Explicit Comparison Theorems for $\mathcal{D}$-modules

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We prove in an explicit way a duality formula between two A_2-modules $M^{\log}$ and $\widetilde{M}^{\log}$ associated to a plane curve and we give an application of this duality to the comparison between $\widetilde{M}^{\log}$ and the A_2-module of rational functions along the curve. We treat the analytic case as well.

1. Introduction

Let $R = \mathbf{C}[x_1, \ldots, x_n]$ be the ring of complex polynomials in n variables. Denote by $\partial_i = \frac{\partial}{\partial x_i}$ the partial derivative with respect to x_i and by $A_n = R\langle \partial_1, \ldots, \partial_n\rangle$ the complex Weyl algebra of order n.

Let $f \in R$ be a non-zero polynomial. The ring of quotients $R_f = \{\frac{g}{f^m} \mid g \in R, m \geq 0\}$ is a left A_n-module by considering $\partial_i(\frac{g}{f^m}) = \frac{\partial_i(g)}{f^m} - m\frac{g\partial_i(f)}{f^{m+1}}$ for $1 \leq i \leq n$. From the existence of the Bernstein polynomial (Bernstein, 1972), R_f is a finitely generated A_n-module. More precisely, there exists an integer $k \geq 1$ such that $R_f = A_n\frac{1}{f^k}$ (i.e. R_f has a single generator $\frac{1}{f^k}$ as left A_n-module). The integer k is related to the roots of the (polynomial) b-function of f (Bernstein, 1972).

A $\mathbf{C}$-derivation of R, $\delta = \sum_i a_i\partial_i$ is said to be logarithmic w.r.t. f if $\delta(f) = \sum_i a_i\partial_i(f) = af$ for some $a \in R$. We denote by $\mathrm{Der}(R, \log f)$ the R-module of logarithmic derivations w.r.t. f and by $I^{\log f}$ (or simply $I^{\log}$) the left ideal of A_n generated by $\mathrm{Der}(R, \log f)$. The quotient module $A_n/I^{\log}$ is denoted by $M^{\log}$.

The set of differential operators $\delta + a \in A_n$, for $\delta \in \mathrm{Der}(R, \log f)$ verifying $\delta(f) = af$, generates a left ideal in A_n that will be denoted by $\widetilde{I}^{\log}$. The quotient module $A_n/\widetilde{I}^{\log}$ is denoted by $\widetilde{M}^{\log}$.

From $\widetilde{I}^{\log} \subset \mathrm{Ann}_{A_n}(1/f)$ we deduce a natural morphism from $\widetilde{M}^{\log}$ to $A_n\frac{1}{f}$ and by composing with the embedding $A_n\frac{1}{f} \subset R_f$ we obtain a natural morphism

$$\phi : \widetilde{M}^{\log} \to R_f.$$

Most results in this paper are concerned with the case $n = 2$. The main goal is to prove in a explicit way a duality formula between $M^{\log}$ and $\widetilde{M}^{\log}$ (in the sense of A_n-modules). The duality between $M^{\log}$ and $\widetilde{M}^{\log}$ extends to the category of analytic $\mathcal{D}$-modules (Theorem 3.1) and we give an application of this duality to the comparison between $\widetilde{M}^{\log}$ and R_f (Section 4).

To achieve these goals we calculate explicit free resolutions of both $M^{\log}$ and $\widetilde{M}^{\log}$. For $M^{\log}$ the free resolution is the Spencer logarithmic resolution of Calderón (1997) (see also Calderón, 1999).

Finally we show that some results can be generalized to special examples in dimension 3.

2. Preliminaries: Analytic $\mathcal{D}$-modules

For each $p = (p_1, \ldots, p_n) \in \mathbf{C}^n$ denote by $\mathcal{O}_p$ the ring of germs of holomorphic functions in p. The ring $\mathcal{O}_p$ is isomorphic to the ring $\mathbf{C}\{x_1 - p_1, \ldots, x_n - p_n\}$ of convergent power series in a neighborhood of p and we have $R \subset \mathcal{O}_p$. Instead of the rings R and A_n we can consider $\mathcal{O}_p$ and $\mathcal{D}_p = \mathcal{O}_p\langle\partial_1, \ldots, \partial_n\rangle$ the ring of linear differential operators with coefficients in $\mathcal{O}_p$. In fact A_n is a subring of $\mathcal{D}_p$ and the elements in $\mathcal{D}_p$ can be written as finite sums $\sum_\alpha a_\alpha \partial^\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, $a_\alpha \in \mathcal{O}_p$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

More generally, we can consider on $X = \mathbf{C}^n$ the sheaf $\mathcal{D}_X$ of linear differential operators with holomorphic coefficients.

