

The number of conformally equivalent maximal graphs

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Abstract

We show that the number of entire maximal graphs with finitely many singular points that are conformally equivalent is a universal constant that depends only on the number of singularities, namely 2^n for graphs with $n + 1$ singularities. We also give an explicit description of the family of entire maximal graphs with a finite number of singularities all of them lying on a plane orthogonal to the limit normal vector at infinity.

1 Introduction

The present paper is devoted to the study of maximal graphs in the Lorentz-Minkowski space $\mathbb{L}^3 = (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$, where $\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1y_1 + x_2y_2 - x_3y_3$. Maximal graphs appear in a natural way when considering variational problems. If $u : \Omega \subset \mathbb{R}^2 \equiv \{x_3 = 0\} \rightarrow \mathbb{R}$ is a smooth function defining a spacelike graph in \mathbb{L}^3 (that is, a graph with Riemannian induced metric), then its area is given by the expression

$$A(u) = \int_{\Omega} \sqrt{1 - |\nabla u|^2},$$

(recall that $|\nabla u| < 1$ since the graph is spacelike). The corresponding equation for the critical points of the area functional in \mathbb{L}^3 is

$$\operatorname{Div} \frac{\nabla u}{\sqrt{1 - |\nabla u|^2}} = 0. \tag{1}$$

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Spacelike graphs satisfying this (elliptic) differential equation are called *maximal graphs*, since they represent local maxima for the area functional. Geometrically, this condition is equivalent to the fact that the mean curvature of the surface in \mathbb{L}^3 vanishes identically. Besides of their mathematical interest, these surfaces, and more generally those having constant mean curvature, have a significant importance in physics [MT].

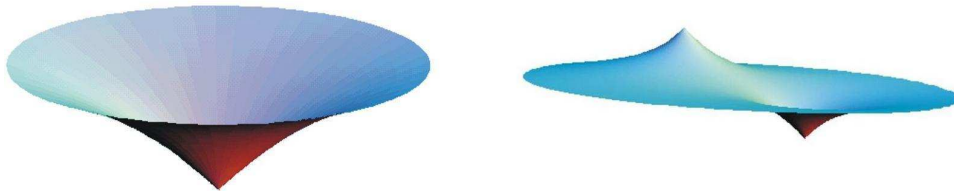


Figure 1: Left: Lorentzian catenoid. Right: Riemann type surface.

From a global point of view, it is known by Calabi's theorem [Ca] that the only everywhere regular complete maximal surface is the plane. In particular, there are no entire maximal graphs besides the trivial one. This motivates to allow the existence of singularities, i.e., points of the surface where the metric degenerates. We will focus here our attention to the case where the singular set is the *smallest* possible, that is, a finite number of points. The first and most known example is the Lorentzian catenoid (Figure 1, left), an entire maximal graph with one singular point, and actually the only one as proved in [Ec], but there are examples with any arbitrary number of singularities. Among them it is worth mentioning the Riemann type maximal graphs (Figure 1, right) obtained in [LLS], with two singular points and characterized by the property of being foliated by circles and lines. Other highly symmetric examples with arbitrary number of singularities (even infinitely many) were constructed in [FL2] (Figure 2). Actually there is a huge amount of such graphs. Indeed, in [FLS] the authors study the moduli space \mathcal{G}_n of entire maximal graphs with $n+1$ singularities, proving that it is an analytic manifold of dimension $3n+4$. A global system of coordinates in this space is given by the position of the singular points in \mathbb{L}^3 and a real number called *the logarithmic growth* that controls the asymptotic behavior.

If $u : \Omega \rightarrow \mathbb{R}$ defines a maximal graph, singular points appear where $|\nabla u| = 1$. At a singular point, the PDE (1) stops being elliptic. Moreover, the tangent plane of the surface becomes lightlike, the normal vector has no well defined limit, and the surface is asymptotic to a half of the light cone of the singular point. For this reason they are called

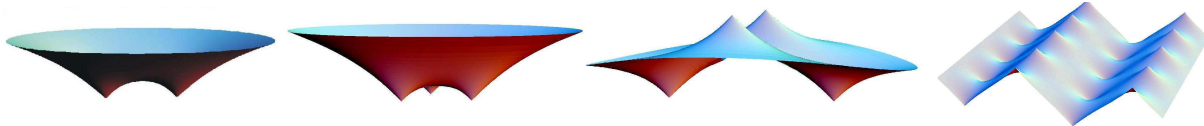


Figure 2: Entire maximal graphs with isolated singularities.

conelike singularities. It should be pointed out that a maximal surface with isolated cone-like singularities is an entire graph if and only if it is complete (that is, divergent curves have infinite length), as proved in [FLS].

If S is a maximal surface with singular set $F \subset S$, its regular part $S \setminus F$ has a natural conformal structure associated to its Riemannian metric. The conformal type of a maximal surface has been widely studied, for example in [FL1, AA] parabolicity criteria for maximal surfaces are given, but there also exist hyperbolic examples, [A11, A12, MUY].

In the case of entire graphs with $n+1$ singularities, it turns out that $S \setminus F$ is conformally equivalent to a n -connected circular domain of the complex plane, that is, the plane with $n+1$ discs removed. Each one of these boundary circles corresponds to a singular point of the graph. Our aim in this paper is to study the space of entire maximal graphs with the same conformal structure, that is

Problem. *Given a n -connected circular domain Ω of the complex plane, how many entire maximal graphs with $n+1$ singularities are there whose conformal structure is biholomorphic to Ω ?*

We will answer this question by proving that the number of (non congruent) maximal graphs supported by a fixed circular domain is finite and does not depend on the circular domain, but only on the number of connected component of the boundary, that is, the number of singularities. This will be the aim of Section 3. Thus, our problem reduces to compute the number of graphs for a fixed conformal structure. In Section 4 we will fix an specific n -connected circular domain (Definition 4.1) and we will find out how many entire graphs are there with this conformal structure, obtaining that there are exactly 2^n non-congruent surfaces. Moreover, the graphs constructed in Section 4 can be characterized by the property of having all their singularities in a plane orthogonal to the limit normal vector at infinity (Theorem 5.1).

