldots, x_n]$ the complex polynomial ring in n variables and by A_n the Weyl algebra of order n. The associative $\mathbf{C}$-algebra A_n is generated by $2n$ symbols $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$ with relations

$$x_i x_j = x_j x_i, \qquad \partial_i \partial_j = \partial_j \partial_i, \qquad \partial_i x_j = x_j \partial_i + \delta_{ij}$$

where δ_{ij} is Kronecker's symbol. A_n is isomorphic to the ring of linear differential operators over the ring R_n.

In the same way let us consider the ring $\mathcal{O}_n = \mathbf{C}\{x_1, \ldots, x_n\}$ of convergent power series in n variables and the ring $\mathcal{D}_n$ of linear differential operators with coefficients in $\mathcal{O}_n$. We have a

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natural inclusion $A_n \subset \mathcal{D}_n$. An element P in A_n (resp. $\mathcal{D}_n$) is called a linear differential operator and it can be written as a finite sum

$$P = \sum_{\beta \in \mathbf{N}^n} p_\beta(x) \partial^\beta$$

where $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}^n$, $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ and $p_\beta(x) \in R_n$ (resp. $\mathcal{O}_n$).

If no confusion is possible we drop the index n and simply write R, A, $\mathcal{O}$ and $\mathcal{D}$. Remember that A (resp. $\mathcal{D}$) is a non-commutative, left and right noetherian ring. Moreover, it is a simple ring (i.e. there are no non-trivial two-sided ideals in A_n; see Björk, 1979).

In this paper we will study some A-modules (and some $\mathcal{D}$-modules) arising in a natural way from Algebraic Geometry and Singularity Theory.

The ring R is a left A-module for the natural action defined as follows:

$$x_i \bullet f = x_i f, \qquad \partial_i \bullet f = \frac{\partial f}{\partial x_i}$$

for any $f \in R$. In fact, R is isomorphic, as a left A-module, to the quotient of A by the left ideal generated by $\partial_1, \ldots, \partial_n$. In the same way $\mathcal{O}$ is a left $\mathcal{D}$-module.

Let us consider $f \in R$ ($f \notin \mathbf{C}$). The localization ring R_f (i.e. the ring of rational functions with poles along f) is the ring of quotients

$$R_f = \left\{ \frac{g}{f^m} \mid g \in R,\ m \in \mathbf{N} \right\}.$$

R_f is a R-module and a left A-module in a natural way: the action $\partial_i \bullet \frac{g}{f^m}$ is just defined as the partial derivative of the rational function g/f^m. Of course R_f is not a finitely generated R-module.

We have an analogous situation in the *analytic* setting, i.e. starting from $f \in \mathcal{O}$ and considering $\mathcal{O}_f$ (the ring of meromorphic functions with poles along f) as a left $\mathcal{D}$-module.

One of the main results in $\mathcal{D}$-module theory is the following theorem:

Theorem 1.1 (*Bernstein, 1972; Björk, 1979*). *We have:*

(i) *For any $f \in R$, the left A_n-module R_f is finitely generated. In fact, there exists a positive integer number k such that R_f is the left A-module generated by the rational function $\frac{1}{f^k}$.*
(ii) *For any $f \in \mathcal{O}$, the left $\mathcal{D}$-module $\mathcal{O}_f$ is finitely generated. In fact, there exists a positive integer number k such that $\mathcal{O}_f$ is the left $\mathcal{D}$-module generated by the meromorphic function $\frac{1}{f^k}$.*

The left A-module generated by $\frac{1}{f^k}$ is just the set

$$A\frac{1}{f^k} = \left\{ P \bullet \frac{1}{f^k},\ P \in A \right\} \subset R_f.$$

The main ingredient in the proof of Theorem 1.1 is the existence of the so-called b-function (or Bernstein–Sato polynomial) attached to f (see Bernstein, 1972; Björk, 1979), which is a non-zero polynomial $b_f(s) \in \mathbf{C}[s]$ with the following property: if $-k$ is the least integer root of $b_f(s)$ then

$$R_f = A\frac{1}{f^k}.$$

Bernstein proved (Bernstein, 1972) that the dimension of the characteristic variety of R_f is n, so R_f is *holonomic*. Kashiwara (1978) proved an analogous result for $\mathcal{O}_f$.

In computational $\mathcal{D}$-module theory a natural problem is the following (for simplicity we only state the polynomial case):

Problem. Given a polynomial $f \in R$:

(a) Compute a positive integer number k such that $R_f = A\frac{1}{f^k}$ and

(b) Compute a system of generators of the annihilator $Ann_A(1/f^k)$, i.e. compute a presentation

$$R_f \simeq \frac{A}{Ann_A\left(\frac{1}{f^k}\right)}.$$

It is well known that there are algorithms to answer both questions (see Oaku, 1997a,b and Oaku and Takayama, 2001). Unfortunately, in many cases the available implementations of these methods cannot obtain the desired results due to the unmanageable size of the Gröbner base computations needed by the algorithms. We propose in this work how to build, using the so-called *logarithmic* $\mathcal{D}$-modules, some natural approximations of the above-mentioned annihilator and how to check whether the approximations are good enough.

2. Gröbner bases in $\mathcal{D}$-module theory

It will not be necessary to make a comprehensive development of the theory of Gröbner bases for $\mathcal{D}$-modules. In Briançon and Maisonobe (1984), Castro (1984) and Castro-Jiménez (1987) it is shown how the division and Buchberger's algorithm (Buchberger, 1965, 1970) can be adapted to the differential operators algebras A and $\mathcal{D}$. So the tools for computing Gröbner bases for left (and right) ideals and submodules of free modules, syzygies and free resolutions are available in this context. The book Saito et al. (2000) is an excellent introduction to Gröbner bases in A and its application to the study of GKZ-hypergeometric systems.

In $\mathcal{D}$ the situation is analogous but, as the coefficients of the differential operators can be convergent power series, the procedures are not algorithmic in its precise sense.

The papers (Oaku and Takayama, 2001; Oaku et al., 2000) contain deep applications of Gröbner bases to the effective computation of the four fundamental operations in $\mathcal{D}$-module theory (localization, local cohomology, restriction and integration). These algorithms use as a main tool the effective computation of b-functions (Oaku, 1997a).

The theory of Gröbner bases for A or $\mathcal{D}$ is part, in fact, of a more general theory of Gröbner bases in a certain family of non-commutative rings, developed in Kandri-Rody and Weispfenning (1990) (see also Bueso et al., 1998).

3. Approximations to $\mathcal{O}_f$

First we recall some results in the context of $\mathcal{D}$-modules and then we go to our approximations using effective methods in A.

3.1. *Logarithmic vector fields. Meromorphic functions*

We compile here K. Saito's definition of logarithmic vector fields and we define some $\mathcal{D}$-modules—which are called logarithmic $\mathcal{D}$-modules—related to the $\mathcal{D}$-module of meromorphic functions $\mathcal{O}_f$.