Let us fix a point $p \in \mathbf{C}^n$. For each $f \in R$ (or more generally $f \in \mathcal{O}_p$) we denote by $\mathrm{Der}(\mathcal{O}_p, \log f)$ the $\mathcal{O}_p$-module of $\mathbf{C}$-derivations of $\mathcal{O}_p$ logarithmic w.r.t. f (i.e. $\mathbf{C}$-derivations δ such that $\delta(f) = af$ for some $a \in \mathcal{O}_p$).

According to Saito (1980) we say that f is free at p if there exists a family $\{\delta_1, \ldots, \delta_n\} \subset \mathrm{Der}(\mathcal{O}_p, \log f)$, $\delta_i = \sum_j c_{ij}\partial_j$, such that the following condition holds:

$$\det((c_{ij})) = u_p f, \qquad \text{for some } u_p \in \mathcal{O}_p,\ u_p(p) \neq 0. \tag{$*$}$$

If f is free at p then $\mathrm{Der}(\mathcal{O}_p, \log f)$ is a free $\mathcal{O}_p$-module of rank n with basis $\{\delta_1, \ldots, \delta_n\}$ verifying the condition $(*)$ (Saito, 1980). As the inclusion $R \subset \mathcal{O}_p$ is flat, for each element $f \in R$, free at p, there exists a basis $\Delta = \{\delta_1, \ldots, \delta_n\}$ of $\mathrm{Der}(\mathcal{O}_p, \log f)$ with coefficients c_{ij} in R. Such a basis can be computed in an algorithmic way by considering a finite system $\mathcal{S}$ of generators of the R-module $Syz_R(\partial_1(f), \ldots, \partial_n(f), f)$ of syzygies among $(\partial_1(f), \ldots, \partial_n(f), f)$. Each syzygy $\sum_i a_i\partial_i(f) + mf = 0$ produces the logarithmic derivation $\sum_i a_i\partial_i$. Indeed, we have $\mathrm{Der}(R, \log f) \simeq Syz_R(\partial_1(f), \ldots, \partial_n(f), f)$. If there is no family of n derivations in $\mathcal{S}$ holding $(*)$, then f is not free at p.

Remark. Suppose now $f \in R$ free at every point in $\mathbf{C}^n$. Then $\mathrm{Der}(R, \log f)$ is a locally free R-module, so it is free (using the theorem of Quillen–Suslin). The basis of this free R-module could be obtained in the general case applying, for example, the algorithms of Logar and Sturmfels (1992).

However, in dimension 2 we have an alternative way of computing a basis of the free $\mathbf{C}[x_1, x_2]$-module $\mathrm{Der}(\mathbf{C}[x_1, x_2], \log f)$ using the Hilbert–Burch theorem (e.g. Eisenbud, 1994). If f defines a smooth plane curve, there is nothing to calculate. Suppose f is not smooth. We apply the Hilbert–Burch theorem to the ideal J generated by the homogenized polynomials $h(f), h(\partial_1(f)), h(\partial_2(f))$ in $S = \mathbf{C}[x_0, x_1, x_2]$ because the variety $V(J)$ has dimension 0 in the projective plane $\mathbf{P}_2(\mathbf{C})$. More precisely, J has a minimal free resolution (that can be computed explicitly) of the form

$$0 \longrightarrow S^2 \xrightarrow{A} S^3 \longrightarrow J \longrightarrow 0,$$

where A is the matrix whose rows are a set of generators of the module of syzygies $Syz_S(h(f), h(\partial_1(f)), h(\partial_2(f)))$. Dehomogenizing (making $x_0 = 1$) the matrix A produces the matrix whose rows generate the module $Syz_R(f, \partial_1(f), \partial_2(f))$. These two rows must be a basis. We think that this argument could be generalized to dimension n.

From now on we assume $p = 0$ in $\mathbf{C}^n$ and we will write $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{O}_0 = \mathcal{O}$. We consider on $\mathcal{D}$ (resp. on A_n) the filtration by the order of the differential operators. The order of $P = \sum_\alpha a_\alpha\partial^\alpha$ is the maximum value of $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $a_\alpha \neq 0$. The graded associated ring is the polynomial ring $\mathrm{gr}(\mathcal{D}) = \mathcal{O}[\xi]$ (resp. $\mathrm{gr}(A_n) = R[\xi]$) where $\xi = (\xi_1, \ldots, \xi_n)$.

