Optimal domain for the Hardy operator

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Abstract. We study the optimal domain for the Hardy operator considered with values in a rearrangement invariant space. In particular, this domain can be represented as the space of integrable functions with respect to a vector measure defined on a δ-ring. A precise description is given for the case of the minimal Lorentz spaces.

1 Introduction

Let S be the Hardy operator defined by

$$Sf(x) = \frac{1}{x}\int_0^x f(y)\,dy\ , \quad x \in (0,\infty)\,,$$

for any function $f \in L^1_{\text{loc}}(\mathbb{R}^+)$. Let X be a *Banach function ideal lattice* (abbreviated $BFIL$), i.e., X is a Banach space of real valued measurable functions on $\mathbb{R}^+$, satisfying that if $g \in X$ and $|f| \le |g|$ a.e., then $f \in X$ and $\|f\|_X \le \|g\|_X$ (see [1, 8] for further information). For such an X, there is a natural space on which S takes values in X, namely,

$$[S,X] = \{f\colon \mathbb{R}^+ \to \mathbb{R} \text{ measurable, } S|f| \in X\}\,.$$

The space $[S,X]$ is a $BFIL$ itself when endowed with the norm $\|f\|_{[S,X]} = \|S|f|\,\|_X$. Obviously, $S\colon [S,X] \to X$ is continuous. Even more, any $BFIL$ Y such that $S\colon Y \to X$ is well defined (and so S is continuous, since it is a positive linear operator between Banach lattices [11, p. 2]), is continuously contained in $[S,X]$. That is, $[S,X]$ is the *optimal domain* for S (considered with values in X) within the class of $BFIL$.

Similar assertions hold for operators T defined by a positive kernel K (i.e., $Tf(x) = \int_0^\infty f(y)K(x,y)\,dy$) such that $T|f| = 0$ a.e. implies $f = 0$ a.e. This general case has been studied in [3, 4], for K defined on $[0,1]\times[0,1]$, where the authors show that the optimal domain $[T,X]$ for T, is closely related to the space $L^1(\nu_X)$ of integrable functions with respect to the vector measure ν_X, defined by $\nu_X(A) = T(\chi_A)$ (assuming K and X satisfy the minimal conditions for ν_X to be a vector measure with values in X). Indeed, under suitable additional conditions, both spaces coincide and a precise description of them is given. The case when K is defined on $\mathbb{R}^+\times\mathbb{R}^+$ has been studied in [6]. Here, the vector measure ν_X associated to

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T is defined on the δ–ring of the bounded measurable sets of $\mathbb{R}^+$ (there are classical kernel operators, like the Hilbert transform, for which ν_X is not defined for sets of infinite measure). Again, under suitable conditions, $[T, X]$ coincides with $L^1(\nu_X)$. However, the Hardy operator does not satisfy these conditions, and we need to find a different argument to describe the space $[S, X]$.

In Section 2 we will study several general properties of $[S, X]$ in the case of *rearrangement invariant spaces* X (abbreviated r.i.; that is, if $g \in X$ and f is equimeasurable with g, then $f \in X$ and $\|f\|_X = \|g\|_X$), and show that the domain is never an r.i. space (Theorem 2.5). In Section 3, we prove that $[S, X]$ admits a vector valued integral representation, and in Section 4 we identify this domain for the minimal Lorentz space Λ_φ.

2 Optimal domain and r.i. spaces

We start with a particular case where we are able to identify the domain for S. We observe that $L^{1,\infty}(\mathbb{R}^+)$ is a quasi-Banach r.i. space.

Proposition 2.1 $[S, L^{1,\infty}(\mathbb{R}^+)] = L^1(\mathbb{R}^+)$, *with equality of norms.*

Proof. Recall that $\|g\|_{L^{1,\infty}(\mathbb{R}^+)} = \sup_{t>0} t\lambda_g(t)$, where $\lambda_g(t) = |\{|g| > t\}|$ is the distribution function of g (see [1]). Let us prove first the following formula for the distribution function of Sf: If $f \in L^1_{\text{loc}}(\mathbb{R}^+)$, $f \geq 0$, and $\{Sf > s\}$ has finite measure for all $s > 0$, then

$$\lambda_{Sf}(t) = \frac{1}{t}\int_{\{Sf>t\}} f(x)\,dx. \tag{1}$$

In fact, since $\{Sf > s\}$ is open and has finite measure, then $\{Sf > s\} = \cup_k (a_k, b_k)$, where $0 \leq a_k < b_k < \infty$ and these intervals are pairwise disjoint. Moreover, if $a_k \neq 0$,

$$\frac{1}{a_k}\int_0^{a_k} f(x)\,dx = \frac{1}{b_k}\int_0^{b_k} f(x)\,dx = s,$$

and hence, for all cases,

$$\int_{a_k}^{b_k} f(x)\,dx = \int_0^{b_k} f(x)\,dx - \int_0^{a_k} f(x)\,dx = s(b_k - a_k).$$

Thus,

$$\begin{aligned}|\{Sf > s\}| &= \sum_k (b_k - a_k) = \frac{1}{s}\sum_k \int_{a_k}^{b_k} f(x)\,dx \\ &= \frac{1}{s}\int_{\cup_k (a_k,b_k)} f(x)\,dx = \frac{1}{s}\int_{\{Sf>s\}} f(x)\,dx.\end{aligned}$$