For each point $p \in \mathbf{C}^n$ let us denote by $\mathcal{O}_p$ the ring of formal power series convergent in a neighborhood of p. Let us consider $Der(\mathcal{O}_p)$ the $\mathcal{O}_p$-module of $\mathbf{C}$-derivations of $\mathcal{O}_p$. The elements in $Der(\mathcal{O}_p)$ are called *vector fields*.

Let $D \subset \mathbf{C}^n$ be the divisor (i.e. the hypersurface) defined by a polynomial $f \in R$ and $p \in D$. A vector field $\delta \in Der(\mathcal{O}_p)$ is said to be *logarithmic* with respect to D if $\delta(f) = af$ for some $a \in \mathcal{O}_p$. The $\mathcal{O}_p$-module of logarithmic vector fields (or logarithmic derivations) is denoted by $Der(\log D)_p$. If there exists a vector field δ such that $\delta(f) = f$ we will say that this divisor is *Euler homogeneous*. Quasi-homogeneous divisors (i.e. divisors defined by weighted homogeneous polynomials) are Euler homogeneous.

From now on, we will suppose that the origin $0 \in \mathbf{C}^n$ is in D and $p = 0$. We will consider some $\mathcal{D}$-modules associated with any divisor D (or more precisely to the germ $(D, 0)$). We will consider the following two families:

1. We call the first family of the following modules *logarithmic* as they arise from the logarithmic derivations:
   - The (left) ideal $I^{\log D} \subset \mathcal{D}$ generated by the logarithmic vector fields $Der(\log D)_0$.
   - The (left) ideal $\widetilde{I}^{\log D} \subset \mathcal{D}$ generated by the set $\{\delta + a \mid \delta \in Der(\log D)_0 \text{ and } \delta(f) = af\}$. More generally, the ideals $\widetilde{I}^{(k)\log D}$ generated by the set
     $$\{\delta + ka \mid \delta \in Der(\log D)_0 \text{ and } \delta(f) = af\}.$$
     It is sensible to consider these ideals: if $\delta(f) = af$ then $(\delta + a) \bullet (1/f) = 0$ and $(\delta + ka) \bullet (1/f^k) = 0$.
   - The modules $M^{\log D} = \mathcal{D}/I^{\log D}$, $\widetilde{M}^{\log D} = \mathcal{D}/\widetilde{I}^{\log D}$ and more generally $\widetilde{M}^{(k)\log D} = \mathcal{D}/\widetilde{I}^{(k)\log D}$.
2. The second set of approximations comes from the following idea: instead of considering the logarithmic derivations (degree one in the derivatives), take elements that annihilate $1/f$ of any order $l \geq 1$ in the derivatives. We will denote as $Ann^l(1/f) \subset Ann_{\mathcal{D}}(1/f)$ the ideal generated by elements $P \in Ann_{\mathcal{D}}(1/f)$ of order $d \leq l$, with $l \geq 1$. So $Ann^1(1/f) = \widetilde{I}^{\log D}$ or, more generally, $Ann^1(1/f^k) = \widetilde{I}^{(k)\log D}$ for $k \geq 1$.

The point is that all these ideals and modules are computable with commutative Gröbner bases as we explain in 3.2.

Logarithmic $\mathcal{D}$-modules are related to the $\mathcal{D}$-module of meromorphic functions $\mathcal{O}_f$ in the following way. The inclusion

$$\widetilde{I}^{(k)\log D} \subset Ann_{\mathcal{D}}(1/f^k)$$

induces a natural morphism

$$\phi_D^k : \widetilde{M}^{(k)\log D} \to \mathcal{O}_f$$

defined by $\phi_D^k(\overline{P}) = P(1/f^k)$ where $\overline{P}$ denotes the class of the operator $P \in \mathcal{D}$ modulo $\widetilde{I}^{(k)\log D}$. The image of ϕ_D^k is $\mathcal{D}\frac{1}{f^k}$, i.e. the $\mathcal{D}$-submodule of $\mathcal{O}_f$ generated by $1/f^k$.

Considering the general ideals $\widetilde{I}^{(k)\log D}$ is a suggestion of Prof. Tajima. The point is the well known chain of inclusions

$$\mathcal{D}f^{-1} \subset \mathcal{D}f^{-2} \subset \cdots \subset \mathcal{D}f^{-k} = \mathcal{D}f^{-k-1} = \cdots = \mathcal{O}_f,$$

where $-k$ is the least integer root of the b-function attached to f (see 1.1).

3.2. Computation of the approximations

From a computational point of view the divisor D will be defined by a polynomial $f \in R$ (more generally, $D \subset \mathbf{C}^n$ could be an *analytic* divisor locally defined by germs of holomorphic functions, i.e. by convergent power series).

We summarize how to build all the ideals defined in the previous section:

- A system of generators of the ideals $I^{\log D}$ and $\widetilde{I}^{\log D}$ is computed using that if an element $P = a_1\partial_1 + \cdots + a_n\partial_n \in A$ verifies that

$$P \bullet (f) = (a_1\partial_1 + \cdots + a_n\partial_n) \bullet (f) = a_0 f$$

  for some $a_0 \in R$, then $(P + a_0)(1/f) = 0$. So computing such a system of generators is equivalent to computing the module of syzygies among f and its derivatives, $Syz(f, f_1, \ldots, f_n)$ (where $f_i = \frac{\partial f}{\partial x_i}$, $1 \le i \le n$). The case of the ideal $\widetilde{I}^{(k)\log D}$ is achieved with the R-module $Syz(kf, f_1, \ldots, f_n)$.

  Example 3.1. Here we perform the computation using Macaulay 2 (Grayson and Stillman, 1999) to obtain $\widetilde{I}^{\log D}$ for $D \equiv (f = xyz(x+y)(x+z) = 0) \subset \mathbf{C}^3$:

```
--loaded Dloadfile.m2

i1 : W = QQ[x,y,z,dx,dy,dz, WeylAlgebra => {x=>dx,y=>dy,z=>dz}]

o1 = W

o1 : PolynomialRing

i2 : F = x*y*z*(x+y)*(x+z);

o2 : W

i3 : K1 = kernel matrix {{F,diff(x,F),diff(y,F),diff(z,F)}}

o3 = image {5} | -5 -x-2z -x-2y |
           {4} | x  0     0     |
           {4} | y  0     xy+y2 |
           {4} | z  xz+z2 0     |

                                4
o3 : W-module, submodule of W

i4 : matrix {{-1,dx,dy,dz}} * gens K1;

             1       3
o4 : Matrix W  <--- W

i5 : I = ideal o4;

o5 : Ideal of W

i6 : toString I

o6 = ideal(x*dx+y*dy+z*dz+5,x*z*dz+z^2*dz+x+2*z,x*y*dy+y^2*dy+x+2*y)
```