The principal symbol of $P \in \mathcal{D}$ (resp. $P \in A_n$) is the element of $\mathrm{gr}(\mathcal{D})$ (resp. $\mathrm{gr}(A_n)$) defined by $\sigma(P) = \sum_{|\alpha|=d} a_\alpha \xi^\alpha$ where d is the order of P.

For each left ideal I in $\mathcal{D}$ (resp. in A_n) we denote by $\mathrm{gr}(I)$ the ideal of $\mathrm{gr}(\mathcal{D})$ (resp. $\mathrm{gr}(A_n)$) generated by the family $\sigma(P)$ for $P \in I$. The characteristic variety of A_n/I is the algebraic set of $\mathbf{C}^{2n}$ (denoted by $Ch(A_n/I)$) defined by the ideal $\mathrm{gr}(I) \subset \mathrm{gr}(A_n)$. In the analytic case the characteristic variety of $\mathcal{D}/I$ is the germ of subvariety in $U \times \mathbf{C}^n$ defined by the ideal $\mathrm{gr}(I) \subset \mathrm{gr}(\mathcal{D})$; here U is a small neighbourhood of the origin in $\mathbf{C}^n$ such that the coefficients of the elements of a finite system of generators of I are holomorphic functions on U.

A left A_n-module A_n/I is said to be holonomic if $\dim Ch(A_n/I) = n$ and we have a similar definition for $\mathcal{D}$-modules.

Remark. For each $f \in R$ we can consider the (left) ideal $I^{\log}_{an}$ generated by $\mathrm{Der}(\mathcal{O}, \log, f)$ in $\mathcal{D}$ and the (left) ideal $\widetilde{I}^{\log}_{an}$ generated by the family $\delta + a$ for vector fields δ such that $\delta(f) = af$ with $a \in \mathcal{O}$. We denote $M^{\log}_{an} = \mathcal{D}/I^{\log}_{an}$ and $\widetilde{M}^{\log}_{an} = \mathcal{D}/\widetilde{I}^{\log}_{an}$. By flatness of the extension $R \subset \mathcal{O}$ we have the equalities $I^{\log}_{an} = \mathcal{D}I^{\log}$ and $\widetilde{I}^{\log}_{an} = \mathcal{D}\widetilde{I}^{\log}$. Then $M^{\log}_{an} = M^{\log} \otimes_R \mathcal{O}$ and $\widetilde{M}^{\log}_{an} = \widetilde{M}^{\log} \otimes_R \mathcal{O}$.

Given a left holonomic A_n-module M, the dual module of M (denoted by M^*) is the left A_n-module associated to the right A_n-module $Ext^n_{A_n}(M, A_n)$ (Björk, 1979). We have the analogous definition for left holonomic $\mathcal{D}$-modules.

3. Duality

In this section we suppose $n = 2$ and f a reduced polynomial in R. For each $p \in \mathbf{C}^2$ let us denote by $\mathcal{O}_p$ the ring of germs of holomorphic functions in the neighborhood of p.

By Saito (1980, 1.7) $\mathrm{Der}(\mathcal{O}_p, \log f)$ is $\mathcal{O}_p$-free of rank 2 for all $p \in \mathbf{C}^2$ and so, according to the results in 2, $\mathrm{Der}(R, \log f)$ is R-free of rank 2.

Let $\{\delta_1, \delta_2\}$ be a basis of $\mathrm{Der}(R, \log f)$ (and hence a basis of $\mathrm{Der}(\mathcal{O}, \log f)$). Let us write

$$\begin{cases} \delta_1 = c_{11}\partial_1 + c_{12}\partial_2, \\ \delta_2 = c_{21}\partial_1 + c_{22}\partial_2 \end{cases}$$

for some polynomials $c_{ij} \in R$.