Let us point out that our main result contrast with the analogous problem in the

related theory of solutions to the Monge-Ampère equation

$$\text{Hess}(u) = 1. \tag{2}$$

Specifically, in [GMM] it is proved that any solution to (2) globally defined on \mathbb{R}^2 with finitely many isolated singularities is uniquely determined by its associated conformal structure, which is also a circular domain of the complex plane.

2 Preliminaries

2.1 Maximal surfaces

A differentiable immersion $X : M \rightarrow \mathbb{L}^3$ from a surface M to \mathbb{L}^3 is said to be spacelike if the tangent plane at any point is spacelike, that is to say, the induced metric on M is Riemannian. The Gauss map of a spacelike surface in \mathbb{L}^3 takes values in the sphere of radius -1 , $\mathbb{H}^2 = \{p \in \mathbb{L}^3 : \langle p, p \rangle = -1\}$. Since \mathbb{H}^2 has two connected components, $\mathbb{H}_+^2 = \mathbb{H}^2 \cap \{x_3 > 0\}$ and $\mathbb{H}_-^2 = \mathbb{H}^2 \cap \{x_3 < 0\}$, spacelike surfaces are always orientable.

A maximal immersion is a spacelike immersion whose mean curvature vanishes. A remarkable property of maximal surfaces in \mathbb{L}^3 is the existence of a Weierstrass-type representation for maximal surfaces, similar to the one of minimal surfaces. Roughly speaking, the Weierstrass representation of a conformal maximal immersion $X : M \rightarrow \mathbb{L}^3$ is a pair (g, ϕ_3) of a meromorphic function and a holomorphic 1-form defined on M such that, up to translation, the immersion can be recovered as

$$X(p) := \text{Real} \int_{p_0}^p \left(\frac{i}{2} \left(\frac{1}{g} - g \right) \phi_3, \frac{-1}{2} \left(\frac{1}{g} + g \right) \phi_3, \phi_3 \right), \tag{3}$$

where $p_0 \in M$ is an arbitrary point. It is worth mentioning that g agrees with the stereographic projection of the Gauss map of the surface. We refer to [Ko, Ec] and Theorem 2.1 below for more details.

We will focus our attention to entire maximal graphs, that is, maximal graphs defined on the whole plane $\{x_3 = 0\}$. As we explained in Section 1, the only everywhere regular example is the plane [Ca], and so singularities (i.e., points where the induced metric converges to zero) appear in a natural way in this setting. The following theorem condense the information regarding the global structure of entire maximal graphs with isolated singularities (also called *conelike singularities*).

Proposition 2.1 (Global behavior, [FLS]) *Let S be a surface with isolated singularities in \mathbb{L}^3 . Then the following two statements are equivalent:*

- (i) S is a complete embedded maximal surface,
- (ii) S is an entire graph over any spacelike plane.

In this case S is asymptotic at infinity to either a half-catenoid or a plane. If we label $F \subset S$ as the singular set, $S \setminus F$ is conformally equivalent to $\Omega_0 := \mathbb{C} \setminus \cup_{p \in F} D_p$, where D_p are pairwise disjoint closed discs. Moreover, the associated conformal reparameterization $X : \Omega_0 \rightarrow \mathbb{L}^3$ extends analytically to $\Omega := \mathbb{C} \setminus \cup_{p \in F} \text{Int}(D_p)$ by putting $X(\partial(D_p)) = X(p)$. The point $p_\infty = \infty$ is called the end of the surface.

2.2 Double surface and representation theorem

As showed in the previous section, the underlying conformal structure of an entire maximal graph with an isolated set of singularities is conformally equivalent to a circular domain in the complex plane. We now go into this aspect in depth to obtain a representation theorem for entire maximal graphs with a finite number of singularities that will be crucial in our study.

For any finitely connected circular domain $\Omega = \mathbb{C} \setminus \cup_{j=1}^k \text{Int}(D_j)$, let Ω^* be its mirror surface and \mathcal{N} the double surface obtained by gluing Ω and Ω^* along their common boundaries as in Figure 3 (see [FK] for an explicit description of this construction). It is clear that \mathcal{N} is a compact Riemann surface of genus $k - 1$ minus two points. We denote by $\overline{\mathcal{N}}$ the compactification of \mathcal{N} by adding these two points.

Finally, we label $J : \mathcal{N} \rightarrow \mathcal{N}$ as the mirror involution mapping a point in Ω into its mirror image and viceversa. Notice that J extends to an antiholomorphic involution on $\overline{\mathcal{N}}$, and its fixed point set of J coincides with $\partial\Omega \equiv \partial\Omega^*$.

This double surface is used in [FLS] to give a characterization of complete maximal surfaces with a finite number of singularities in terms of their Weierstrass data:

Theorem 2.1 (Representation) *Let $X : \Omega \rightarrow \mathbb{L}^3$ be a conformal immersion of an entire maximal graph with $n + 1$ conelike singularities, where $\Omega = \mathbb{C} \setminus \cup_{j=1}^{n+1} \text{Int}(D_j)$, D_j pairwise disjoint closed discs. Label $\overline{\mathcal{N}}$ as the compactification of the double surface of Ω . Then the Weierstrass data of X , (g, ϕ_3) , satisfy:*

- (i) g is a meromorphic function on $\overline{\mathcal{N}}$ of degree $n + 1$, $|g| < 1$ on Ω , and $g \circ J = \frac{1}{\bar{g}}$,