- The ideals $Ann^l(1/f)$ are computed analogously. If $l \ge 1$ is fixed, any expression

$$\left(\sum_{\substack{i_1+\cdots+i_n\leq l\\ a_{i_1,\ldots,i_n}\in R}} a_{i_1,\ldots,i_n}\partial_1^{i_1}\cdots\partial_n^{i_n}\right)\bullet(1/f)=0$$

produces—once you multiply by f^{l+1}—a syzygy among f and a set of expressions in the partial derivatives of f up to degree l. For example, for $n=2, l=2$ we have

$$(a_{00}+a_{10}\partial_x+a_{01}\partial_y+a_{20}\partial_x^2+a_{11}\partial_x\partial_y+a_{02}\partial_y^2)\bullet\left(\frac{1}{f}\right)=0$$

$$\Rightarrow \frac{a_{00}}{f}+a_{10}\frac{-f_x}{f^2}+a_{01}\frac{-f_y}{f^2}$$

$$+a_{20}\left(\frac{-f_{xx}}{f^2}+2\frac{f_x^2}{f^3}\right)+a_{11}\left(\frac{-f_{xy}}{f^2}+2\frac{f_x f_y}{f^3}\right)+a_{02}\left(\frac{-f_{yy}}{f^2}+2\frac{f_y^2}{f^3}\right)=0.$$

So we have

$$a_{00}f^2+a_{10}(-f_x f)+a_{01}(-f_y f)$$

$$+a_{20}(-f_{xx}f+2f_x^2)+a_{11}(-f_{xy}f+2f_x f_y)+a_{02}(-f_{yy}f+2f_y^2)=0.$$

Therefore, in this case, the module of syzygies needed is

$$Syz(f^2,-f_x f,-f_y f,-f_{xx}f+2f_x^2,-f_{xy}f+2f_x f_y,-f_{yy}f+2f_y^2).$$

Example 3.2. We use again Macaulay to compute $Ann^2(1/f)$ where $f=x^4+y^5+xy^4$. First, we load the following file `ann2reiffen.txt`:

```
W = QQ[x,y,dx,dy, WeylAlgebra => {x=>dx,y=>dy}]

F = x^4 + y^5 + x*y^4
F1 = diff(x,F)
F2 = diff(y,F)
F11 = diff(x,F1)
F12 = diff(y,F1)
F22 = diff(y,F2)
P = F^2
Px = -F*F1
Py = -F*F2
Pxx = -F*F11 + 2*F1^2
Pxy = -F*F12 + 2*F1*F2
Pyy = -F*F22 + 2*F2^2

M2 = matrix {{1,dx,dy,dx^2,dx*dy,dy^2}}
```

And we compute $Ann^2(1/f)$:

```
--loaded Dloadfile.m2

i1 : load "ann2reiffen.txt"

--loaded ann2_reiffen.txt

i2 : K2 = kernel matrix {{P,Px,Py,Pxx,Pxy,Pyy}};

i3 : M2 * gens K2;
```

```
                1       5
o3 : Matrix W  <--- W

i4 : Ann2 = ideal o3;

o4 : Ideal of W

i5 : toString Ann2

ideal(143/700*x^2*y*dx^2+879/2800*x*y^2*dx^2-3/2800*y^3*dx^2+...

143/700*x^3*dx^2-377/70000*x^2*y*dx^2-447507/910000*x*y^2*dx^2+...

1243/3800*x^3*dx^2+121991873/34580000*x^2*y*dx^2+...

-11/7*x^3*dx^2-1607/364*x^2*y*dx^2-1033/364*x*y^2*dx^2+...

99/28*x^3*dx^2+1405291/182000*x^2*y*dx^2+636537/182000*x*y^2*dx^2-...)
```

The calculations above obtain generators of the respective ideals in the *analytic* case, due to the flatness of $\mathcal{D}$ over A: we have only computed modules of syzygies, in short. Unfortunately, the comparison between two $Ann^l(1/f)$ and $Ann^{l'}(1/f)$ when $l \neq l'$ is made in the Weyl algebra. The inclusion $A \subset \mathcal{D}$ is not faithfully flat. To distinguish these ideals in the analytic case it is necessary to make local calculations or—indirectly—compare associated objects like the characteristic variety of the respective modules. We do not develop this issue in this work.

4. Comparison tests

We propose in this section methods for comparing the logarithmic modules presented above with annihilating ideals.

4.1. Direct comparison

The first method is complete but needs the calculation of the b-function and the annihilator of f^k with $k \leq -1$.

Experimental evidence shows that for many divisors the b-function is hard to compute. As soon as the dimension is greater than, say, 4 or the degrees of the polynomials that define the divisor are relatively high the calculations become unmanageable. More precisely, the problem seems to rest in the calculation of the annihilator $Ann_{\mathcal{D}[s]}(f^s)$ and the use of certain elimination orders during the calculation of Gröbner bases (here $\mathcal{D}[s]$ stands for the polynomial ring, with the indeterminate s commuting with $\mathcal{D}$).

Anyway, the following test—that uses Oaku's algorithm (Oaku, 1997a) for the computation of b-functions and the Oaku–Takayama algorithm for computing annihilators (Oaku and Takayama, 2001)—can be applied in many interesting situations.

Test 4.1. Comparison of $\mathcal{O}_f$ and $\widetilde{M}^{(\alpha_0)\log D}$.
INPUT: *A polynomial equation $f = 0$ of a divisor $D \subset \mathbf{C}^n$;*

1. *Compute the b-function of f. Let $-\alpha_0$ be its least integer root.*
2. *Compute the ideal $Ann_{\mathcal{D}}(1/f^{\alpha_0})$.*

3. *Compute a set of generators* $\{\mathbf{s}_1, \ldots, \mathbf{s}_r\}$ *of* $Syz(f_1, \ldots, f_n, f)$. *The ideal* $\widetilde{I}^{(\alpha_0)\log D}$ *is generated by the elements*

$$\mathbf{s}_j \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \\ -\alpha_0 \end{pmatrix} \in \mathcal{D}, \qquad j = 1, \ldots, r.$$

OUTPUT:
IF $Ann_{\mathcal{D}}(1/f^{\alpha_0}) = \widetilde{I}^{(\alpha_0)\log D}$
THEN RETURN $\mathcal{O}_f \simeq \widetilde{M}^{(\alpha_0)\log D}$
OTHERWISE RETURN $\mathcal{O}_f \not\simeq \widetilde{M}^{(\alpha_0)\log D}$

The correctness of the algorithm is obvious as

$$\mathcal{O}_f \simeq \mathcal{D}\frac{1}{f^{\alpha_0}} \simeq \frac{\mathcal{D}}{Ann_{\mathcal{D}}(1/f^{\alpha_0})}.$$

4.2. Indirect approach: A sufficient condition

We present here an indirect method for deducing that $\mathcal{O}_f$ and the modules of type $\widetilde{M}^{(k)\log D}$ or $\mathcal{D}/Ann^l(1/f)$ do not coincide. The method is strongly based on the following result (cf. Mebkhout, 1989) which is a deep result in $\mathcal{D}$-module theory:

Theorem 4.2. *The vector space* $Ext^i_{\mathcal{D}}(\mathcal{O}_f, \mathcal{O})$ *is zero for* $i \geq 0$.