Using (1) we now have that if $Sf \in L^{1,\infty}(\mathbb{R}^+)$, $f \geq 0$, then

$$\begin{aligned} \|Sf\|_{L^{1,\infty}(\mathbb{R}^+)} &= \sup_{s>0} s\lambda_{Sf}(s) = \sup_{s>0} \int_{\{Sf>s\}} f(x)\,dx \\ &= \int_{\{Sf>0\}} f(x)\,dx = \|f\|_{L^1(\mathbb{R}^+)}. \end{aligned}$$

Conversely, if $0 \leq f \in L^1(\mathbb{R}^+)$, then $\lambda_{Sf}(s) < \infty$ for all $s > 0$ and so, the equalities above hold, i.e., $\|f\|_{L^1(\mathbb{R}^+)} = \|Sf\|_{L^{1,\infty}(\mathbb{R}^+)}$. □

We are going to consider the case of the $L^p(\mathbb{R}^+)$ spaces. It is very easy to show that $[S, L^1(\mathbb{R}^+)] = \{0\}$. For the other indexes we have the following:

Proposition 2.2 $L^p(\mathbb{R}^+) \subsetneq [S, L^p(\mathbb{R}^+)]$, $1 < p \leq \infty$.

Proof. Hardy's inequality proves that $L^p(\mathbb{R}^+) \subset [S, L^p(\mathbb{R}^+)]$. Now, fix $\alpha \in (-1, 0)$, and define the unbounded function $f_\alpha(t) = (1-t)^\alpha \chi_{(0,1)}(t)$. Observe that $f_{-1/p} \in L^1(\mathbb{R}^+) \setminus L^p(\mathbb{R}^+)$, $1 < p < \infty$. An easy calculation gives,

$$Sf_{-1/p}(t) = \begin{cases} \dfrac{1-(1-t)^{1-1/p}}{(1-1/p)t}, & 0 < t < 1 \\ \dfrac{p}{p-1}\dfrac{1}{t}, & t \geq 1. \end{cases}$$

Therefore, we get the counterexample since $Sf_{-1/p}(t) \in L^q(\mathbb{R}^+)$, for all $1 < q \leq \infty$. Observe that $f^*_{-1/p} \notin [S, L^p(\mathbb{R}^+)]$ and hence $[S, L^p(\mathbb{R}^+)]$ is not r.i. □

For a $BFIL$ X, if we define

$$\Gamma_X = \{f\colon \mathbb{R}^+ \to \mathbb{R} \text{ measurable, } Sf^* \in X\},$$

with norm $\|f\|_{\Gamma_X} = \|Sf^*\|_X$, then Γ_X is the largest r.i. $BFIL$ space contained in $[S, X]$. In fact, if $f \in \Gamma_X$, then $S|f| \leq Sf^* \in X$ and so $f \in [S, X]$, and if Y is an r.i. $BFIL$ contained in $[S, X]$, then for $f \in Y$ we have that $f^* \in Y$ and so $Sf^* \in X$, that is $f \in \Gamma_X$.

Proposition 2.3 *Given a $BFIL$ X, we have the following:*

(a) If $S\colon X \to X$, then $X \subset [S, X]$.

(b) If X is r.i., then $\Gamma_X \subset X \cap [S, X]$.

(c) If $S\colon X \to X$ and X is r.i., then $\Gamma_X = X$.

(d) If X is an r.i., the following conditions are equivalent:

(d1) $\Gamma_X \neq \{0\}$.

(d2) $\chi_{(0,1)} \in \Gamma_X$.

(d3) $\chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t) \in X$.

(d4) $(L^\infty \cap L^{1,\infty})(\mathbb{R}^+) \subset X$.

Proof. (a) is obvious. To prove (b), given $f \in \Gamma_X$, since $f^* \leq Sf^* \in X$, then $f^* \in X$ and so $f \in X$. (c) follows from (a), (b), and the fact that Γ_X is the largest r.i. contained in $[S, X]$. Finally, observe that for $f = \chi_{(0,1)}$, we have $Sf(t) = \chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t)$, and the equivalences (d1)-(d4) follow easily. For example, if $g \in (L^\infty \cap L^{1,\infty})(\mathbb{R}^+)$, then $g^*(t) \leq C\min(1, 1/t) = C(\chi_{(0,1)}(t) + \frac{1}{t}\chi_{(1,\infty)}(t))$. Thus, (d3) implies (d4). □

We observe that we only need X to be an r.i. to prove that (d3) implies (d4). Proposition 2.2 shows that the embedding in Proposition 2.3-(a) may be strict. Let us see now an example of an r.i. $BFIL$ space for which the embedding in Proposition 2.3-(b) is also strict (see also Example 4.1).

Proposition 2.4 $\Gamma_{(L^1+L^\infty)(\mathbb{R}^+)} \subsetneq (L^1 + L^\infty)(\mathbb{R}^+) \cap [S, (L^1 + L^\infty)(\mathbb{R}^+)]$.

