Finding multiplies solutions for non-linear integer programming

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Abstract

We explain how to compute all the solutions of a nonlinear integer problem using the algebraic test-sets associated to some linear subproblem. These test-sets are obtained using Gröbner bases. We compare our method with previous approaches .

Keywords: Integer non-lineal optimization, multiplies solutions, test-sets, Groebner basis

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1 Introduction

In many real-life combinatorial optimization problems is of great interest for the decision-maker to have not only one solution, but the set of *all* optimal solutions (see [10] or [7], for example). The information provided by this set can give some additional information about the solutions, and sometimes is a first step for multi-objective optimization. On the other hand, sometimes these problems requires non-linear constraints to be modeled properly.

A method for problems of the form

$$\begin{aligned} &\min\ cx^t \\ &\text{s.t. } Ax^t \leq b^t \\ &x \in \Omega \\ &x \in \mathbb{N}^n \end{aligned} \quad (P)$$

where $A \in \mathbb{Z}^{m \times n}, c \in \mathbb{Z}^n, b \in \mathbb{Z}^m$ and the region Ω is defined by linear and nonlinear constraints was proposed in [9]. This method makes use of the test-sets associated to the linear subproblem

$$\begin{aligned} &\min\ cx^t \\ &\text{s.t. } Ax^t \leq b^t \\ &x \in \mathbb{N}^n \end{aligned} \quad (P_L)$$

A set $T \subset \mathbb{Z}^n$ is a test-set associated to (P_L) if $T \subset \ker(A)$, and for any non optimal x feasible for P_L there exists a $t \in T$ such that $x - t$ is feasible and $c(x-t) < c(x)$. As a consequence, starting from a optimal point $\hat{x}$ of (P_L) you can *recover* the set of all the feasible points, adding elements of the test-set until you eventually complete all the feasible region. In this way you can obtain the optimal points of (P) *walking back* from the linear optimal point until you reach the region Ω. Technical details can be found in [9]. There are several ways of computing test-sets, and one of the most efficient is using Gröbner bases (see for example [3]) with the software `4ti2` (see [5]).

In [1] and [6] the method of [9] is applied to real-life size problems with very competitive results. In this work: 1) we explain how to modify the walk-back method to obtain all the optimal points and 2) we compare its performance with the natural generalization of the algorithm presented in [10] using the computational system COUENNE (see [2]).

2 An algebraic algorithm to find all optimal points.

The main idea of our method is quite simple: starting from optimal point of the linear problem (P_L), we add test-set vectors until we find the points inside the non-linear region Ω. We save *all* points that has the best cost into Ω. The pseudocode is the following one:

```
INPUT: $c, A, b; \Omega$; optimal point $\beta$ of $P_L$; $T$ associated test-set of $P_L$.

$Opt := \emptyset;$
$Leaves := \{\beta + t | \forall t \in T\} \cap \mathbb{N}^n$
$costOpt = \infty$

IF $\beta \in \Omega$
THEN $Opt := \{\beta\};$
        $costOpt := c\beta^t$

WHILE $(Leaves \neq \emptyset)$ DO
    FOR $h \in Leaves$ DO
        IF $c(h) < costOpt$
                $Leaves = (Leaves \setminus \{h\}) \cup (\{h + t | \forall t \in T\} \cap \mathbb{N}^n)$
                IF $h \in \Omega$
                THEN $Opt = \{h\};$
                        $costOpt = ch^t;$
                        $Leaves = (Leaves \setminus \{h\}) \cup (\{h + t | \forall t \in T\} \cap \mathbb{N}^n)$

                        ♯ the list of old candidates is deleted
                        ♯ and updated with a new candidate

        ELSE IF $c(h) > costOpt$
        THEN $Leaves = Leaves \setminus \{h\}$
        ♯ these branches are discarded

        ELSE IF $c(h) = costOpt$
        THEN $Leaves = (Leave \setminus \{h\}) \cup (\{h + t | \forall t \in T\} \cap \mathbb{N}^n)$
                IF $h \in \Omega$
                THEN $Opt = Opt \cup \{h\};$
                ♯ a new candidate to be an optimal point has been obtained

END WHILE

OUTPUT: $Opt$ the set of all optimal points with cost $costeOpt$
```

3 Computational experiments: reliability of series-parallel systems

Reliability problems are considered an important measure in the design of engineering processes, in which a series of systems is similar to a chain in which all the components must operate, since the failure of one of these components will suppose the failure of the complete system.

A mathematical model to minimize the cost of the system is

$$\begin{aligned} \min\ & \sum_{i=0}^{n}\sum_{j=1}^{k} c_{ij}x_{ij} \\ s.t.\ \ & R(x) \geq R_0 \\ & \sum_{j=1}^{k} x_{ij} \geq 1,\ \forall i = 1,\dots,n \\ & 0 \leq x_{ij} \leq u_{ij},\ \forall i = 1,\dots,n,\ j = 1,\dots,k \\ & x_{ij} \in \mathbb{Z}^{+} \end{aligned} \tag{1}$$

with $R(x) = \prod_{i=1}^{n}\left(1 - \prod_{j=1}^{k}(1-r_{ij})^{x_{ij}}\right)$. Where:

- n: number of subsystems
- k_i: number of different kinds of components in subsystem i.
- r_{ij}: reliability of component j in the subsystem i.
- c_{ij}: cost of component j in the subsystem i.
- l_{ij}, u_{ij}: upper and lower dimensions of the numbers of components j in the subsystem i. We can suppose that $l_{ij} = 0$.
- R_0: Minimum reliability required return for the whole system.
- x_{ij}: numbers of components j in the subsystem i.

In our experiments, we have considered that all subsystem have equal numbers of components, that is, $k_i = k,\ \forall i$.

- $r_{ij} \in (0.9, 0.99), \forall i = 1,\dots,n,\ j = 1,\dots,k$
- $c_{ij} \in \{1,2,\dots,10\}, \forall i = 1,\dots,n,\ j = 1,\dots,k$
- $R_0 = 0.9$
- $u_{ij} = 4,\ \forall i = 1,\dots,n,\ j = 1,\dots,k$
- $n \in \{3, 4\}$

- $k \in \{2, 3\}$

We have run about 120 examples to test our algorithm coded in Python in a computer with an Intel Core i5, 3.5 Ghz, 8 Gb of RAM, under Ubuntu. The General Cut row stands for the natural generalization of the method proposed in [10], we have been used COUENNE ([2]) for the example which have been sent to `neos-server.org`.

	Number of examples	% examples
Test-set	84	70 %
General cut	36	30 %

Table 1
Reliability examples $n = 3, 4,\ k = 2, 3$ with all solutions.

We can observe that:

	Number of examples	% examples
Test-set	42	97,67 %
General cut	1	2,33 %

Table 2
Reliability examples $n = 3, 4,\ k = 2, 3$ with multiplies solutions.

	Number of examples	% examples
Test-set	42	54,55 %
General cut	35	45,45 %

Table 3
Reliability examples $n = 3, 4,\ k = 2, 3$ an unique solution.

	3 Systems		4 Systems	
	2 components	3 components	2 components	3 components
Test-set	0,07	0,19	0,37	0,51
General cut	0,43	1,23	0,84	1,50

Table 4
Average CPU times Reliability examples $n = 3, 4,\ k = 2, 3$.

- The Test-Set method and the General Cut are both exact
- The General Cut is worse in CPU time than the test-set method.

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