We establish how, under certain algorithmic conditions, some cohomology groups are not zero. This is the strategy used in Ucha (1999), Castro-Jiménez and Ucha-Enríquez (2001) and Castro-Jiménez and Ucha (2002).

To compare $\widetilde{M}^{(\alpha)\log D}$ or $\mathcal{D}/Ann^l(1/f)$—for some $\alpha, l \geq 1$—with $\mathcal{O}_f$ we will only use a free resolution of the approximation. As the algorithm looks for a technical condition in some step of the free resolution, in many examples it is not necessary to compute the whole resolution.[1]

We need an auxiliary concept:

Definition. For $P = \sum p_\beta(x)\partial^\beta \in A$, the *coefficient ideal* of P, $C(P) \subset R$, is the ideal generated by the elements $p_\beta(x) \in R$.
For an element $\mathbf{P} = (P_1, \ldots, P_m) \in A^m$, the coefficient ideal $C(\mathbf{P})$ is $C(\mathbf{P}) = C(P_1) + \cdots + C(P_m)$.

The coefficient ideal of a vector of differential operators contains (in general, strictly) the set of elements h obtainable by applying the vector to any $h_1, \ldots, h_m \in R$, that is

$$(P_1, \ldots, P_m) \bullet \begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix} = h.$$

The next condition—the core of this work—is based on the impossibility of obtaining some elements in a suitable ideal:

[1] Taking into account that computing a complete free resolution can be a problem of great complexity, this option is very interesting.

Definition. If

$$0 \to \mathcal{D}^{r_s} \xrightarrow{\varphi_s} \cdots \to \mathcal{D}^{r_2} \xrightarrow{\varphi_2} \mathcal{D}^{r_1} \xrightarrow{\varphi_1} \mathcal{D}^{r_0} \xrightarrow{\pi} M \to 0$$

is a free resolution of a $\mathcal{D}$-module M, we say that the *Successive Matrices Condition (SMC) holds at level i* if the two successive morphisms φ_i, φ_{i+1} have matrices verifying:

1. There exists a row $\mathbf{P}$, say the j-th row, in the matrix of φ_i such that $C(\mathbf{P}) \neq R$.
2. There exists an element $p \in R$ with $p \notin C(\mathbf{P})$ such that, for every element Q of the column j of the matrix φ_{i+1}, we have $Q(p) = 0$.

We will say that SMC holds (for the given free resolution) if it holds at some level i.

The first condition in the above definition is computable with Gröbner bases. The second condition becomes computable in any of the following ways:

- Check the elements of $Ker(\varphi_{i+1})$ obtained using the algorithms of Tsai and Walther (2001).
- Check the elements p that are linear combinations of the power products of a suitable basis of $R/C(\mathbf{P})$ (see for example Cox et al., 1996).

The first option needs holonomicity of the corresponding logarithmic module (see Example 6.4). The second option is used when the implementations of the methods of Tsai and Walther (2001) cannot manage some concrete examples.

Remark. It is clear that the condition (2) of the SMC would have to be changed to the more adequate

(2′) There exists an element $p \in R$ with $p \notin Im(\mathbf{P})$—where $Im(\mathbf{P})$ denotes the image of the morphism with matrix $\mathbf{P}$—such that, for every element Q of the column j of the matrix φ_{i+1}, we have $Q(p) = 0$.

We will call this alternative condition SMC′. In general the inclusion $Im(\mathbf{P}) \subset C(\mathbf{P})$ is strict and it is more precise to look for elements in $Im(\mathbf{P})$. Unfortunately, SMC′ is difficult to verify in a computational way. We treat an interesting example of this situation in 6.3.

Test 4.3. INPUT: *A polynomial equation $f = 0$ of a divisor $D \subset \mathbf{C}^n$;*

1. *Compute the desired approximation $M = \widetilde{M}^{(\alpha)\log D}$ of $\mathcal{O}_f$ (as in 3.2).*
2. *Compute a free resolution of M:*

$$0 \to \mathcal{D}^{r_s} \xrightarrow{\varphi_s} \cdots \to \mathcal{D}^{r_2} \xrightarrow{\varphi_2} \mathcal{D}^{r_1} \xrightarrow{\varphi_1} \mathcal{D} \xrightarrow{\pi} M \to 0.$$

OUTPUT:
IF *SMC holds* OR *M is not holonomic* THEN RETURN $\mathcal{O}_f \neq M$.
OTHERWISE RETURN *"The test does not decide"*

We need a lemma to justify the test. It explains the role of the SMC.

Lemma 4.1. *Let D be a divisor, M a finitely generated left $\mathcal{D}$-module and*

$$0 \to \mathcal{D}^{r_s} \xrightarrow{\varphi_s} \cdots \to \mathcal{D}^{r_2} \xrightarrow{\varphi_2} \mathcal{D}^{r_1} \xrightarrow{\varphi_1} \mathcal{D}^{r_0} \xrightarrow{\pi} M \to 0 \qquad (*)$$

a free resolution of M that satisfies SMC at level i. Then

$$Ext^i_{\mathcal{D}}(M, \mathcal{O}) \neq 0.$$

Proof. To obtain the *Ext* groups, we have to apply the functor $Hom_{\mathcal{D}}(-, \mathcal{O})$ to the resolution $(*)$. Using that

$$Hom_{\mathcal{D}}(\mathcal{D}^r, \mathcal{O}) \simeq \mathcal{O}^r$$

we obtain the complex

$$0 \to \mathcal{O}^{r_0} \xrightarrow{\varphi_1^t} \mathcal{O}^{r_1} \xrightarrow{\varphi_2^t} \mathcal{O}^{r_2} \to \cdots \xrightarrow{\varphi_{s-1}^t} \mathcal{O}^{r_{s-1}} \xrightarrow{\varphi_s^t} \mathcal{O}^{r_s} \to 0,$$

where φ_i^t denotes the morphism with matrix the transpose of φ_i. The derivatives now act naturally.
Then

$$Ext^i_{\mathcal{D}}(M, \mathcal{O}) = Ker\ \varphi_{i+1}^t / Im\ \varphi_i^t.$$

As we have said, the key of this lemma is that SMC looks for an element of the kernel that does not belong to the required image. This element yields a non-zero element of $Ext^i_{\mathcal{D}}(M, \mathcal{O})$.