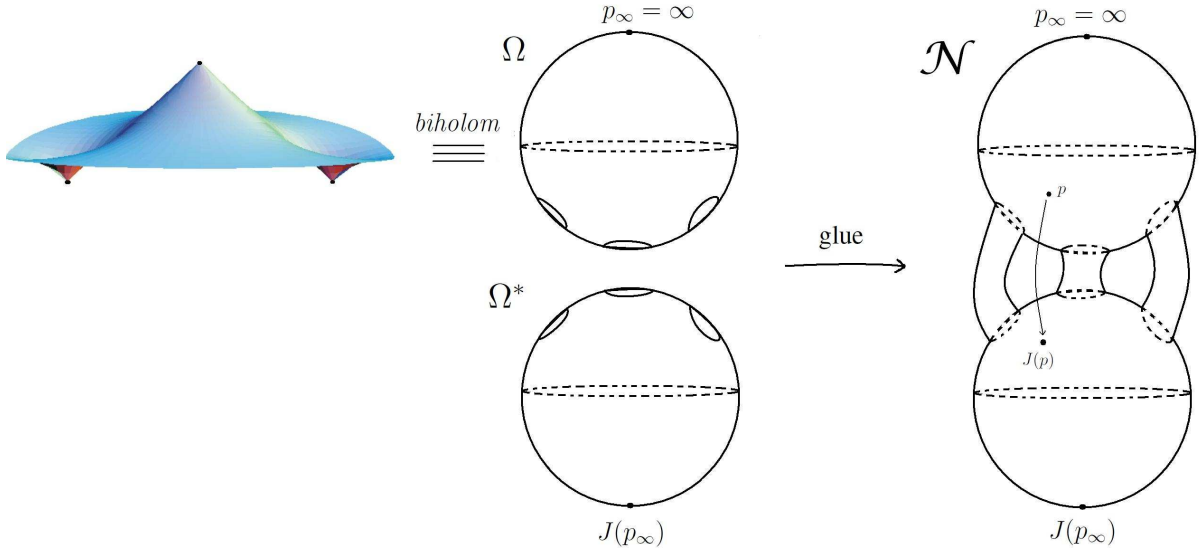


Figure 3: The double surface associated to a maximal surface with singularities

- (ii) ϕ_3 is a holomorphic 1-form on $\overline{\mathcal{N}} \setminus \{p_\infty, J(p_\infty)\}$, where $p_\infty = \infty \in \Omega$, with poles of order at most two at p_∞ and $J(p_\infty)$, and satisfying $J^*(\phi_3) = -\overline{\phi_3}$,
- (iii) the zeros of ϕ_3 in $\overline{\mathcal{N}} \setminus \{p_\infty, J(p_\infty)\}$ coincide (with the same multiplicity) with the zeros and poles of g .

Conversely, let $\overline{\mathcal{N}}$ be a compact genus n Riemann surface. Suppose that there exists an antiholomorphic involution $J : \overline{\mathcal{N}} \rightarrow \overline{\mathcal{N}}$ such that the fixed point set of J consists of $n + 1$ pairwise disjoint analytic Jordan curves γ_j , $j = 0, 1, \dots, n$, and that $\overline{\mathcal{N}} \setminus \bigcup_{j=0}^n \gamma_j = \Omega_0 \cup J(\Omega_0)$, where $\overline{\Omega_0}$ is topologically equivalent (and so conformally) to $\overline{\mathbb{C}}$ minus a finite number of pairwise disjoint open discs.

Then, for any (g, ϕ_3) satisfying (i), (ii) and (iii) the map $X : \overline{\Omega_0} \setminus \{p_\infty\} \rightarrow \mathbb{L}^3$ given by Equation (3) is well defined and $S = X(\overline{\Omega_0} \setminus \{p_\infty\})$ is an entire maximal graph with conelike singularities corresponding to the points $q_j := X(\gamma_j)$, $j = 0, 1, \dots, n$.

2.3 Divisors on a Riemann surface.

An important part of our work in this paper deals with classical properties of divisors on compact Riemann surfaces. We recall here the notation and basic results that will be used in the sequel (see [FK] for more details).

Let Σ be a Riemann surface. A (multiplicative) divisor on Σ is a formal symbol $\mathcal{D} = p_1^{k_1} \cdot \dots \cdot p_h^{k_h}$, where $p_{k_j} \in \Sigma$ and $k_j \in \mathbb{Z}$. We can also write the divisor \mathcal{D} as

$$\mathcal{D} = \prod_{p \in \Sigma} p^{k_p},$$

where $k_p \neq 0$ only for finitely many. We call $\text{Div}(\Sigma)$ to the multiplicative group of divisors on Σ . We can define an order in $\text{Div}(\Sigma)$, indeed, given $\mathcal{D}_1 = \prod_{p \in \Sigma} p^{k_p^1}$ and $\mathcal{D}_2 = \prod_{p \in \Sigma} p^{k_p^2} \in \text{Div}(\Sigma)$, we say that $\mathcal{D}_1 \geq \mathcal{D}_2$ if $k_p^1 \geq k_p^2$ for all $p \in \Sigma$.

The degree of the divisor \mathcal{D} is defined as the integer $\deg(\mathcal{D}) = \sum_{p \in \Sigma} k_p$. $\mathcal{D} \in \text{Div}(\Sigma)$ is an *integral* divisor if $k_p \geq 0$ for any $p \in \Sigma$. We denote by $\text{Div}_k(\Sigma)$ the set of integral divisors of degree k .

Let f be a meromorphic function on Σ . The associated divisor of f is defined as $(f) = \prod_{p \in \Sigma} p^{k_p}$, where for any zero (resp. pole) p of f of order α we have $k_p = \alpha > 0$ (resp. $k_p = -\alpha < 0$), and $k_p = 0$ in other case. Likewise we define the associated divisor of a meromorphic 1-form. Classical theory of Riemann surfaces give that both functions and 1-forms are determined by their divisors up to a multiplying constant. Moreover, the degree of a meromorphic function on a compact Riemann surface is 0, whereas the associated divisor of a 1-form has degree $2n - 2$, where n is the genus of the surface.

3 A first approach to the problem

Let G be an entire maximal graph with $n + 1$ conelike singularities. When $n = 0$, Ecker [Ec] characterized the Lorentzian catenoid (Figure 1, left) as the unique entire maximal graph with 1 singular point, so we will assume from now on that $n \geq 1$.

As showed in Section 2.1, the underlying conformal structure of a maximal graph is conformally equivalent to a circular domain $\Omega \subset \mathbb{C}$ with $n + 1$ boundary components. Moreover, if we rotate the surface so that the end is horizontal, as a consequence of Theorem 2.1 the divisors of the Weierstrass data (g, ϕ_3) of G must be of the form

$$(g) = \frac{D \cdot p_\infty}{D^* \cdot p_\infty^*}, \quad (\phi_3) = \frac{D \cdot D^*}{p_\infty \cdot p_\infty^*}, \quad (4)$$

where $p_\infty = \infty \in \overline{\Omega}$ is the end of the surface, $D \in \text{Div}_n(\overline{\Omega})$, and $*$ denotes the mirror involution. Notice that the divisor D determines uniquely the Weierstrass data (g, ϕ_3) up to replacing by $(e^{i\theta} g, A \phi_3)$, for any $\theta, A \in \mathbb{R}$.