According to the first remark of Section 2 we can suppose that

$$\det((c_{ij})) = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = f.$$

According to Calderón (1997) (see also Calderón, 1999, Corollary 4.2.2), $\{\sigma(\delta_1), \sigma(\delta_2)\}$ is a regular sequence in $\mathrm{gr}(\mathcal{D})$ and then also in $\mathrm{gr}(A_2)$. In particular

$$\langle \sigma(\delta_1), \sigma(\delta_2) \rangle = \mathrm{gr}(I^{\log}) = \mathrm{gr}(\widetilde{I}^{\log}),$$

where $\langle \sigma(\delta_1), \sigma(\delta_2) \rangle$ denote the ideal of $\mathrm{gr}(A_2)$ generated by $\{\sigma(\delta_1), \sigma(\delta_2)\}$. From this equality one can deduce $Ch(\widetilde{M}^{\log}) = Ch(M^{\log})$ and $\dim(Ch(A_2/I^{\log})) = 2$. So, both modules $M^{\log}$ and $\widetilde{M}^{\log}$ are holonomic. From the last remark of Section 2 we obtain $Ch(M^{\log}_{an}) = Ch(M^{\log})$ and $Ch(\widetilde{M}^{\log}_{an}) = Ch(\widetilde{M}^{\log})$.

Now we will compute free resolutions of $M^{\log}$ and $\widetilde{M}^{\log}$.

Remember we have an explicitly computed basis $\{\delta_1, \delta_2\}$ of $\mathrm{Der}(R, \log f)$, $\delta_i = c_{i1}\partial_1 + c_{i2}\partial_2$, with $\det((c_{ij})) = f$ and explicitly computed polynomials m_i verifying $\delta_i(f) = m_i f$. Let us write $[\delta_1, \delta_2] = \alpha_1\delta_1 + \alpha_2\delta_2$ for some (explicitly computed) α_1, α_2 in R.

From Calderón (1997) (see also Calderón, 1999) a free resolution of $M^{\log}$ is

$$0 \longrightarrow A_2 \xrightarrow{\psi_2} A_2^2 \xrightarrow{\psi_1} A_2 \longrightarrow M^{\log} \longrightarrow 0$$

where ψ_2 is defined by the matrix $(-\delta_2 - \alpha_1, \delta_1 - \alpha_2)$ and ψ_1 by $\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$. So, one has the following proposition.

Corollary 3.1. $\mathrm{Ext}^2_{A_2}(M^{\log}, A_2) \simeq A_2/J$ *where* J *is the right ideal of* A_2 *generated by* $\{\delta_1 - \alpha_2, \delta_2 + \alpha_1\}$.

Proposition 3.1. *A free resolution of* $\widetilde{M}^{\log}$ *is*

$$0 \longrightarrow A_2 \xrightarrow{\phi_2} A_2^2 \xrightarrow{\phi_1} A_2 \longrightarrow \widetilde{M}^{\log} \longrightarrow 0 \qquad (**)$$

where ϕ_2 *is defined by the matrix*

$$(-\delta_2 - m_2 - \alpha_1, \delta_1 + m_1 - \alpha_2),$$

and ϕ_1 *by* $\begin{pmatrix} \delta_1 + m_1 \\ \delta_2 + m_2 \end{pmatrix}$.

Proof. It is easy to prove that $(**)$ is a complex of A_2-modules. To check its exactness, it is enough to consider the order filtration on that complex and to verify the exactness of the resulting complex (see Björk, 1979, Chapter 2, Lemma 3.13). We are using here the same argument of Calderón (1997), (see also Calderón, 1999, 4.1.3).

The graded associated complex to $(**)$ is precisely

$$0 \longrightarrow \mathrm{gr}(A_2) \xrightarrow{M_1} \mathrm{gr}(A_2)^2 \xrightarrow{M_2} \mathrm{gr}(A_2) \longrightarrow \mathrm{gr}(\widetilde{M}^{\log}) \longrightarrow 0,$$

where the matrices are

$$M_1 = (-\sigma(\delta_2), \sigma(\delta_1)), \quad M_2 = \begin{pmatrix} \sigma(\delta_1) \\ \sigma(\delta_2) \end{pmatrix},$$

which is exact because $\{\sigma(\delta_1), \sigma(\delta_2)\}$ is a regular sequence in $\mathrm{gr}(A_2)$ (see Calderón, 1999). □

Theorem 3.1. *We have* $(M^{\log})^* \simeq \widetilde{M}^{\log}$ *and the same result holds in the analytic case.*

Proof. According to 3.1 it is enough to prove the equalities

$$-\delta_1^T + \alpha_2 = \delta_1 + m_1, \; -\delta_2^T - \alpha_1 = \delta_2 + m_2,$$

where $()^T$ means the corresponding adjoint operator.