Proof. Let us see that S is not bounded on $(L^1 + L^\infty)(\mathbb{R}^+)$. In fact, if

$$g(t) = \frac{1}{t\log^2(\frac{e^2}{t})}\chi_{(0,1)}(t),$$

then g is a decreasing function in $(L^1+L^\infty)(\mathbb{R}^+)$. Now set $f(t) = g(t-1)\chi_{(1,2)}(t)$. Then, $f^* = g$, $Sf \in (L^1 + L^\infty)(\mathbb{R}^+)$ (observe that since $f \in L^1$ and it is bounded at zero, then $Sf \in L^\infty$), and $Sf^* \notin (L^1 + L^\infty)(\mathbb{R}^+)$:

$$\|Sf^*\|_{(L^1+L^\infty)(\mathbb{R}^+)} = \int_0^1 (Sg)^*(t)\, dt = \int_0^1 \frac{1}{t\log(\frac{e^2}{t})}\, dt = \infty.$$

Hence, we have shown that $\Gamma_{(L^1+L^\infty)(\mathbb{R}^+)} \subsetneq (L^1 + L^\infty)(\mathbb{R}^+) \cap [S, (L^1 + L^\infty)(\mathbb{R}^+)]$. □

We are going to show that Proposition 2.2 can be extended to any r.i. space:

Theorem 2.5 *If X is an r.i. $BFIL$ Banach space, and $S : X \to X$, then $X \subsetneq [S, X]$. Hence $[S, X]$ is not r.i. (in fact $[S, X] \not\subset (L^1 + L^\infty)(\mathbb{R}^+)$).*

Proof. Let us prove that we can find a function in $[S, X]$ which is not in $(L^1 + L^\infty)(\mathbb{R}^+)$, and hence not in X either. We start with the following observation: If $f \geq 0$,

$$f \notin (L^1 + L^\infty)(\mathbb{R}^+) \iff \text{for every } c > 0,\ f\chi_{\{f>c\}} \notin L^1(\mathbb{R}^+). \qquad (2)$$

It is clear that if for some $c > 0$, $f\chi_{\{f>c\}} \in L^1(\mathbb{R}^+)$, then

$$f = f\chi_{\{f>c\}} + f\chi_{\{f\leq c\}} \in (L^1 + L^\infty)(\mathbb{R}^+).$$

Conversely, assume $f = g + h$, $h \in L^\infty(\mathbb{R}^+)$. Take $c = 2\|h\|_{L^\infty(\mathbb{R}^+)} > 0$. Then,

$$f\chi_{\{f>c\}} = (g+h)\chi_{\{g+h>2\|h\|_{L^\infty(\mathbb{R}^+)}\}} \leq (g+h)\chi_{\{|g|>\|h\|_{L^\infty(\mathbb{R}^+)}\}} \leq 2|g|.$$

If $g \in L^1(\mathbb{R}^+)$, then $f\chi_{\{f>c\}} \in L^1(\mathbb{R}^+)$.

If $X \subset L^1(\mathbb{R}^+)$, we have that $[S, X] \subset [S, L^1(\mathbb{R}^+)] = \{0\}$, and so, by Proposition 2.3-(a), $X = \{0\}$. Hence, $X \not\subseteq L^1(\mathbb{R}^+)$. Thus, we can find a positive and decreasing function $f \in X$ such that if $F(t) = \int_0^t f(x)\,dx$, then F is strictly increasing and not bounded: take $f_1 \in X \backslash L^1(\mathbb{R}^+)$, f_1 decreasing (and hence $f_1 \geq 0$). Choose $f_2 \in (L^1 \cap L^\infty)(\mathbb{R}^+)$, decreasing and positive everywhere (e.g. $f_2(t) = (1+t^2)^{-1}$). Note that, since X is an r.i. $BFIL$, $(L^1 \cap L^\infty)(\mathbb{R}^+) \subset X$ (see [8, Theorem II.4.1]) and so $f_2 \in X$. Then $f = f_1 + f_2$ satisfies the required conditions. Now take t_1=1, and by induction, choose $t_{k+1} > t_k$ satisfying that $F(t_{k+1}) = 2F(t_k) = 2^k F(1)$. We are now going to modify F on each interval (t_k, t_{k+1}) in such a way that we obtain a new absolutely continuous, positive and increasing function G satisfying that $F(t) \approx G(t)$, and if $g(t) = G'(t)$, a.e. $t > 0$, then $g \notin (L^1 + L^\infty)(\mathbb{R}^+)$. Hence, $g \in [S, X]$ (observe that $S(g) \approx S(f) \in X$), and $g \notin X$.

On the interval $[0, t_1)$, we set $G(t) = F(t)$. Now we observe the following: since

$$\int_{t_k}^{t_{k+1}} f(x)\,dx = F(t_k) \geq F(t_{k-1}) = \int_{t_{k-1}}^{t_k} f(x)\,dx,$$

and f is decreasing, then $t_{k+1} - t_k \geq t_k - t_{k-1} \geq t_2 - 1$. Therefore, the right triangle T_k determined by the vertices $(t_{k+1} - t_2 + 1, F(t_{k+1}) - F(1))$, $(t_{k+1}, F(t_{k+1}) - F(1))$, and $(t_{k+1}, F(t_{k+1}))$ (which is congruent to the triangle T_1: $(1, F(1))$, $(t_2, F(1))$, and $(t_2, F(2))$) is contained in the right triangle $(t_k, F(t_k))$, $(t_{k+1}, F(t_k))$, and $(t_{k+1}, F(t_{k+1}))$, for each $k \geq 1$ (observe that T_k has side lengths independent of k).

On the interval $[t_k, t_{k+1} - t_2 + 1]$, we define $G(t)$ to be the line joining the points $(t_k, F(t_k))$ and $(t_{k+1} - t_2 + 1, F(t_{k+1}) - F(1))$. To define G on the interval $(t_{k+1} - t_2 + 1, t_{k+1})$ we use the following argument: fix a convex function h on $[1, t_2]$, such that $h(1) = F(1)$, $h(t_2) = F(t_2)$, and $h'(t_2^-) = \infty$ (thus, the graph of h is contained in T_1). Now, using the congruence between T_1 and T_k (call it A_k, so that $A_k(T_1) = T_k$) we translate the graph of h to T_k, and define $G(t)$, if $t \in (t_{k+1} - t_2 + 1, t_{k+1})$, by means of the equality

$$(t, G(t)) = A_k(t - t_{k+1} + t_2, h(t - t_{k+1} + t_2))$$

(thus, $G(t) = h(t)$ if $t \in (1, t_2)$). We observe that G is a continuous, increasing function on $[0, \infty)$. Moreover $G(t) \leq F(t)$ since, by concavity, the graph of F is above the line through the points $(t_k, F(t_k))$ and $(t_{k+1}, F(t_{k+1}))$, while G is below that line, by construction. On the other hand, if $t \in (t_k, t_{k+1})$ then

$$G(t) \geq G(t_k) = F(t_k) = F(t_{k+1})/2 \geq F(t)/2,$$

and we get the other estimate.