Conversely, for any integral divisor D of degree n on $\overline{\Omega}$ such that there exist a meromorphic function g and 1-form ϕ_3 satisfying (4), it is immediate to check that (g, ϕ_3) fulfill conditions (i) to (iii) in Theorem 2.1. Thus by means of Equation (3) we can obtain an entire maximal graph with $n + 1$ conelike singularities, horizontal end, and conformal structure Ω . Moreover, this graph is unique up to homotheties and vertical rotations.

The problem of finding out whether exists a pair (g, ϕ_3) satisfying (4) for a given divisor D is closely related with the Abel-Jacobi map of the corresponding compact Riemann surface $\overline{\mathcal{N}}$, $\varphi : \text{Div}(\overline{\mathcal{N}}) \rightarrow \mathcal{J}(\overline{\mathcal{N}})$, where $\mathcal{J}(\overline{\mathcal{N}})$ denotes the Jacobian bundle of $\overline{\mathcal{N}}$ (see [FK] for its definition). Abel Theorem states that $\mathcal{D} \in \text{Div}(\overline{\mathcal{N}})$ is the divisor associated to a meromorphic function (resp. 1-form) on $\overline{\mathcal{N}}$ if and only if $\varphi(\mathcal{D}) = 0$ (resp. $\varphi(\mathcal{D}) = T$, where $T \in \mathcal{J}(\overline{\mathcal{N}})$ is a fixed element in the Jacobian bundle). Thus, in our case the divisors D coming from Weierstrass data are precisely those satisfying:

$$\varphi(D) + \varphi(p_\infty) - \varphi(D^*) - \varphi(p_\infty^*) = 0, \quad \varphi(D) + \varphi(D^*) - \varphi(p_\infty) - \varphi(p_\infty^*) = T.$$

This set of divisors is deeply studied in [FLS], proving that the previous two equations are equivalent to

$$2\varphi(D) - 2\varphi(p_\infty^*) = T. \tag{5}$$

Before going into the properties of this set, let us fix some notation. Let Ω be a n -connected circular domain and write $\partial\Omega = \cup_{j=0}^n \gamma_{c_j}(r_j)$, with $\gamma_{c_j}(r_j) = \{z \in \mathbb{C}, |z - c_j| = r_j\}$. Up to a Möbius transformations we can assume that $c_0 = 0$, $r_0 = 1$ and $c_1 \in \mathbb{R}^+$. Thus, we can parameterize the space \mathcal{T}_n of marked (i.e., with an ordering in the boundary components) n -connected circular domains (up to biholomorphisms) by their corresponding uplas $v = (c_1, r_1, \dots, c_n, r_1, \dots, r_n) \in \mathbb{R}^+ \times \mathbb{C}^{n-1} \times (\mathbb{R}^+)^n$, of centers and radii, with the convention $c_0 = 0$ and $r_0 = 1$. By this identification, \mathcal{T}_n can be considered as an open subset of $\mathbb{R}^+ \times \mathbb{C}^{n-1} \times (\mathbb{R}^+)^n$, and therefore it inherits a natural analytic structure of manifold of dimension $3n - 1$. We label as $\Omega(v)$ the circular domain defined by $v \in \mathcal{T}_n$. Now define the *spinorial bundle*

$$\mathcal{S}_n = \{(v, D) : v \in \mathcal{T}_n, 2\varphi_v(D) - 2\varphi_v(p_\infty^*) = T_v\},$$

where the subscript v refers to the double surface of $\Omega(v)$, then

Theorem 3.1 ([FLS]) *The spinorial bundle \mathcal{S}_n defined above is an analytical manifold of dimension $3n - 1$. Moreover, the map*

$$\nu : \mathcal{S}_n \rightarrow \mathcal{T}_n$$

$$\nu(v, D) = v$$

is a finitely sheeted covering.

Thus, the number of divisors $D \in \text{Div}_n(\overline{\Omega(v)})$ satisfying Equation (5) is a universal constant that depends not on the conformal structure $\Omega(v)$, but only on the number of boundary components (equivalently, the number of singularities of the maximal graph). As explained above, each divisor corresponds to a unique congruence class of entire maximal graphs with $n + 1$ singularities and conformal structure $\Omega(v)$. Thus we have the following

Corollary 3.1 *For each $n \in \mathbb{N}$ there exists a constant $C(n) \in \mathbb{N}$ such that, for any n -connected circular domain Ω , the number of non-congruent entire maximal graph with conformal structure biholomorphic to Ω is exactly $C(n)$.*

Remark 3.1 *Since the space \mathcal{T}_n is simply-connected, it follows from Corollary 3.1 that the number of connected components of \mathcal{S}_n is $C(n)$. In particular, the number of connected components of the moduli space of entire maximal graphs with $n + 1$ singularities is also $C(n)$.*

Indeed, label \mathcal{G}_n as the space of marked entire maximal graph with horizontal end and $n+1$ singularities, where a mark means an ordering $m = (q_0, \dots, q_n)$ of the singular points of the graph. As we commented in Section 1, \mathcal{G}_n can be endowed with a differentiable structure of manifold of dimension $3n + 4$ with coordinates given by $(G, m) \mapsto (m, c)$, being c the logarithmic growth at the end. On the other hand, we can consider the map

$$\begin{aligned} \epsilon : \mathcal{G}_n &\rightarrow \mathcal{S}_n \times \mathbb{L}^3 \times \mathbb{S}^1 \times \mathbb{R} \\ \epsilon((G, m)) &= ((v, D), q_0, g(1), h(1)) \end{aligned}$$

where, if (g, ϕ_3) denote the Weierstrass data of the graph, then

- $(v, D) \in \mathcal{S}_n$ is given by the conformal structure of G (with the order in $v \in \mathcal{T}_n$ given by the order in m), and the divisor D defined as in Equation (5),
- q_0 is the first singular point in m ,
- $h := \frac{\phi_3}{dz}$ (here z means the natural conformal parameter in $\Omega(v) \subset \mathbb{C}$, recall that $1 \in \partial\Omega(v)$ for all $v \in \mathcal{T}_n$).