We denote by C the matrix (c_{ij}) and by $\underline{\delta}$ (resp. $\underline{\partial}$) the vector (δ_1, δ_2) (resp. (∂_1, ∂_2)). We can write

$$[\delta_1, \delta_2] = (\alpha_1, \alpha_2)\underline{\delta}^t = (\alpha_1, \alpha_2)C\underline{\partial}^t$$

where $()^t$ means transpose.

On the other hand,

$$[\delta_1, \delta_2] = (\delta_1(c_{21}) - \delta_2(c_{11}), \delta_1(c_{22}) - \delta_2(c_{12}))\underline{\partial}^t.$$

From the last two equalities and multiplying by $Adj(C)^t$ we obtain

$$(\alpha_1, \alpha_2)f = (\delta_1(c_{21}) - \delta_2(c_{11}), \delta_1(c_{22}) - \delta_2(c_{12}))Adj(C)^t.$$

It follows that

$$(\alpha_1, \alpha_2) = (m_2 + \partial_1(c_{21}) + \partial_2(c_{22}), m_1 - \partial_1(c_{11}) - \partial_2(c_{12}))$$

using $\delta_i(f) = (c_{i1}\partial_1 + c_{i2}\partial_2)(c_{11}c_{22} - c_{12}c_{21}) = m_i f$.

The same method can be applied to establish that $(M_{an}^{\log})^* \simeq \widetilde{M}_{an}^{\log}$. $\square$

Remark. In dimension n, if $M^{\log}$ admits an analogous free resolution as in 3.1, we have a proof for the last theorem that generalizes the ideas above. See Castro and Ucha (2000).

4. An Application: Comparing Modules

In this section $n = 2$.

A polynomial $f \in R$ is said to be quasi-homogeneous if there exists $w = (w_1, w_2) \in \mathbf{N}^2$ such that $f(x_1^{w_1}, x_2^{w_2})$ is an homogeneous polynomial and $w_i > 0$ for $i = 1, 2$.

The duality formula has an interesting application in order to compare $\widetilde{M}^{\log}$ to R_f and $\widetilde{M}_{an}^{\log}$ to $\mathcal{O}[1/f]$. We need two previous technical propositions to give the theorem.

Proposition 4.1. *If f is a quasi-homogeneous (reduced) polynomial, then* $\widetilde{I}^{\log} = \mathrm{Ann}_{A_2}(1/f)$ *and* $\widetilde{I}_{an}^{\log} = \mathrm{Ann}_{\mathcal{D}}(1/f)$

Proof. By flatness we only have to consider the first case. We have

- Let s be an indeterminate and let us denote $A_2[s] = A_2 \otimes_{\mathbf{C}} \mathbf{C}[s]$. Let α_0 be the smallest root of the global b-function of f. If $\alpha \notin \alpha_0 + 1 + \mathbf{N}$ then
$$\mathrm{Ann}_{A_2}(f^\alpha) = \{P(\alpha) | P(s) \in \mathrm{Ann}_{A_2[s]}(f^s)\}.$$
See Kashiwara (1976, 6) or Saito *et al.* (2000, 5.3.13) for a proof.
- $\mathrm{Ann}_{A_2[s]}(f^s) = \langle \chi - s, \partial_1(f)\partial_2 - \partial_2(f)\partial_1 \rangle$ where $\chi(f) = f$. See Yano (1978, 2.24).
- If f is a plane curve then the local b-function has no integer roots less than -1 (Varchenko, 1982). From this fact, as the global b-function is the least common multiple of the b-functions localized at any point (Mebkhout and Narváez-Macarro, 1991), then the global b-function has the same property.

We deduce that

$$\mathrm{Ann}_{A_2}(1/f) = \langle \chi + 1, \partial_1(f)\partial_2 - \partial_2(f)\partial_1 \rangle,$$

where χ is an Euler vector field associated to the quasi-homogeneous curve f. Clearly, these elements of the annihilator generate the ideal $\widetilde{I}^{\log}$. $\square$

Proposition 4.2. *If f is not a quasi-homogeneous (reduced) plane curve, then*

$$\mathrm{Ext}^2_{\mathcal{D}}(\widetilde{M}_{an}^{\log}, \mathcal{O}) \neq 0.$$

PROOF. The proof of this statement contains, as an essential ingredient, a re-reading of the demonstration of Calderón *et al.* (1999, Theorem 3.7).