Suppose that the condition is verified for the j-th row $\mathbf{P}$. If all the components of the column j of the matrix $(K_{lm})_{lm}$ of φ_{i+1} applied to some $p \in R$ with $p \notin C(\mathbf{P})$ produce 0, then $\mathbf{p} = (0, \ldots, p, \ldots, 0)$—where p is in the j-th position—is in $Ker\ \varphi_{i+1}^t$:

$$\begin{pmatrix} & K_{1j} & \\ & & \\ \vdots & \vdots & \vdots \\ & & \\ & K_{s_j j} & \end{pmatrix} \bullet \begin{pmatrix} 0 \\ \vdots \\ p \\ \vdots \\ 0 \end{pmatrix} = 0.$$

Obviously, this $\mathbf{p}$ cannot be in $Im\ \varphi_i^t$, since applying the row $\mathbf{P}$ to any column of elements in $\mathcal{O}$ we only obtain elements in $C(\mathbf{P})$.

That is, if $\mathbf{P} = (P_{j1}, \ldots, P_{js_j})$ then the equation

$$\begin{pmatrix} & \cdots & \\ P_{j1} & \cdots & P_{js_j} \\ & \cdots & \end{pmatrix} \bullet \begin{pmatrix} h_1 \\ \vdots \\ h_{s_j} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ p \\ \vdots \\ 0 \end{pmatrix}$$

has no solution for any $h_1, \ldots, h_{s_j} \in R$. □

We summarize in a theorem the results of this section. This shows the correctness of the Test 4.3.

Theorem 4.4. *Let $D \equiv (f = 0) \subset \mathbf{C}^n$ be a divisor with a free resolution of some approximation M of $\mathcal{O}_f$ that satisfies SMC at some level. Then $\mathcal{O}_f \not\simeq M$.*

Proof. Using Lemma 4.1 we have that

$$Ext^i_{\mathcal{D}}(M, \mathcal{O}) \neq 0.$$

So, using 4.2, the approximation M and $\mathcal{O}_f$ are not isomorphic. □

It is well known that $Ext^i_{\mathcal{D}}(M, \mathcal{O}) = 0$ for $i > n$. A simpler case of SMC would appear if you had a free resolution of length n (for a given $\mathcal{D}$-module M), that is, a free resolution of type

$$0 \to \mathcal{D}^{r_n} \xrightarrow{\varphi_n} \cdots \to \mathcal{D}^{r_2} \xrightarrow{\varphi_2} \mathcal{D}^{r_1} \xrightarrow{\varphi_1} \mathcal{D}^{r_0} \xrightarrow{\pi} M \to 0$$

of length n. Then

$$Ext^n_{\mathcal{D}}(M, \mathcal{O}) = \frac{\mathcal{O}^{r_n}}{Im\ \varphi_n^t},$$

and SMC means, at level n, that there exists in the matrix of φ_n a row with coefficient ideal not equal to R. This situation is very easy to test and—in principle—it can be obtained with the algorithms of Gago-Vargas (2003) that produce a free resolution of at most length n.

Remark. We underline again that, as we have only used free resolutions of modules to apply the indirect method, our test is applicable to the analytic case, due to the flatness of $\mathcal{D}$ over A.

5. Spencer-free divisors

In this section we summarize a bunch of results about a special case for which the methods of this work have turned out to be good, as we will show in the final section of examples.

Definition (*Saito, 1980*). Let $D \subset \mathbf{C}^n$ be a divisor and suppose $0 \in D$. D is said to be *free* (at the origin) if the $\mathcal{O}$-module $Der(\log D)_0$ is free.

Smooth divisors and normal crossing divisors are free. By Saito (1980) any reduced germ of plane curve $D \subset \mathbf{C}^2$ is a free divisor. By Saito's criterion (Saito, 1980), $D \equiv (f = 0) \subset \mathbf{C}^n$ is free if and only if there exist n vector fields $\delta_i = \sum_{j=1}^n a_{ij}\partial_j$, $i = 1, \ldots, n$, such that $\det(a_{ij}) = uf$ where det means determinant, u is a unit in $\mathcal{O}$ (i.e. $u(0) \neq 0$) and a_{ij} is a holomorphic function in $\mathcal{O}$.

Definition. We say that a free divisor D is *of Spencer type* if the complex

$$\mathcal{D} \otimes_{\mathcal{O}} \wedge^{\bullet} Der(\log D) \to M^{\log D} \to 0$$

is a (locally) free resolution of $M^{\log D}$ and if this last $\mathcal{D}$-module is holonomic.

There are analogous resolutions for the family of modules $\widetilde{M}^{(k)\log D}$. The complex in the above definition has been introduced in Calderón-Moreno (1999). The differential for this complex is

$$\begin{aligned} \mathrm{d}(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) &= \sum_{i=1}^{p} (-1)^{i-1} P\delta_i \otimes (\delta_1 \wedge \cdots \hat{\delta_i} \cdots \wedge \delta_p) \\ &\quad + \sum_{1 \le i < j \le p} (-1)^{i+j} P \otimes ([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \hat{\delta_i} \cdots \hat{\delta_j} \cdots \wedge \delta_p). \end{aligned}$$

For Spencer type divisors, the solution complex $Sol(M^{\log D})$ (that is, the complex $\mathbf{R}\mathcal{H}om_{\mathcal{D}}(M^{\log D}, \mathcal{O})$) is naturally quasi-isomorphic to $\Omega^{\bullet}(\log D)$, as we pointed out in Castro-Jiménez and Ucha (2002) as a deduction of Calderón-Moreno (1999). Here $\Omega^{\bullet}(\log D)$ is the complex of logarithmic differential forms with respect to D (see Saito, 1980). On the other hand, the duality—in the sense of $\mathcal{D}$-modules—$(M^{\log D})^* \simeq \widetilde{M}^{\log D}$ proved in Castro-Jiménez and Ucha (2002) has important consequences for comparing $\widetilde{M}^{\log D}$ and $\mathcal{O}_f$:

Theorem 5.1 (*Ucha, 1999; Castro-Jiménez and Ucha-Enríquez, 2001*). *In dimension 2, the morphism* $\phi_D^1 : \widetilde{M}^{\log D} \to \mathcal{O}_f$ *(see 3.1) is an isomorphism if and only if* $D \equiv (f = 0)$ *is a quasi-homogeneous plane curve.*

In the proof of the above theorem, we used that the SMC condition can be applied at level 2 for any non-quasi-homogeneous plane curve.

Theorem 5.2 (*Castro-Jiménez and Ucha, 2002*). *Suppose the divisor* $D \subset \mathbf{C}^n$ *is free and locally quasi-homogeneous. Then the morphism* $\phi_D^1 : \widetilde{M}^{\log D} \to \mathcal{O}_f$ *(see 3.1) is an isomorphism (so,* $\widetilde{M}^{\log D}$ *and* $\mathcal{O}_f$ *are isomorphic as* $\mathcal{D}$*-modules).*

As a consequence of the last result, it can be deduced that, for free central arrangements, we have $\widetilde{I}^{\log D} = Ann_{\mathcal{D}}(1/f)$. This equality could be related to the final conjecture in Walther (2005). The equality extends to the algebraic case too.