Then, it is clear from the above explanation that ϵ is bijective. Moreover, the induced topology in \mathcal{G}_n by ϵ agree with the one given by its before mentioned differentiable structure, as proved in [FLS]. Thus, the number of connected components of \mathcal{G}_n is $C(n)$.

4 Counting maximal graphs on a given circular domain

As it was showed in the previous section, the number of maximal graphs that share the same underlying conformal structure only depends on the number of boundary components of the conformal support. Thus, in this section we will fix an specific circular domain and we will find out how many non-congruent maximal graphs are defined on that surface.

Let $n \in \mathbb{N}$, and $a_1 < a_2 < \dots < a_{2n+2} \in \mathbb{R}$. Throughout this section, $\overline{\mathcal{N}}_0$ will denote the (hyperelliptic) compact genus n Riemann surface associated to the function $\sqrt{\prod_{j=1}^{2n+2} (z - a_j)}$, that is,

$$\overline{\mathcal{N}}_0 := \{(z, w) \in \overline{\mathbb{C}}^2 : w^2 = \prod_{j=1}^{2n+2} (z - a_j)\}.$$

And we will also define $\mathcal{N}_0 = \overline{\mathcal{N}}_0 \setminus \{z^{-1}(\infty)\}$.

The surface $\overline{\mathcal{N}}_0$ can be realized as a two sheeted covering of the Riemann sphere. Indeed, consider two copies of $\overline{\mathbb{C}}$. Following [FK], we label these copies as sheet *I* and sheet *II*. We "cut" each copy along curves joining a_{2j+1} with a_{2j+2} , for any $j = 1, \dots, n$. We assume that these cuts does not intersect each others (see Figure 4). Each cut has two banks: a N-bank and a S-bank. We recover the surface $\overline{\mathcal{N}}_0$ by identifying the N-bank (resp. S-bank) of a cut in the sheet *I* with the corresponding S-bank (resp. N-bank) in the sheet *II*.

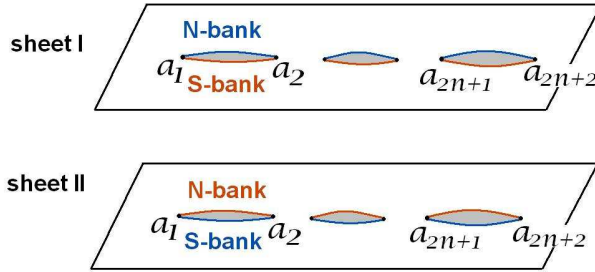


Figure 4: A model for the Riemann surface \mathcal{N}_0 .

We denote by $z, w : \overline{\mathcal{N}}_0 \rightarrow \overline{\mathbb{C}}$ the two canonical projections, whose associated divisors

are

$$(w) = \frac{a_1 \cdot \dots \cdot a_{2n+2}}{(p_\infty)^{n+1} \cdot (p_\infty^*)^{n+1}} \quad \text{and} \quad (dz) = \frac{a_1 \cdot \dots \cdot a_{2n+2}}{(p_\infty)^2 \cdot (p_\infty^*)^2},$$

where $a_j \equiv (a_j, 0)$ and $\{p_\infty, p_\infty^*\} = z^{-1}(\{\infty\})$. We will label p_∞ as the one where the coefficient of degree $-(n+1)$ of the Laurent series of w is -1 .

Finally we define $J_0 : \overline{\mathcal{N}}_0 \rightarrow \overline{\mathcal{N}}_0$ as the antiholomorphic involution given by $J_0(z, w) = (\bar{z}, -\bar{w})$. The fixed points of J_0 are the Jordan curves $\gamma_j = \{(z, w) \in \overline{\mathcal{N}}_0 : z \in [a_{2j-1}, a_{2j}]\}$, $j = 1, \dots, n+1$. Moreover, $\mathcal{N}_0 \setminus \cup_{j=1}^{n+1} \gamma_j$ has two connected components, each one of them corresponding to a single-valued branch of w , and biholomorphic to a n -connected circular domain.

Definition 4.1 *Let $n \in \mathbb{N}$, and $a_1 < a_2 < \dots < a_{2n+2} \in \mathbb{R}$. Consider the above defined compact Riemann surface*

$$\overline{\mathcal{N}}_0 := \{(z, w) \in \overline{\mathbb{C}}^2 : w^2 = \prod_{j=1}^{2n+2} (z - a_j)\},$$

with the antiholomorphic involution $J_0(z, w) = (\bar{z}, -\bar{w})$. Label Δ as the set of fixed points of J_0 . We will define $\bar{\Omega}_0$ as the closure of the connected component of $\overline{\mathcal{N}}_0 \setminus \Delta$ containing p_∞ , and Ω_0 will denote the circular domain $\Omega_0 := \bar{\Omega}_0 \setminus \{p_\infty\}$.

Proposition 4.1 *Let (g, ϕ_3) be Weierstrass data on Ω_0 of an entire maximal graph with $n+1$ singularities and horizontal end. Then there exists $n+1$ distinct points $\{b_1, \dots, b_{n+1}\} \subset \{a_1, \dots, a_{2n+2}\}$, such that*

$$g = e^{i\theta} \frac{w + P(z)}{w - P(z)} \quad \text{and} \quad \phi_3 = A \left(\frac{w}{P(z)} - \frac{P(z)}{w} \right) dz, \quad (6)$$

where $P(z) = \prod_{j=1}^{n+1} (z - b_j)$, $\theta \in \mathbb{R}$, and $A \in \mathbb{R}^*$.

Proof: By Theorem 2.1, the associated divisors to (g, ϕ_3) are given by

$$(g) = \frac{D \cdot p_\infty}{J(D) \cdot p_\infty^*} \quad \text{and} \quad (\phi_3) = \frac{D \cdot J(D)}{p_\infty \cdot p_\infty^*} \quad (7)$$

where $D \in \text{Div}_n(\bar{\Omega}_0)$. Here, p_∞ denotes the point in $\bar{\Omega}_0 \cap z^{-1}(\infty)$, and $p_\infty^* = J(p_\infty)$.