By proposition 3.1, a free resolution of $\widetilde{M}^{\log}_{an}$ is

$$0 \longrightarrow \mathcal{D} \xrightarrow{\phi_2} \mathcal{D}^2 \xrightarrow{\phi_1} \mathcal{D} \longrightarrow \widetilde{M}^{\log}_{an} \longrightarrow 0,$$

where ϕ_2 is the matrix

$$(-\delta_2 - m_2 - \alpha_1, \delta_1 + m_1 - \alpha_2).$$

Now we apply to the complex above the functor $\mathrm{Hom}_{\mathcal{D}}(-, \mathcal{O})$. Hence,

$$\mathrm{Ext}^2_{\mathcal{D}}(\widetilde{M}^{\log}, \mathcal{O}) \simeq \mathcal{O}/Img(\phi_2^*).$$

Here ϕ_2^* denotes the associated mapping to ϕ_2 by applying the functor Hom. To guarantee that this vector space has dimension greater than zero, it is enough to show that a pair of functions $h_1, h_2 \in \mathcal{O}$ such that

$$(-\delta_2 - m_2 - \alpha_1, \delta_1 + m_1 - \alpha_2) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 1, \qquad (***)$$

does not exist, that is to say, that $1 \notin Img(\phi_2^*)$.

Let us take $\delta_1 = c_{11}\partial_1 + c_{12}\partial_2$. As $m_1 - \alpha_2 = \partial_1(c_{11}) + \partial_2(c_{12})$, (from the proof of 3.1) we will show either c_{11} and c_{12} have no linear parts, or after derivation these linear parts become 0.

Of course f has no quadratic part: in that case, because of the classification of the singularities in two variables, f would be equivalent to a quasi-homogeneous curve $x_1^2 + x_2^{k+1}$, for some k. Then we can suppose that

$$f = f_n + f_{n+1} + \cdots = \sum_{k \geq n} f_k = \sum_{k \geq n} \sum_{i+j=k} a_{ij} x_1^i x_2^j,$$

where $n \geq 3$ and $f_n \neq 0$.

We will write

$$\delta_1 = c_{11}\partial_1 + c_{12}\partial_2 = \delta_0^1 + \delta_1^1 + \cdots = \sum_{k \geq 0} \sum_{i+j=k+1} (\beta_{ij}^1 x_1^i x_2^j \partial_1 + \gamma_{ij}^1 x_1^i x_2^j \partial_2),$$

where the linear part δ_0^1 is $(x_1 x_2) A_0 (\partial_1 \partial_2)^t$, and A_0 is a 2×2 matrix with complex coefficients.

If $A_0 = 0$, we have finished. Otherwise, the possibilities of the Jordan form of A_0 are

$$A_0 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad A_0 = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}.$$

As δ_1 is not an Euler vector (because f is not quasi-homogeneous), we deduce:

- If we take the first Jordan form, then (see the cited demonstration of Calderón *et al.*, 1999) $f_n = x_1^p x_2^q$ and $\delta_0 = q x_1 \partial_1 - p x_2 \partial_2$. After a sequence of changes of coordinates we have that $f = x_1^p x_2^q$ with $p + q = n \geq 3$, that contradicts that f is reduced.
- For the second Jordan form with $\lambda_1 \neq 0$, it has to be $f_n = 0$, that contradicts that f has its initial part of degree n.
- For the second option with $\lambda_1 = 0$ we have $\delta_0^1 = x_2 \partial_1$ and, in this situation, the linear part of c_{11} is x_2. If we precisely apply ∂_1, we obtain 0.

In a similar way, we prove the same for $m_2 + \alpha_1$.

So the minimal order of monomials in

$$(-\delta_2 - m_2 - \alpha_1)(h_1) + (\delta_1 + m_1 - \alpha_2)(h_2)$$

is greater or equal to 1, for any $h_1, h_2 \in \mathcal{O}$. So $(***)$ has no solution. □

THEOREM 4.1. *The natural morphism* $\widetilde{M}^{\log}_{an} \xrightarrow{\psi} \mathcal{O}[\frac{1}{f}]$ *is an isomorphism if and only if* f *is a quasi-homogeneous (reduced) polynomial.*