SMC provides a strategy for testing whether the so-called *Logarithmic Comparison Theorem (LCT)* holds, that is, whether the complex $\Omega^\bullet(\star D)$ of meromorphic differential forms and the complex $\Omega^\bullet(\log D)$ of logarithmic differential forms (both with respect to D) are quasi-isomorphic (see Calderón Moreno et al. (2002) and Castro-Jiménez et al. (1996)). We have

Theorem 5.3 (*Castro-Jiménez and Ucha-Enríquez, 2004*). *A Spencer free divisor* $D \subset \mathbf{C}^n$ *verifies LCT if and only if* $\phi_D^1 : \widetilde{M}^{\log D} \to \mathcal{O}_f$ *is an isomorphism.*

This result was proved (Torrelli, 2004) for the case of Koszul-free divisors.

6. Examples

It is very important to point out that all the calculations needed in this section are *calculations of Gröbner bases*, namely

- Computations of syzygies among a polynomial and its derivatives to present $I^{\log D}$ or $\widetilde{I}^{\log D}$.
- Testing whether a divisor is Euler homogeneous: the property holds if the ideal of first components of elements in $Syz(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ is the whole ring R.
- Computations of free resolutions (so syzygies again, essentially) of modules over the corresponding Weyl algebra.
- Equality of left ideals in A.
- Calculation of $Ann_{\mathcal{D}}(1/f)$ and the b-function for a polynomial f: Gröbner bases with elimination orders in the corresponding Weyl algebra with an additional variable s (see Saito et al. (2000) for example).
- Testing holonomicity of a $\mathcal{D}$-module, i.e. testing whether the associated characteristic variety has dimension n.

Throughout the section, the computations have been made with `kan/sm1` and the $\mathcal{D}$-modules package of `Macaulay 2` (respectively Takayama (1991) and Grayson and Stillman (1999)). Finally, some computations of syzygies among polynomials have been made with `CoCoA` (see Capani et al., 1995).

6.1. *Example 1:* $D \equiv (f = x_1^n + x_2^n + \cdots + x_n^n = 0) \subset \mathbf{C}^n$

It is well known that the least integer root of $f = x_1^n + x_2^n + \cdots + x_n^n$ is $-n+1$. So in this case

$$\mathcal{O}_f = \mathcal{D} \cdot \frac{1}{f^{n-1}}.$$

On the other hand, it is not a free divisor for $n \geq 2$ (because D has an isolated singularity) so the theorems of Section 5 are not applicable. Although for this example the direct approach can be used, we will illustrate the Successive Matrices Condition in this case.

The direct method in this case works like this:

- The ideal $Ann_{\mathcal{D}[s]}(f^s)$ is generated by

$$\left\{ -ns + x_1\partial_1 + \cdots + x_n\partial_n,\ \frac{\partial f}{\partial x_i}\partial_j - \frac{\partial f}{\partial x_j}\partial_i \text{ for } 1 \leq i < j \leq n \right\},$$

  as you can easily check using the algorithm of Oaku[2] (see Oaku, 1997a). Specializing s to the value $s = -n+1$ we obtain

$$\begin{aligned} Ann(1/f^{n-1}) &= \langle -n(-n+1) + x_1\partial_1 + \cdots + x_n\partial_n \rangle \\ &\quad + \left\langle \frac{\partial f}{\partial x_i}\partial_j - \frac{\partial f}{\partial x_j}\partial_i \text{ for } 1 \leq i < j \leq n \right\rangle. \end{aligned}$$

  One set of generators of $Syz(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, f)$ is

$$\{(x_1, \ldots, x_n, -n)\} \cup \left\{ \frac{\partial f}{\partial x_i}\mathbf{e}_{j+1} - \frac{\partial f}{\partial x_j}\mathbf{e}_{i+1} \text{ for } 1 \leq i < j \leq n \right\},$$

  (where $\mathbf{e}_l$ is the element of A_n^{n+1} with 1 in the position l and 0 in the rest, for $1 \leq l \leq n$).

  We have that

$$\mathcal{O}_f \simeq \mathcal{D}\frac{1}{f^{n-1}} = \widetilde{M}^{(n-1)\log D},$$

  as can be deduced by comparing the annihilator $Ann_{\mathcal{D}}(1/f^{n-1})$ with $\widetilde{I}^{(n-1)\log D}$. They are the same ideal.
- Let us illustrate the SMC for $n = 3$ (i.e. for $f = x^3 + y^3 + z^3$) at levels 2 and 3 in order to compare $Ann_{\mathcal{D}}(1/f)$ with $\widetilde{I}^{\log D}$.

Remark. It is remarkable that sometimes the SMC is hidden. If you apply the command `Dres` of the $\mathcal{D}$-modules package of `Macaulay 2`, the resolution obtained does not verify the SMC. Instead of the resolution being provided directly, the command `kernel` is used to control exactly which generators are chosen in each step; this is very helpful for looking for the conditions needed.

In this case the following free resolution of $\widetilde{M}^{\log D}$ can be obtained:

$$0 \to \mathcal{D}^3 \xrightarrow{\varphi_3} \mathcal{D}^6 \xrightarrow{\varphi_2} \mathcal{D}^4 \xrightarrow{\varphi_1} \mathcal{D} \xrightarrow{\pi} \widetilde{M}^{\log D} \to 0.$$

The matrix φ_2 of the first module of syzygies is

$$\begin{pmatrix} 0 & \partial_x & -\partial_y - \partial_z & 0 \\ 0 & x^2 & -y^2 & z^2 \\ z^2\partial_y - y^2\partial_z & -3y\partial_y - 3z\partial_z - 6 & -3x\partial_y & x\partial_z \\ z^2\partial_x - x^2\partial_z & 0 & -3x\partial_x - 3y\partial_y - 3z\partial_z - 6 & 0 \\ y^2\partial_x - x^2\partial_y & 0 & 0 & -3x\partial_x - 3y\partial_y - 3z\partial_z - 6 \\ 0 & -2x & y^2\partial_x - x^2\partial_y & -z^2\partial_x + x^2\partial_z \end{pmatrix}.$$

[2] The calculation of $Ann_{\mathcal{D}[s]}(f^s)$ and the b-function can be carried out by hand with Gröbner bases for the general case.

The matrix of the second module of syzygies is

$$\varphi_3 = \begin{pmatrix} -x^2 & \partial_x & 0 & 0 & 0 & 1 \\ -3y\partial_y - 3z\partial_z - 6 & 0 & -\partial_x & \partial_y & -\partial_z & 0 \\ 0 & 3y\partial_y + 3z\partial_z & x^2 & -y^2 & z^2 & -3x \end{pmatrix}.$$

As you can easily detect, the second row of φ_2 is

$$\mathbf{P} = (0, x^2, -y^2, z^2),$$

so $C(\mathbf{P}) = (x^2, y^2, z^2)$. In φ_3 the corresponding second column is

$$\begin{pmatrix} \partial_x \\ 0 \\ 3y\partial_y + 3z\partial_z \end{pmatrix}.$$

It is clear that $(0, 1, 0, 0, 0, 0)^t \in Ker\ \varphi_3$. The SMC holds at level 2 so $Ext^2_{\mathcal{D}}(\widetilde{M}^{\log D}, \mathcal{O}) \neq 0$.