We will denote by $F : \overline{\mathcal{N}}_0 \rightarrow \overline{\mathcal{N}}_0$ the holomorphic involution given by $F(z, w) = (z, -w)$.

Claim 4.1 *In the above conditions there exist $n+1$ distinct points $\{b_1, \dots, b_{n+1}\} \subset \{a_1, \dots, a_{2n+2}\}$, such that $g = \frac{G_1}{G_2}$ for two meromorphic functions G_1, G_2 on $\overline{\mathcal{N}}_0$ satisfying*

$$a) (G_1) \geq \frac{p_\infty^*}{b_1 \cdot \dots \cdot b_{n+1}}$$

$$b) (G_2) \geq \frac{p_\infty}{b_1 \cdot \dots \cdot b_{n+1}}$$

Since g has degree $n+1$ and $\overline{\mathcal{N}}_0$ is hyperelliptic, the two meromorphic functions g and z satisfy a relation $P(g, z) = 0$, where P is a polynomial in two variables with algebraic degree two in the first one and $n+1$ in the second (see [FK]). We can rewrite this relation as $P_2(z)g^2 + P_1(z)g + P_0(z) = 0$, with P_i polynomials whose maximum algebraic degree is $n+1$. Solving this equation we obtain

$$g = \frac{-P_1 \pm \sqrt{P_1^2 - 4P_0P_2}}{2P_2}.$$

Consider the meromorphic function $f = \sqrt{P_1^2 - 4P_0P_2} = \pm(2gP_2 + P_1)$. Let us check that $f = cw$, for some constant $c \in \mathbb{R}^*$. Indeed, any meromorphic function on the hyperelliptic surface $\overline{\mathcal{N}}_0$ can be expressed as $f = R_1(z) + R_2(z)w$, with R_i rational functions (see [FK]). In our case, f^2 is a polynomial function in z , and so it follows that either $R_1 = 0$ or $R_2 = 0$. The last case would imply that g is a rational function of z , which is impossible from Equation (7) so $f = R_2(z)w$. Now observe that f has poles only at p_∞ and p_∞^* with order at most $n+1$, which implies that f/w is a holomorphic function on $\overline{\mathcal{N}}_0$, and therefore constant. Thus, $f = cw$ for some $c \in \mathbb{R}^*$. Up to replace P_i by $\pm cP_i$, $i = 1, 2$, we can suppose that

$$g = \frac{P_1 + w}{2P_2}.$$

We will also assume that the leading coefficient of P_1 is one. Since P_1 and P_2 are meromorphic functions of degree $\leq 2(n+1)$ that only depend on z , it is not hard to realize that (7) implies that

$$(P_1 + w) = \frac{D \cdot E}{(p_\infty)^{n-1} \cdot (p_\infty^*)^{n+1}} \quad \text{and} \quad (P_2) = \frac{J(D) \cdot E}{(p_\infty)^n \cdot (p_\infty^*)^n},$$

where $E := F(J(D)) \in \text{Div}_n(\overline{\Omega}_0)$. Thus, the meromorphic function

$$h = \frac{P_2(P_1 + w)}{w \prod_{e \in E} (z - z(e))} \frac{dz}{\phi_3}$$

satisfies that $(h) = \frac{E \cdot p_\infty}{F(E) \cdot p_\infty^*} = (\frac{1}{h} \circ F)$, and therefore up to a multiplying constant $h \circ F = 1/h$. On the other hand, $\deg(h) = n + 1$, and reasoning as before we can deduce that $h = (\hat{P}_1(z) + w)/\hat{P}_2(z)$, for some $\hat{P}_i(z)$ polynomial functions in z with algebraic degree less than or equal to $n + 1$. Since $h \circ F = 1/h$, we infer that $w^2 = \hat{P}_1^2 - \hat{P}_2^2$ and so, setting $S = -\hat{P}_1 - \hat{P}_2$ we can write $h = (S - w)/(S + w)$.

Looking at the divisor of h is immediate to realize that there exists an integral divisor B with $\deg B = n + 1$ such that:

$$(S - w) = \frac{E \cdot B}{p_\infty^n \cdot (p_\infty^*)^{n+1}} \quad \text{and} \quad (S + w) = \frac{F(E) \cdot B}{p_\infty^{n+1} \cdot (p_\infty^*)^n}.$$

Since points in B are zeros of both $S + w$ and $S - w$, they must be $n + 1$ distinct (recall that w only has simple zeroes) points of $\{a_1 \dots a_{2n+2}\}$. Setting $G_1 = \frac{P_1 + w}{S - w}$ and $G_2 = \frac{2P_2}{S - w}$ the claim is proved.

Claim 4.2 *Up to multiplicative constants, the functions G_1 and G_2 in Claim 4.1 are given by $G_1 = \frac{w}{P(z)} + 1$ and $G_2 = \frac{w}{P(z)} - 1$, being $P(z) = \prod_{j=1}^n (z - b_j)$.*

Call B to the integral divisor given by $B = b_1 \cdot \dots \cdot b_{n+1}$. By Riemann-Roch Theorem, the dimension of the linear space of meromorphic functions on $\overline{\mathcal{N}}_0$ satisfying condition *a*) (resp. *b*) in Claim 4.1 is $1 + d$ where d is the dimension of the linear space of meromorphic 1-forms ν on $\overline{\mathcal{N}}_0$ satisfying $(\nu) \geq \frac{B}{p_\infty}$ (resp. $(\nu) \geq \frac{B}{p_\infty}$). Let us see that $d = 0$.

Indeed, observe first that by the residues theorem, both spaces agree with the space $L(B)$ of holomorphic 1-forms ν with $(\nu) \geq B$. But since $\{\frac{dz}{w}, z\frac{dz}{w}, \dots, z^{n-1}\frac{dz}{w}\}$ is a basis for the space of holomorphic 1-forms on $\overline{\mathcal{N}}_0$, any $\nu \in L(B)$ must be of the form $\nu = P(z)\frac{dz}{w}$, where P is a polynomial with algebraic degree less than n . Thus, if a Weierstrass point a_{j_0} is a zero of ν then its order is at least two. It follows that the number of zeroes of the holomorphic 1-form ν is at least $2(n + 1)$ which is impossible because $\overline{\mathcal{N}}_0$ has genus n .