PROOF. If f is quasi-homogeneous then $\widetilde{I}^{\log}_{an} = \mathrm{Ann}_{\mathcal{D}}(1/f)$ because of Proposition 4.1 and therefore ψ is an isomorphism. Reciprocally, if ψ is an isomorphism, then

$$\mathrm{Ext}^2_{\mathcal{D}}(\mathcal{O}[1/f], \mathcal{O}) \simeq \mathrm{Ext}^2_{\mathcal{D}}(\widetilde{M}^{\log}_{an}, \mathcal{O}).$$

Because of a result of Mebkhout (1989), $\mathrm{Ext}^2_{\mathcal{D}}(\mathcal{O}[1/f], \mathcal{O}) = 0$ and, if we take into account proposition 4.2, we obtain that f has to be quasi-homogeneous. □

REMARK. Another argument could be used in the second part of the last proof applying a result of Torrelli (1998, 3.2.2.3): if f is not a quasi-homogeneous (reduced) plane curve then $\mathrm{Ann}_{\mathcal{D}}(1/f)$ cannot be generated by elements of degree one in ∂ and then $\mathrm{Ann}_{\mathcal{D}}(1/f) \neq \widetilde{I}^{\log}_{an}$.

REMARK. In the polynomial case we have an analogous theorem to 4.1. Suppose first that $\widetilde{M}^{\log}$ is isomorphic to R_f. Then by the last remark of Section 2 follows $\widetilde{M}^{\log}_{an} \simeq \mathcal{O}[1/f]$ so f is a quasi-homogeneous polynomial in R. Reciprocally if f is quasi-homogeneous we use Proposition 4.1.

5. An Explicit Example in Dimension 3

Let $R = \mathbf{C}[x, y, z]$. In Calderón (1997, 4.1.3), the condition of being Koszul-free (that is, the principal symbols of the generators of $I^{\log}_{an}$ form a regular sequence) is a sufficient condition to assure the existence of the free resolution of $M^{\log}_{an}$. We illustrate in this section that this condition is not necessary to have the duality formula.

We will consider the surface defined by Calderón (1997) with

$$h = xy(x + y)(xz + y) = 0.$$

We obtain in this case that:

- $\mathrm{Ann}_{A_3}(1/h) = \widetilde{I}^{\log}$.
- $\widetilde{M}^{\log} \simeq (M^{\log})^*$.

and the results are valid in the analytic case as well. The calculation is as follows:

1. We can compute a basis of Der $(R, \log h)$ with a set of generators of the syzygies among $h, \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}$. We obtain

$$\delta_1 = x\partial_x + y\partial_y$$
$$\delta_2 = xz\partial_z + y\partial_z$$
$$\delta_3 = x^2\partial_x - y^2\partial_y - xz\partial_z - yz\partial_z$$

with

$$\delta_1(h) = 4h, \quad \delta_2(h) = xh, \quad \delta_3(h) = (2x - 3y)h,$$

and

$$\begin{vmatrix} x & y & 0 \\ 0 & 0 & xz + y \\ x^2 & -y^2 & -xz - yz \end{vmatrix} = h.$$

2. The global b-function of h in A_3 is

$$b(s) = (4s + 5)(2s + 1)(4s + 3)(s + 1)^3.$$

This polynomial has no integer roots smaller than -1, so

$$R_h \simeq A_3 \frac{1}{h}.$$

3. We check that $\mathrm{Ann}_{A_3}(1/h)$ is equal to $\widetilde{I}^{\log}$ using Groebner bases obtained from the corresponding sets of generators. The computations of the b-function and the annihilating ideal of h^s have been made using the algorithms of Oaku (1997), implemented in Maekawa *et al.* (2000). The same Groebner basis computation shows that $M^{\log} = A_3/I^{\log}$ (where $I^{\log} = (\delta_1, \delta_2, \delta_3)$) is holonomic.
4. We calculate a free resolution of the module $M^{\log}$. The first module of syzygies is generated in this case by the relations deduced from the expressions of the $[\delta_i, \delta_j]$ with $i \neq j$:

$$\begin{aligned} [\delta_1, \delta_2] &= \delta_2 \\ [\delta_1, \delta_3] &= \delta_3 \\ [\delta_2, \delta_3] &= -x\delta_2. \end{aligned}$$

The second module of syzygies is generated by only one element $\mathbf{s} = (s_1, s_2, s_3)$:

$$\begin{aligned} s_1 &= -y^2\partial_y + x^2\partial_x - zy\partial_z - zx\partial_z - x \\ s_2 &= -y\partial_z - xz\partial_z \\ s_3 &= y\partial_y + x\partial_x - 2. \end{aligned}$$