Define now $g(t) = G'(t)$, a.e. $t > 0$. Let us show that $g \notin (L^1 + L^\infty)(\mathbb{R}^+)$: Using (2), if we fix $c > 0$, and $k \in \mathbb{N}$, we can find $s \in (1, t_2)$ such that $g(t) > c$, if $t \in (s, t_2)$ (observe that $g(t_2^-) = G'(t_2^-) = h'(t_2^-) = \infty$). Then,

$$\int_{\{x\in(1,t_{k+1}):g(x)>c\}} g(x)\,dx \geq \sum_{j=2}^{k+1} \int_{s-t_2+t_j}^{t_j} g(x)\,dx = k\int_s^{t_2} h'(x)\,dx \underset{k\to\infty}{\longrightarrow} \infty\,.$$

□

Remark 2.6 We observe that without the hypothesis on X, Theorem 2.5 is false. In fact, as we have proved in Proposition 2.1, $[S, L^{1,\infty}(\mathbb{R}^+)] = L^1(\mathbb{R}^+)$, which is an r.i. space.

3 Vector integral representation for the Hardy operator

The representation of a linear operator T between function spaces, as an integration operator with respect to a vector measure ν, is always interesting since allows to study the properties of T and its domain through the properties of ν and the space of integrable functions with respect to ν. However, this representation may be not possible. In this section, we give conditions which guarantee that the Hardy operator S has an integral representation.

Associated to S we have the finitely additive set function

$$A \;\longrightarrow\; \nu(A) = S(\chi_A)\,.$$

Depending on the family of measurable sets $\mathcal{R}$ on which we define ν, and the space X where we want ν to take values, $\nu\colon \mathcal{R} \to X$ may (or may not) be a vector measure (i.e., well defined and countably additive). For instance, if $X = L^1(\mathbb{R}^+)$ no family of measurable sets $\mathcal{R}$ satisfies that $\nu\colon \mathcal{R} \to X$ is a vector measure. Consider another example: the set function $\nu\colon \mathcal{B}(\mathbb{R}^+) \to (L^1+L^\infty)(\mathbb{R}^+)$, where $\mathcal{B}(\mathbb{R}^+)$ is the σ–algebra of all Borel subsets of $\mathbb{R}^+$. This set function is well defined but it is not a vector measure, since taking $A_j = [j, j+1)$ we have $\|\nu(\cup_{j\geq k} A_j)\|_{L^1+L^\infty} = 1$, for all k. Then, for any r.i. $BFIL$ X, we have that $\nu\colon \mathcal{B}(\mathbb{R}^+) \to X$ is not a vector measure, since X is continuously contained in $(L^1 + L^\infty)(\mathbb{R}^+)$ ([8, Theorem II.4.1]).

We now consider the case when X is a Lorentz space. Recall that for an increasing concave function $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$, with $\varphi(0) = 0$, the Lorentz space Λ_φ is defined by

$$\Lambda_\varphi = \left\{ f\colon \mathbb{R}^+ \to \mathbb{R} \text{ measurable},\ \int_0^\infty f^*(t)d\varphi(t) < \infty \right\},$$

where f^* is the decreasing rearrangement of f. The space Λ_φ endowed with the norm $\|f\|_{\Lambda_\varphi} = \int_0^\infty f^*(t)\,d\varphi(t)$, is an r.i. $BFIL$ space. Choosing $\mathcal{R}$ as the δ–ring (ring closed under countable intersections)

$$\mathcal{R} = \{A \in \mathcal{B}(\mathbb{R}^+) :\ |A| < \infty \ \text{ and } \ \exists\, \varepsilon > 0\,,\ |A \cap [0, \varepsilon]| = 0\}\,, \tag{3}$$

where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}^+$, we have the following result.

Proposition 3.1 *$\nu(A) \in \Lambda_\varphi$ for every $A \in \mathcal{R}$ if and only if*

$$\theta_\varphi(y) = \int_y^\infty \frac{\varphi'(t)}{t}\, dt < \infty\,, \quad \text{for all } y > 0\,, \tag{4}$$

where φ' is the derivative of φ. Moreover, if (4) holds, then $\nu\colon \mathcal{R} \to \Lambda_\varphi$ is a vector measure.