Therefore the dimension of the linear space of meromorphic functions satisfying condition *a*) (resp. *b*) in the Claim 4.1 is 1. It is easy to show that the function $\frac{w}{\prod_{j=1}^n (z - b_j)} + 1$ (resp. $\frac{w}{\prod_{j=1}^n (z - b_j)} - 1$) is a basis for this space, so Claim 4.2 is proved.

As a consequence of the previous claims, we can write:

$$g = \frac{G_1}{G_2} = c \frac{w + P(z)}{w - P(z)},$$

for a suitable constant $c \in \mathbb{C}^*$. As $g \circ J = 1/\bar{g}$ we infer that $c = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

To finish observe that the divisor of ϕ_3 coincides with the divisor for the 1-form $(\frac{w}{P(z)} - \frac{P(z)}{w})dz$, and as a consequence

$$\phi_3 = A \left(\frac{w}{P(z)} - \frac{P(z)}{w} \right) dz,$$

since $J^*(\phi_3) = -\overline{\phi_3}$ we get $A \in \mathbb{R}$. This concludes the proof. \square

To finish the classification of the entire maximal graphs on the given circular domain Ω_0 we need to find out when the pair given by (6) are actually Weierstrass data. This is done in the following proposition. Figure 4 shows two examples of the surfaces given by these Weierstrass representation.

Proposition 4.2 *Choose $b_1 < b_2 < \dots < b_{n+1}$ points in $\{a_1, \dots, a_{2n+2}\}$, and define $P(z) = \prod_{j=1}^{n+1} (z - b_j)$.*

Then the pair (g, ϕ_3) given by Equation (6) are Weierstrass data on Ω_0 of an entire maximal graph with $n+1$ singularities if and only if $b_j \in \{a_{2j-1}, a_{2j}\}$ for all $j = 1, \dots, n+1$.

Proof: We just have to check the conditions stated in Theorem 2.1. Recall that $J(z, w) = (\bar{z}, -\bar{w})$, and define $Q(z) = w^2/P(z) = \prod_{j=1}^{n+1} (z - c_j)$. For simplicity, we will assume that $\theta = 0$ and $A = 1$.

Conditions (ii) and (iii) are straightforward for all the possible values of b_1, \dots, b_{n+1} . Let us show when (i) is accomplished.

First, notice that $g^{-1}(1) = \{b_1, \dots, b_{n+1}\}$. In particular, $\deg(g) = n+1$. In particular, in order to be g the Gauss map of a maximal surface with conelike singularities, any connected component in $\partial\Omega_0$ must have exactly one point with $g = 1$, and so $b_j \in \{a_{2j-1}, a_{2j}\}$ for every $j = 1, \dots, n+1$.

Conversely, assume that $b_j \in \{a_{2j-1}, a_{2j}\}$, $j = 1, \dots, n+1$, and let us show that g has no critical points on $\partial\Omega_0 \equiv \cup_{j=1}^{n+1} [a_{2j-1}, a_{2j}]$. After some computations one easily gets that

$$dg = \frac{QdP - PdQ}{w(Q + P - 2w)}.$$

Thus for critical points in $\mathcal{N}_0 = \overline{\mathcal{N}_0} \setminus \{z^{-1}(\infty)\}$ we have $QdP = PdQ$, or equivalently,

$$\sum_{j=1}^{n+1} \frac{1}{z - b_j} = \sum_{j=1}^{n+1} \frac{1}{z - c_j}.$$

If we assume that $b_j \in \{a_{2j-1}, a_{2j}\}$ for all $j = 1, \dots, n+1$, and we have a point $p_0 \in [a_{2j_0-1}, a_{2j_0}] \subset \partial\Omega_0$, with $a_{2j_0-1} = b_{j_0}$ and $a_{2j_0} = c_{j_0}$ (the case $a_{2j_0-1} = c_{j_0}$ and $a_{2j_0} = b_{j_0}$ is similar) then we have that

$$\frac{1}{z(p_0) - c_j} < \frac{1}{z(p_0) - b_{j+1}} < 0, \quad j = 1, \dots, n+1,$$

(here we use the convention $b_{n+2} = b_1$), and this gives that p_0 cannot be a critical point of g .

To finish just notice that $g \circ J = 1/\bar{g}$ and therefore $|g| = 1$ on the $n+1$ connected components of $\partial\Omega_0$. Since g is injective on each one of these curves, and $\deg(g) = n+1$, then $|g| \neq 1$ on $\mathcal{N}_0 \setminus \partial\Omega_0$. Taking into account that $g(p_\infty) = 0$ we have that $|g| < 1$ on Ω_0 . \square

Definition 4.2 Let Ω_0 the circular domain given in Definition 4.1 for some real numbers $a_1 < \dots < a_{2n+2}$. For each subset $\tau = \{b_1, \dots, b_{n+1}\} \subset \{a_1, \dots, a_{2n+2}\}$ with $b_j \in \{a_{2j-1}, a_{2j}\}$, $j = 1, \dots, n+1$, we will define the G_τ as the entire maximal graph with $n+1$ singularities with Weierstrass data (g_τ, ϕ_3^τ) on Ω_0 given by

$$g_\tau = \frac{w + P(z)}{w - P(z)} \quad \text{and} \quad \phi_3^\tau = \left(\frac{w}{P(z)} - \frac{P(z)}{w} \right) dz,$$

where $P(z) = \prod_{j=1}^{n+1} (z - b_j)$.

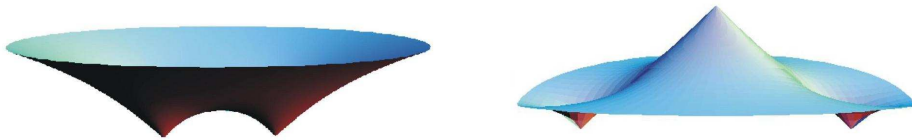


Figure 5: Two examples of the surfaces obtained for $n = 1$ and $n = 2$

Theorem 4.1 Let Ω_0 be the n -connected circular domain given in Definition 4.1. Then the number of non-congruent entire maximal graphs whose underlying conformal structure is Ω_0 is exactly 2^n .