The computation of this free resolution is performed using Groebner bases.
5. With a similar procedure to the one used in 3.1 we obtain that $(M^{\log})^*$ is the left A_3-module associated with the right A_3-module $A_3/(s_1, s_2, s_3)A_3$. Then

$$(M^{\log})^* \simeq A_3/(s_1^t, s_2^t, s_3^t).$$

It is enough to compute s_1^t, s_2^t, s_3^t and check that they generate $\widetilde{I}^{\log}$. Hence

$$(M^{\log})^* = (A_3/I^{\log})^* \simeq A_3/\widetilde{I}^{\log} = \widetilde{M}^{\log}.$$

Remark. As we pointed, it is interesting that $\{\sigma(\delta_1), \sigma(\delta_2), \sigma(\delta_3)\}$ does not form a regular sequence in $\mathrm{gr}(A_3) = \mathbf{C}[x, y, z, \xi, \eta, \zeta]$. We have $z\eta\zeta - \xi\zeta \notin \langle\sigma(\delta_1), \sigma(\delta_2)\rangle$ such that

$$(z\eta\zeta - \xi\zeta)\sigma(\delta_3) \in \langle\sigma(\delta_1), \sigma(\delta_2)\rangle.$$

So h is not Koszul free and nevertheless duality holds.

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References

Bernstein, I. N. (1972). Analytic continuation of generalized functions with respect to a parameter. *Funct. Anal. Appl.*, **6**, 273–285.

Björk, J-E. (1979). *Rings of Differential Operators*. Amsterdam, North-Holland.

Calderón-Moreno, F. J. (1997). Operadores diferenciales logarítmicos con respecto a un divisor libre, Ph.D. Thesis, Universidad de Sevilla.

Calderón-Moreno, F. J. (1999). Logarithmic differential operators and logarithmic De Rham complexes relative to a free divisor. *Ann. Sci. E.N.S.*, $\mathbf{4}^e$ **série, t. 32**, 701–714.

Calderón-Moreno, F. J., Mond, D., Narváez-Macarro, L., Castro-Jiménez, F. J. (1999). Logarithmic Cohomology of the Complement of a Plane Curve. Preprint, University of Warwick, 3/1999.

Castro-Jiménez, F. J., Ucha, J. M. (2000). Free divisors and duality for D-modules. Preprint math.AG/0103085.

Eisenbud, D. (1994). *Commutative Algebra with a View toward Algebraic Geometry*. New York, Springer.

Kashiwara, M. (1976). B-functions and holonomic systems. *Invent. Math.*, **38**, 33–53.

Logar, A., Sturmfels, B. (1992). Algorithms for the Quillen–Suslin theorem. *J. Algebra*, **145**, 231–239.

Maekawa, M., Ohara, K., Okutani, Y., Noro, M., Tamura, K., Takayama, N. (2000). OpenXM project, `http://www.openxm.org`.

Mebkhout, Z. (1989). Le théorème de comparaison entre cohomologies de De Rham d'une variété algébrique complexe et le théorème d'existence de Riemann. *Publications Mathematiques*, **69**, 47–89.

Mebkhout, Z., Narváez-Macarro, L. (1991). La théorie du polynôme de Bernstein-Sato pour les algèbres de Tate et de Dwork-Monsky-Washnitzer. *Ann. Sci. École Norm. Sup.*, $\mathbf{4}^e$ **série, t. 24**, 227–256.

Oaku, T. (1997). Algorithms for the b-function and $\mathcal{D}$-modules associated with a polynomial. *J. Pure Appl. Algebra*, **117 & 118**, 495–518.

Saito, K. (1980). Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo*, **27**, 256–291.

Saito, M., Sturmfels, B., Takayama, N. (2000). *Gröbner Deformations of Hypergeometric Differential Equations*. New York, Springer.

Torrelli, T. (1998). Équations Fonctionelles Pour Une Fonction Sur Un Espace Singulier, Ph. D. Thesis, Université de Nice-Sophia Antipolis.

Varchenko, A. N. (1982). Asymptotic Hodge structure in the vanishing cohomology. *Math. USSR Izvestija*, **3**, 469–512.

Yano, T. (1978). On the theory of b-Functions. *Publ. RIMS, Kyoto Univ.*, **14**, 111–202.