As the third row of φ_3 is

$$\mathbf{Q} = (0, 3y\partial_y + 3z\partial_z, x^2, -y^2, z^2, -3x)$$

and $1 \notin C(\mathbf{Q})$, $Ext^3_{\mathcal{D}}(\widetilde{M}^{\log D}, \mathcal{O}) \neq 0$. It is the SMC at level 3.

6.2. Example 2: $D \equiv (f = x(x^2 - y^3)(x^2 - zy^3) = 0) \subset \mathbf{C}^3$

We prove here, using the direct method of 4.1, that $\widetilde{M}^{\log D} \simeq \mathcal{O}_f$.

This example belongs to an interesting family: it is not locally quasi-homogeneous[3] but *Euler homogeneous* and verifies that $Ann_{\mathcal{D}}(1/f) = \widetilde{I}^{\log D}$. Remember that in dimension 2 the last equality and being quasi-homogeneous are equivalent.

This divisor is free and $\delta_1, \delta_2, \delta_3$ form a (global) basis of $Der(\log D)$, where

$$\begin{aligned} \delta_1 &= \frac{3}{2}x\partial_x + y\partial_y \\ \delta_2 &= (y^3z - x^2)\partial_z \\ \delta_3 &= \left(-\frac{1}{2}xy^2\right)\partial_x - \frac{1}{3}x^2\partial_y + (y^2z^2 - y^2z)\partial_z, \end{aligned}$$

whose coefficients verify that

$$\begin{vmatrix} \frac{3}{2}x & y & 0 \\ 0 & 0 & y^3z - x^2 \\ -\frac{1}{2}xy^2 & -\frac{1}{3}x^2 & y^2z^2 - y^2z \end{vmatrix} = -\frac{1}{2}f.$$

This example uses the direct approach: we can calculate the annihilator of $1/f$—because it is manageable—and compare it with $\widetilde{I}^{\log D}$. They are the same ideal. We calculate the b-function of f too. Its least integer root is -1, so

$$\mathcal{O}_f \simeq \mathcal{D} \cdot f^{-1} \simeq \frac{\mathcal{D}}{Ann_{\mathcal{D}}(1/f)}.$$

[3] Beyond the scope of this work, there is an indirect proof of this fact using that D is not a *Koszul free* divisor (see Calderón-Moreno and Narváez-Macarro, 2002).

To complete this example, we will explain how the duality and the Spencer type condition mentioned in 5 are checked. We use for this purpose a free resolution of $M^{\log D}$.

Our work is divided into two steps:

- Verify that $M^{\log D}$ is holonomic.[4] In this case the dimension of the characteristic variety of $M^{\log D}$ is 3, so it is holonomic.
- Compute a free resolution of $M^{\log D}$ and check whether it is of Spencer type. If this happens then duality holds by Castro-Jiménez and Ucha (2002).

Here are some details of the resolution:

1. The module $Syz(\delta_1, \delta_2, \delta_3)$ is generated by the syzygies obtained from the commutators $[\delta_i, \delta_j]$. We have $Syz(\delta_1, \delta_2, \delta_3) = \langle \mathbf{s}_{12}, \mathbf{s}_{13}, \mathbf{s}_{23} \rangle$ where

$$\begin{aligned} \mathbf{s}_{12} &= (-\delta_2, \delta_1 - 3, 0) \\ \mathbf{s}_{13} &= (-\delta_3, 0, \delta_1 - 2) \\ \mathbf{s}_{12} &= (0, -\delta_3 - y^2 z, \delta_2). \end{aligned}$$

2. The module $Syz(\mathbf{s}_{12}, \mathbf{s}_{13}, \mathbf{s}_{23})$ is generated by the element $\mathbf{r}$:

$$\begin{aligned} \mathbf{r} = \Big(& -y^2 z^2 \partial_z + y^2 z \partial_z + \frac{1}{2} x y^2 \partial_x - y^2 z + \frac{1}{3} x^2 \partial_y, \\ & y^3 z \partial_z - x^2 \partial_z,\ -y \partial_y - \frac{3}{2} x \partial_x + 5 \Big) \end{aligned}$$

This is the element required to have the Spencer type resolution, so duality holds.

6.3. *Example 3:* $D \equiv (f = (xz + y)(x^3 - y^4) = 0) \subset \mathbf{C}^3$

In this case we will show how to deduce that some cohomology group is not zero with an ad hoc procedure that generalizes the computational SMC. This example belongs to a family covered in Castro-Jiménez and Ucha-Enríquez (2005) with an alternative method.

This divisor is free. One basis of $Der(\log D)$ is

$$\begin{aligned} \delta_1 &= 4x\partial_x + 3y\partial_y - z\partial_z + 15 \\ \delta_2 &= 4x^2\partial_x + 3xy\partial_y + y\partial_z + 16x \\ \delta_3 &= \frac{4}{3} x y^2 z \partial_x + \frac{1}{3} y^3 \partial_x + y^3 z \partial_y + \frac{1}{4} x^2 \partial_y - \frac{1}{4} x \partial_z + \frac{16}{3} y^2 z. \end{aligned}$$

The divisor is of Spencer type because it is Koszul free (remember that you have to prove that the symbols of the elements of a basis form a regular sequence). We will prove that $Ext^3_{\mathcal{D}}(\widetilde{M}^{\log D}, \mathcal{O}) \neq 0$.

To begin with, the Spencer resolution looks as follows:

$$0 \to \mathcal{D} \xrightarrow{\varphi_3} \mathcal{D}^3 \xrightarrow{\varphi_2} \mathcal{D}^3 \xrightarrow{\varphi_1} \mathcal{D} \xrightarrow{\pi} \widetilde{M}^{\log D} \to 0.$$

We have

$$Ext^3_{\mathcal{D}}(\widetilde{M}^{\log D}, \mathcal{O}) = \mathcal{O}/im\ \varphi_3^t.$$

[4] If it is not holonomic, the computation of its dual cannot be managed as we do.

So we are in a comfortable situation: we do not need to find elements of some kernel, only to look for elements that are not obtainable with the matrix of φ_3^t. The elements of the matrix of φ_3^t are of the form

$$(-1)^i(\delta_i + m_i) + (-1)^i \sum_{l \neq i} \alpha_l^{il},$$

where the m_i verifies $\delta_i(f) = m_i f$ and the α_l^{il} are the coefficients of the Poisson brackets $[\delta_i, \delta_l]$ expressed as combinations of the δ_l.

Let us see how to calculate these α_l^{il} using Macaulay.