Proof: From Propositions 4.1 and 4.2 we know that any maximal graph G with horizontal end defined on Ω_0 have Weierstrass data $(g = e^{i\theta} g_\tau, \phi_3 = A \phi_3^\tau)$, where $\theta \in \mathbb{R}$, $A \in \mathbb{R}^*$ and (g_τ, ϕ_3^τ) are given by Definition 4.2.

Observe that replacing the set τ by its complementary $\{a_1, \dots, a_{2n+2}\} \setminus \tau$ gives congruent surfaces (more specifically, (g, ϕ_3) are transform into $(-g, -\phi_3)$). So, we can assume without loss of generality that $b_1 = a_1$. To avoid congruences, we will also normalize so that $g(a_1) = h(a_1) = 1$, where $h = \frac{\phi_3}{dz}$. Looking at the expressions for g and ϕ_3 this means that $\theta = 0$, $A = 1$. Thus, the number of non-congruent maximal graphs defined on Ω_0 is the number of possible choices of $b_j \in \{a_{2j-1}, a_{2j+1}\}$, $j = 2, \dots, n+1$, which is 2^n . \square

Taking into account our previous discussion in Section 3, we can conclude that:

Theorem 4.2 *The number of non-congruent entire maximal graphs with the same conformal structure is 2^n , where $n+1$ is the number of (conelike) singularities.*

Equivalently, the number of connected components of the space \mathcal{G}_n of entire marked maximal graphs with $n+1$ singularities and horizontal end is 2^n .

5 Maximal graphs with coplanar singularities

We will prove now that the surfaces constructed in the previous section are characterized by the property of having all its singularities on a plane orthogonal to the limit normal vector at infinity. In particular, for $n = 1$, surfaces obtained in Section 4 describe the whole moduli space of the entire maximal graphs with two singular points.

Theorem 5.1 *Let $G \subset \mathbb{L}^3$ be an entire maximal graph with $n+1$ conelike singularities. Then G has all its singularities lying on a timelike plane in \mathbb{L}^3 orthogonal (in the Lorentzian sense) to the normal vector at the end if and only if G is congruent to one of the examples given in Definition 4.2.*

Proof: Assume that G has all its singularities in an orthogonal plane to the normal vector at the end. Up to a rigid motion in \mathbb{L}^3 we can assume that the end is horizontal and the singularities lie in the plane $\{x_1 = 0\}$. Let $X : \Omega \rightarrow G \subset \mathbb{L}^3$ a conformal reparameterization of G . By the uniqueness result in [Kly] (see also [FLS] Remark 2.5), the surface is symmetric with respect to the plane $\{x_1 = 0\}$. This symmetry induces an antiholomorphic involution $T : \overline{\Omega} \rightarrow \overline{\Omega}$ leaving $\partial\Omega$ globally fixed. It follows that T extends to an antiholomorphic involution $T : \overline{\mathcal{N}} \rightarrow \overline{\mathcal{N}}$, where \mathcal{N} is the mirror surface, by putting $T \circ J = J \circ T$ (J is the mirror involution). Moreover, if (g, ϕ_3) are the Weierstrass data of

the immersion, $g \circ T = \bar{g}$ and $T^*(\phi_3) = \bar{\phi}_3$. It is straightforward that T must have exactly two fixed points on every connected component of the circular domain $\partial\bar{\Omega}$. We call these points p_1, \dots, p_{2n+2} . Observe that the end $p_\infty \in \bar{\Omega}$ is also fixed by T .

Consider the holomorphic involution $F = J \circ T$, whose fixed points are exactly p_1, \dots, p_{2n+2} . Therefore, $\bar{\mathcal{N}}$ is a compact genus n Riemann surface with $2n+2$ fixed points, this means that $\bar{\mathcal{N}}$ is hyperelliptic with Weierstrass points p_1, \dots, p_{2n+2} (see [FK]),

$$\bar{\mathcal{N}} \equiv \{(z, w) \in \bar{\mathbb{C}}^2 : w^2 = \prod_{i=1}^{2n+2} (z - a_i)\},$$

where $(a_j, 0)$ corresponds to p_j for any j (and so $a_j \neq a_k$ for $k \neq j$). With this identification we have $F(z, w) = (z, -w)$. Up to a Möbius transformation we can suppose that $z(p_{2n+1}) = 1$, $z(p_{2n+2}) = -1$, and $z(p_\infty) = \infty$.

In what follows we will identify $a_j = (a_j, 0) \in \bar{\mathcal{N}}$. To prove $a_j \in \mathbb{R}$ notice that the divisor associated to the meromorphic 1-form $d(\overline{z \circ J})$ coincides with the one of dz and therefore $\overline{z \circ J} = kz + \lambda$, for some $k, \lambda \in \mathbb{R}$. Since $a_{2n+1} = 1$ and $a_{2n+2} = -1$ are fixed by J it follows that $\overline{z \circ J} = z$ which implies that $a_j \in \mathbb{R}$. Moreover, since $w^2 \circ J = \bar{w}^2$, then $w \circ J = \pm \bar{w}$. Taking into account that J interchanges the two points with $z = \infty$, namely p_∞ and $p_\infty^* = J(p_\infty)$, then $w \circ J = -\bar{w}$. Therefore $J(z, w) = (\bar{z}, -\bar{w})$ and $T(z, w) = (\bar{z}, \bar{w})$. In particular, Ω agrees with the circular domain Ω_0 defined in Definition 4.1 and by Propositions 4.1 and 4.2 we are done.

Conversely, let G_τ one of the graphs defined in Definition 4.2. Consider the involution $T(z, w) = (\bar{z}, \bar{w})$ on $\bar{\mathcal{N}}_0$ that fix globally any component of $\partial\Omega_0$. Moreover, $g_\tau \circ T = \bar{g}_\tau$ and $T^*(\phi_3^\tau) = \bar{\phi}_3^\tau$, thus, T induces an isometry on the resulting surface, namely $I(x_1, x_2, x_3) = (-x_1, x_2, x_3)$. Since $\{a_1, \dots, a_{2n+2}\}$ are fixed by T it follows that all the singularities lie in the plane $\{x_1 = 0\}$. \square

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