Proof. We first observe that (4) is equivalent to saying that θ_φ is integrable near 0, since

$$\int_0^\varepsilon \theta_\varphi(y)\, dy = \varphi(\varepsilon) - \varphi(0^+) + \varepsilon\theta_\varphi(\varepsilon).$$

Now, given $A \in \mathcal{R}$ we have

$$\int_0^\infty \nu(A)^*(t)\, d\varphi(t) = \varphi(0^+)\, \nu(A)^*(0^+) + \int_0^\infty \nu(A)^*(t)\, \varphi'(t)\, dt\,,$$

where

$$\nu(A)^*(0^+) = \|\nu(A)\|_\infty = \sup_{0<x<\infty} \frac{1}{x} \int_0^x \chi_A(y)\, dy = \sup_{0<x<\infty} \frac{1}{x}\, |[0,x] \cap A| \leq 1,$$

and since $(S|f|)^* \leq Sf^*$,

$$\begin{aligned} \int_0^\infty \nu(A)^*(t)\, \varphi'(t)\, dt &\leq \int_0^\infty \frac{\varphi'(t)}{t} \int_0^t \chi_{[0,|A|)}(y)\, dy\, dt \\ &= \int_0^{|A|} \int_y^\infty \frac{\varphi'(t)}{t}\, dt\, dy\,. \end{aligned}$$

Then, if (4) holds, $\nu(A) \in \Lambda_\varphi$, for all $A \in \mathcal{R}$.

Conversely, if $\nu(A) \in \Lambda_\varphi$ for every $A \in \mathcal{R}$, then, taking $A = [\frac{a}{2}, a]$ for any $a > 0$ we have $A \in \mathcal{R}$ and

$$\frac{a}{2}\, \theta_\varphi(a) \leq \int_{\frac{a}{2}}^a \theta_\varphi(y) dy = \int_0^\infty \nu(A)(t) \varphi'(t) dt \leq \int_0^\infty \nu(A)^*(t) \varphi'(t) dt < \infty\,,$$

since θ_φ is decreasing. So, $\theta_\varphi(y) < \infty$ for all $y > 0$. Hence, φ satisfying (4) is equivalent to $\nu\colon \mathcal{R} \to \Lambda_\varphi$ is well defined. Let us see that in this case ν is countably additive:

Given a disjoint sequence $(A_j) \subset \mathcal{R}$, with $A = \cup_{j\geq 1} A_j \in \mathcal{R}$, and taking $\varepsilon > 0$ such that $|A \cap [0,\varepsilon]| = 0$, we have

$$\sup_{0<x<\infty} \frac{1}{x}\, |[0,x] \cap \cup_{j\geq k} A_j| \leq \frac{1}{\varepsilon}\, |\cup_{j\geq k} A_j|\,.$$

Then

$$\|\nu(\cup_{j\geq k} A_j)\|_{\Lambda_\varphi} \leq \frac{\varphi(0^+)}{\varepsilon}\, |\cup_{j\geq k} A_j| + \int_0^{|\cup_{j\geq k} A_j|} \theta_\varphi(y)\, dy \;\longrightarrow\; 0$$

as $k \to \infty$, since $|A| < \infty$ and condition (4) holds. □

From Proposition 3.1 we deduce conditions for a general space X, under which $\nu\colon \mathcal{R} \to X$ is a vector measure. Let X be an r.i. $BFIL$ space and φ_X the fundamental function of X defined by $\varphi_X(t) = \|\chi_{[0,t]}\|_X$, for $t \in \mathbb{R}^+$. Taking an equivalent norm in X if necessary, we have that φ_X is concave ([1, 8]). Then, since Λ_{φ_X} is continuously contained in X (see [8, Theorem II.5.5]), we have that a measure with values in Λ_{φ_X} is also a measure with values in X.

Corollary 3.2 *If φ_X satisfies* (4)*, then $\nu\colon \mathcal{R} \to X$ is a vector measure.*

Remark 3.3 If X has fundamental function φ_X satisfying (4) and $\varphi_X(0^+) = 0$, it is sufficient to take $\tilde{\mathcal{R}} = \{A \in \mathcal{B}(\mathbb{R}^+) : |A| < \infty\}$ for $\nu : \tilde{\mathcal{R}} \to X$ to be a vector measure.

From now on we will assume that X is an r.i. $BFIL$, with fundamental function φ_X satisfying (4). Thus, $\nu : \mathcal{R} \to X$ is a vector measure, which will be denoted by ν_X to indicate the space where the values are taken. We will make use of the integration theory for vector measures defined on δ-rings, due to Lewis [10] and Masani and Niemi [12, 13]. So, we consider the space $L^1(\nu_X)$ of integrable functions with respect to ν_X, namely, measurable functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that

(i) f is integrable with respect to $|x^*\nu_X|$, for all $x^* \in X^*$, and

(ii) for each $A \in \mathcal{B}(\mathbb{R}^+)$, there is a vector, denoted by $\int_A f d\nu \in X$, such that

$$x^*\left(\int_A f d\nu\right) = \int_A f dx^*\nu, \quad \text{for all } x^* \in X^*,$$

where $|x^*\nu_X|$ is defined on $\mathcal{B}(\mathbb{R}^+)$ as the variation of the real measure $x^*\nu_X$. Noting that $|A| = 0$ if and only if $\nu(A) = 0$ a.e., the space $L^1(\nu_X)$ endowed with the norm

$$\|f\|_{\nu_X} = \sup_{x^* \in B_{X^*}} \int |f| d|x^*\nu_X|,$$

is a $BFIL$ space, in which the $\mathcal{R}$–simple functions (i.e., simple functions with support in $\mathcal{R}$) are dense. Moreover, $L^1(\nu_X)$ is order continuous (i.e., order bounded increasing sequences are norm convergent). Since X is a Banach lattice and ν_X is a positive vector measure, it can be proved that $\|f\|_{\nu_X} = \|\int |f| d\nu_X\|_X$, for all $f \in L^1(\nu_X)$ (see the discussion after the proof of [3, Theorem 5.2]). For results concerning the space L^1 of a vector measure defined on a δ–ring, see [5].