Example 6.1. First we load the file `ideal-34.txt`:

```
W = QQ[x,y,z,dx,dy,dz, WeylAlgebra =>{x=>dx,y=>dy,z=>dz}]

F = (x*z + y)*(x^3 - y^4)

P1 = 4*x*dx + 3*y*dy - z*dz + 15

P2 = 4*x^2*dx + 3*x*y*dy + y*dz + 16*x

P3 = 4/3*x*y^2*z*dx + 1/3*y^3*dx + y^3*z*dy + 1/4*x^2*dy -1/4*x*dz + 16/3*y^2*z
```

To obtain the α_l^{il} we ask for suitable syzygies[5]:

```
i3 : kernel matrix {{P1*P2-P2*P1,P1,P2,P3}}

o3 = image {3} | -1/4 ... |
           {2} | 0    ... |
           {3} | 1    ... |
           {5} | 0    ... |

                                   4
o3 : W-module, submodule of W

i4 : kernel matrix {{P1*P3-P3*P1,P1,P2,P3}}

o4 = image {5} | 0                         -1/5 ...|
           {2} | 4/3x2dx+xydy+1/3ydz+16/3x 0    ...|
           {3} | -4/3xdx-ydy+1/3zdz-11/3   0    ...|
           {5} | 0                         1    ...|

                                   4
o4 : W-module, submodule of W

i5 : kernel matrix {{P2*P3-P3*P2,P1,P2,P3}}

o5 = image {6} | 0                         1   ...|
           {2} | 4/3x2dx+xydy+1/3ydz+16/3x 0   ...|
           {3} | -4/3xdx-ydy+1/3zdz-11/3   y2z ...|
           {5} | 0                         -5x ...|

                                   4
o5 : W-module, submodule of W
```

[5] You can use the formulas provided in Castro-Jiménez and Ucha (2002, Lemma 4.2.) too.

So the elements in the matrix φ_3 are

$$\begin{aligned} Q_1 &= 4x\partial_x + 3y\partial_y - z\partial_z + 6 \\ Q_2 &= 4x^2\partial_x + 3xy\partial_y + y\partial_z + 11x \\ Q_3 &= \frac{4}{3}xy^2z\partial_x + \frac{1}{3}y^3\partial_x + y^3z\partial_y + \frac{1}{4}x^2\partial_y - \frac{1}{4}x\partial_z + \frac{19}{3}y^2z. \end{aligned}$$

It is immediately clear that, if we denote by $\mathbf{Q}$ the matrix (Q_1, Q_2, Q_3) of φ_3 in the resolution, then $C(\mathbf{Q})$ covers all the elements in $\mathcal{O}$. So the computational SMC could not be applied in this case. Fortunately we can deduce that the cohomology group is not zero, obtaining an element that is not in $Im(\mathbf{Q})$. So in fact we use SMC' in this case. We prove that the equation

$$(Q_1, Q_2, Q_3) \bullet \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = z^6$$

has no solution:

- As $Q_2(h_2) + Q_3(h_3)$ is an element of the ideal (x, y), it is enough to show that the equation $Q_1(h_1) = z^6 + \alpha x + \beta y$ with $\alpha, \beta \in \mathcal{O}$ has no solution for any α, β.
- The key is that the monomial z^6 does not appear in the expansion of $Q_1(h_1)$. So the equation

$$\begin{aligned} &(4x\partial_x + 3y\partial_y - z\partial_z + 6) \bullet \left(\sum h_{ijk}x^iy^jz^k\right) \\ &= \sum (4i + 3j - k + 6)h_{ijk}x^iy^jz^k) = z^6 + \alpha x + \beta y \end{aligned}$$

  has no solution. More precisely, it is necessary to have

$$(4 \cdot 0 + 3 \cdot 0 - 6 + 6)h_{006} = 1.$$

6.4. *Example 4:* $D \equiv (f = (xz + y)(x^4 + y^5 + xy^4) = 0) \subset \mathbf{C}^3$

The divisor is free with $\delta_1, \delta_2, \delta_3$ as a global basis of $Der(\log D)$:

$$\begin{aligned} \delta_1 &= xz\partial_z + y\partial_z + x \\ \delta_2 &= -8x^2\partial_x - 10xy\partial_x - 6xy\partial_y - 8y^2\partial_y + 2yz\partial_z - 2y\partial_z - 40x - 48y \\ \delta_3 &= -\frac{1}{4}y^2z^2\partial_z - xy^2\partial_x - \frac{1}{4}y^3\partial_x - \frac{3}{4}y^3\partial_y + \frac{1}{4}y^2z\partial_z - \frac{1}{4}y^2z \\ &\quad - \frac{5}{4}x^2\partial_x + \frac{25}{4}xy\partial_x + \frac{1}{4}x^2\partial_y - \frac{5}{4}xy\partial_y + 5y^2\partial_y - \frac{5}{4}yz\partial_z - \frac{19}{4}y^2 \\ &\quad - \frac{1}{4}x\partial_z - \frac{25}{4}x + 30y. \end{aligned}$$

In this case, the module $M^{\log D}$ is not holonomic, so $\mathcal{O}_f \neq M^{\log D}$ directly.

6.5. *Example 5:* $D \equiv (f = (xz + y^2)(y^4 + x^5 + yx^4) = 0) \subset \mathbf{C}^3$

The divisor is free of Spencer type with $\delta_1, \delta_2, \delta_3$ as a global basis of $Der(\log D)$:

$$\begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{pmatrix} = \begin{pmatrix} -x^2 - 3/4xy & -5/4xy - y^2 & 1/2y^2 - xz - 5/4yz \\ -3/4x^3 - 5xy + 1/4y^2 & -1/4x^3 - x^2y - 25/4y^2 & 1/2x^2y - 5/4x^2z - 15/2yz + 1/4z^2 \\ 5x^2 + 15/4xy & 25/4xy + 5y^2 & 15/2xz + 25/4yz \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}.$$

We have

$$\delta_1(f) = m_1 f = (-7x - 6y) f,$$
$$\delta_2(f) = m_2 f = (-6x^2 - 75/2y + 1/4z) f,$$
$$\delta_3(f) = m_3 f = (75/2x + 30y) f.$$

Again we check whether $Ext^3_{\mathcal{D}}(\widetilde{M}^{\log D}, \mathcal{O})$ is zero using SMC at level 3, where

$$Ext^3_{\mathcal{D}}(\widetilde{M}^{\log D}, \mathcal{O}) = \mathcal{O}/im\ \varphi_3^t$$

and

$$\varphi_3 = (\delta_3 + m_3 + p_3, -\delta_2 - m_2 - p_2, \delta_1 + m_1 + p_1)\,,$$

with $p_1 = -17/4x - 4y$, $p_2 = -22/4x^2 - 25y + 1/2z$, $p_3 = 95/4x + 20y$.

Now it is clear that $1 \in \mathcal{O}$ but $1 \notin im\ \varphi_3$, as $im\ \varphi_3 \in (x, y)$.

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