For every $f \in L^1(\nu_X)$ it can be proved that $Sf = \int f d\nu_X \in X$, see [6, Proposition 3.1.(b)]. Thus, S coincides on $L^1(\nu_X)$ with the integration operator with respect to ν_X and $L^1(\nu_X) \hookrightarrow$

$[S,X]$, with $\|f\|_{[S,X]} = \|f\|_{\nu_X}$. Even more, $L^1(\nu_X)$ is the largest order continuous $BFIL$ space contained in $[S,X]$. Let us prove this fact: Let Y be an order continuous $BFIL$ such that Y is continuously contained in $[S,X]$. Given $0 \le f \in Y$, there are simple functions ψ_n such that $0 \le \psi_n \uparrow f$. We take the $\mathcal{R}$–simple functions $\varphi_n = \psi_n \chi_{[\frac{1}{n},n]}$ for which $0 \le \varphi_n \uparrow f$. For all $A \in \mathcal{B}(\mathbb{R}^+)$ we have $0 \le \varphi_n \chi_A \uparrow f\chi_A \in Y$. Since Y is order continuous it follows that $\varphi_n\chi_A \to f\chi_A$ in Y and then $\varphi_n\chi_A \to f\chi_A$ in $[S,X]$. So $\|S(f\chi_A) - S(\varphi_n\chi_A)\|_X = \|S|f\chi_A - \varphi_n\chi_A|\,\|_X \to 0$ as $n \to \infty$. Thus, $S(\varphi_n\chi_A) = \int_A \varphi_n d\nu_X$ converges in X, for every $A \in \mathcal{B}(\mathbb{R}^+)$. Using [5, Proposition 2.3], we have that $f \in L^1(\nu_X)$. Therefore $Y \subset L^1(\nu_X)$ and the inclusion is positive and continuous.

If X is order continuous, then it is easy to see that $[S,X]$ is also order continuous, and thus $L^1(\nu_X) = [S,X]$.

Now, let us consider the larger space

$$L^1_w(\nu_X) = \left\{ f : \mathbb{R}^+ \to \mathbb{R} \text{ measurable} : \int |f| d|x^*\nu_X| < \infty \text{ for all } x^* \in X^* \right\},$$

which is a $BFIL$ space with the norm $\|\cdot\|_{\nu_X}$, satisfying the Fatou property (i.e., $(f_n) \subset L^1_w(\nu_X)$, $\sup_n \|f_n\|_{\nu_X} < \infty$, $0 \le f_n \uparrow f$ a.e. implies $f \in L^1_w(\nu_X)$ and $\|f_n\|_{\nu_X} \uparrow \|f\|_{\nu_X}$). Note that $L^1(\nu_X) \hookrightarrow L^1_w(\nu_X)$.

In a similar way to [4, Proposition 3.2.(ii)], it can be proved that $[S,X] \hookrightarrow L^1_w(\nu_X)$ with $\|f\|_{\nu_X} \le \|f\|_{[S,X]}$. Even more, $L^1_w(\nu_X)$ is the smallest $BFIL$ space with the Fatou property containing $[S,X]$.

If X has the Fatou property, then $[S,X]$ also has the Fatou property and thus $L^1_w(\nu_X) = [S,X]$.

Summarizing, the following result has been established.

Proposition 3.4 *Let X be an r.i. $BFIL$ space whose fundamental function φ_X satisfies* (4). *For the δ–ring $\mathcal{R}$ given in* (3) *we have:*

(a) $\nu_X : \mathcal{R} \to X$ *is a vector measure, where* $\nu_X(A) = S(\chi_A)$.

(b) $L^1(\nu_X) \hookrightarrow [S,X] \hookrightarrow L^1_w(\nu_X)$.

(c) $L^1(\nu_X)$ *is the largest order continuous $BFIL$ space contained in* $[S,X]$.

(d) $L^1_w(\nu_X)$ *is the smallest $BFIL$ space with the Fatou property containing* $[S,X]$.

(e) If X is order continuous, then $L^1(\nu_X) = [S,X]$.

(f) If X has the Fatou property, then $L^1_w(\nu_X) = [S,X]$.

Example 3.5 For $1 < p \leq \infty$, the space $X = L^p(\mathbb{R}^+)$ satisfies the hypothesis of Proposition 3.4. Since for $1 < p < \infty$ the space L^p is order continuous and has the Fatou property, we have

$$[S, L^p] = L^1(\nu_{L^p}) = L^1_w(\nu_{L^p}) \; .$$

For $p = \infty$ we have

$$L^1(\nu_{L^\infty}) \hookrightarrow [S, L^\infty] = L^1_w(\nu_{L^\infty}) \; ,$$

since L^∞ has the Fatou property. Observe that $L^1(\nu_{L^\infty}) \subsetneq [S, L^\infty]$. For instance, $\chi_{\mathbb{R}^+} \in [S, L^\infty] \backslash L^1(\nu_{L^\infty})$. Indeed, if $\chi_{\mathbb{R}^+} \in L^1(\nu_{L^\infty})$, then by [5, Corollary 3.2.b)], ν_{L^∞} is strongly additive (i.e., $\nu_{L^\infty}(A_n) \to 0$ whenever (A_n) is a disjoint sequence in $\mathcal{R}$), but taking $A_n = [2^n, 2^{n+1})$ we obtain $\|\nu_{L^\infty}(A_n)\|_\infty = 1/2$, for all $n \geq 1$ and this is a contradiction.

Example 3.6 Let X be a Lorentz space Λ_φ with φ satisfying (4); that is, satisfying the hypothesis of Proposition 3.4. Since Λ_φ has the Fatou property, we have

$$L^1(\nu_{\Lambda_\varphi}) \hookrightarrow [S, \Lambda_\varphi] = L^1_w(\nu_{\Lambda_\varphi}) \; .$$

In the case when $\varphi(0^+) = 0$ and $\varphi(\infty) = \infty$ we have that Λ_φ is order continuous (see [8, Corollary 1 to Theorem II.5.1]) and so

$$L^1(\nu_{\Lambda_\varphi}) = [S, \Lambda_\varphi] = L^1_w(\nu_{\Lambda_\varphi}) \; .$$

4 Optimal domain for the Lorentz spaces Λ_φ

Let X be a $BFIL$ space. Recall the definition of the space

$$\Gamma_X = \{\, f : \mathbb{R}^+ \to \mathbb{R} \;\text{ measurable},\; Sf^* \in X \} \, .$$

In general, Γ_X is not a closed subspace of $[S, X]$. For instance, if we take $X = L^p$ for $1 < p < \infty$, we have (see Proposition 2.2):

$$\mathcal{S}(\mathcal{R}) \subset \Gamma_{L^p} = L^p \subsetneq [S, L^p] = L^1(\nu_{L^p}) \, ,$$

where $\mathcal{S}(\mathcal{R})$ is the space of $\mathcal{R}$–simple functions. Then, Γ_{L^p} is not closed in $[S, L^p]$, since $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\nu_{L^p})$.

Example 4.1 Consider the Lorentz space Λ_φ. For any measurable function f, noting that Sf^* is decreasing, it follows

$$\begin{aligned}
\int_0^\infty (Sf^*)^*(t)\,d\varphi(t) &= \int_0^\infty Sf^*(t)\,d\varphi(t)\\
&= \varphi(0^+)Sf^*(0^+) + \int_0^\infty Sf^*(t)\,\varphi'(t)\,dt\\
&= \varphi(0^+)\|Sf^*\|_\infty + \int_0^\infty \frac{\varphi'(t)}{t}\int_0^t f^*(s)\,ds\,dt\\
&= \varphi(0^+)\|f\|_\infty + \int_0^\infty f^*(s)\int_s^\infty \frac{\varphi'(t)}{t}\,dt\,ds\\
&= \varphi(0^+)\|f\|_\infty + \int_0^\infty f^*(s)\,\theta_\varphi(s)\,ds\,.
\end{aligned}$$

Therefore,

$$\Gamma_{\Lambda_\varphi} = L^\infty \cap \Lambda_{\int_0^t \theta_\varphi(s)ds}\,.$$

In the case when $\varphi(0^+) = 0$, we have $\Gamma_{\Lambda_\varphi} = \Lambda_{\int_0^t \theta_\varphi(s)ds}$. Moreover, in this case, $\Gamma_{\Lambda_\varphi} = \Lambda_\varphi$ if and only if $\int_0^t \theta_\varphi(s)\,ds$ and φ are equivalent (e.g. $\varphi(t) = t^{1/p}$, for $1 < p < \infty$), and this holds if and only if there exists a constant $C > 0$ such that

$$t\,\theta_\varphi(t) \le C\,\varphi(t), \quad \text{for all } t \in (0,\infty)\,, \tag{5}$$

since

$$\begin{aligned}
\int_0^t \theta_\varphi(s)\,ds &= \int_0^t\int_s^\infty \frac{\varphi'(y)}{y}\,dy\,ds = \int_0^\infty \frac{\varphi'(y)}{y}\int_{[0,t]\cap[0,y]} ds\,dy\\
&= \int_0^\infty \frac{\varphi'(y)}{y}\min\{t,y\}\,dy = \int_0^t \varphi'(y)\,dy + t\int_t^\infty \frac{\varphi'(y)}{y}\,dy\\
&= \varphi(t) + t\,\theta_\varphi(t)\ .
\end{aligned}$$

Condition (5) is also equivalent to saying that $\varphi' \in B_1$ (see [2]).

The function $\varphi(t) = \min\{1,t\}$ (for which $\Lambda_\varphi = L^1 + L^\infty$) does not satisfy condition (5), so $\Gamma_{L^1+L^\infty} \subsetneq L^1 + L^\infty$. (For more information about this kind of embeddings and the boundedness of the Hardy operator see [2].)

Now we will describe the space $[S,\Lambda_\varphi]$ in the case when $\varphi(0^+) = 0$. Observe that

$$\begin{aligned}
\int_0^\infty (S|f|)^*(t)\,\varphi'(t)\,dt &\ge \int_0^\infty S|f|(t)\,\varphi'(t)\,dt = \int_0^\infty \frac{\varphi'(t)}{t}\int_0^t |f(s)|\,ds\,dt\\
&= \int_0^\infty |f(s)|\int_s^\infty \frac{\varphi'(t)}{t}\,dt\,ds = \int_0^\infty |f(s)|\,\theta_\varphi(s)\,ds\,.
\end{aligned}$$

Then, we always have that

$$[S,\Lambda_\varphi] \hookrightarrow L^1(\theta_\varphi(t)\,dt), \tag{6}$$

where $L^1(\theta_\varphi(t)\,dt)$ denotes the space of integrable functions with respect to the Lebesgue measure with density θ_φ.

We will use the following result for an r.i. $BFIL$ X, with the Fatou property. In this case, X' (the Köthe dual of X) is a norming subspace of X^*, that is

$$\|f\|_X = \sup_{g\in B_{X'}} |< g, f >| = \sup_{g\in B_{X'}} \left| \int_0^\infty g(x)f(x)\,dx \right| ,$$

[11, Proposition 1.b.18]. Note that if f is positive, the supremum above can be taken for positive functions in $B_{X'}$.

Lemma 4.2 *Let X be an r.i. $BFIL$ space, with the Fatou property. Suppose X satisfies*

$$h_y \in X \quad a.e.\ y > 0\ , \quad where\ h_y(x) := \frac{1}{x}\chi_{[y,\infty)}(x)\,. \tag{7}$$

Then $L^1(\phi_X(t)\,dt) \hookrightarrow [S, X]$, for $\phi_X(y) = \|h_y\|_X$.

Proof. Note that, since X is and r.i., from Proposition 2.3-(d) we have that condition (7) is equivalent to $\Gamma_X \neq \{0\}$, and this happens if and only if $(L^1 \cap L^\infty)(\mathbb{R}^+) \subset [S, X]$, since Γ_X is the largest r.i. $BFIL$ contained in $[S, X]$. In particular, any simple function f with finite support is in $[S, X]$ and

$$\begin{aligned}
\|f\|_{[S,X]} &= \|S|f|\|_X = \sup_{0\le g\in B_{X'}} \int_0^\infty g(x)\, S|f|(x)\,dx \\
&= \sup_{0\le g\in B_{X'}} \int_0^\infty \frac{g(x)}{x} \int_0^x |f(y)|\,dy\,dx \\
&= \sup_{0\le g\in B_{X'}} \int_0^\infty |f(y)| \int_y^\infty \frac{g(x)}{x}\,dx\,dy \\
&\le \int_0^\infty |f(y)|\,\|h_y\|_X\,dy = \int_0^\infty |f(y)|\,\phi_X(y)\,dy\,.
\end{aligned}$$

For $f \in L^1(\phi_X(t)\,dt)$ we can take simple functions (f_n) with finite support, such that $0 \le f_n \uparrow |f|$. Then

$$\sup_{n\ge1} \|f_n\|_{[S,X]} \le \sup_{n\ge1} \int_0^\infty |f_n(y)|\,\phi_X(y)\,dy = \int_0^\infty |f(y)|\,\phi_X(y)\,dy < \infty\,.$$

Thus, $f \in [S, X]$ and $\|f\|_{[S,X]} = \sup_{n\ge1} \|f_n\|_{[S,X]} \le \int_0^\infty |f(y)|\,\phi_X(y)\,dy$. We have used that $[S, X]$ has the Fatou property since X has this property. $\square$

Remark 4.3 (a) If X is an r.i. $BFIL$ space, with fundamental function satisfying (4), then we have that $\mathcal{S}(\mathcal{R}) \subset [S, X]$. In particular, $S\chi_A \in X$ for $A = (a, b)$, with $0 < a < b < \infty$. Then, since $S\chi_A(x) = (1 - \frac{a}{x})\chi_{(a,b)}(x) + (b - a)\frac{1}{x}\chi_{[b,\infty)}(x)$ and $(1 - \frac{a}{x})\chi_{(a,b)}(x) \in (L^1 \cap L^\infty)(\mathbb{R}^+) \subset X$, condition (7) holds for X.

(b) Let $X = \Lambda_\varphi$, with φ satisfying (4) and $\varphi(0^+) = 0$. From (a) we have that $h_y \in \Lambda_\varphi$ and

$$\phi_{\Lambda_\varphi}(y) = \int_0^\infty h_y^*(s)\,\varphi'(s)\,ds = \int_0^\infty \frac{\varphi'(s)}{y+s}\,ds\,.$$

Actually, in this case, (4) and (7) are equivalent. Then, by Lemma 4.2, $L^1(\phi_{\Lambda_\varphi}(t)\,dt) \hookrightarrow [S, \Lambda_\varphi]$. Note that ϕ_{Λ_φ} is equivalent to the function given by $\theta_\varphi(t) + \frac{\varphi(t)}{t}$. Indeed,

$$\phi_{\Lambda_\varphi}(t) = \int_t^\infty \frac{\varphi'(s)}{t+s}\,ds + \int_0^t \frac{\varphi'(s)}{t+s}\,ds$$

where

$$\frac{1}{2}\,\theta_\varphi(t) = \frac{1}{2}\int_t^\infty \frac{\varphi'(s)}{s}\,ds \le \int_t^\infty \frac{\varphi'(s)}{t+s}\,ds \le \int_t^\infty \frac{\varphi'(s)}{s}\,ds = \theta_\varphi(t)$$

$$\frac{1}{2}\,\frac{\varphi(t)}{t} = \frac{1}{2t}\int_0^t \varphi'(s)\,ds \le \int_0^t \frac{\varphi'(s)}{t+s}\,ds \le \frac{1}{t}\int_0^t \varphi'(s)\,ds = \frac{\varphi(t)}{t}.$$

So, $\phi_{\Lambda_\varphi}(t) \le \theta_\varphi(t) + \frac{\varphi(t)}{t} \le 2\phi_{\Lambda_\varphi}(t)$.

Theorem 4.4 *A Lorentz space Λ_φ with φ satisfying (4), $\varphi(0^+) = 0$ and for which there exists a constant $C > 0$ such that*

$$\frac{\varphi(t)}{t} \le C\,\theta_\varphi(t), \quad \text{for all} \quad t \in (0, \infty)\,, \tag{8}$$

satisfies

$$[S, \Lambda_\varphi] = L^1(\theta_\varphi(t)\,dt) = L^1(\phi_{\Lambda_\varphi}(t)\,dt)\,.$$

Proof. Using (6) and Lemma 4.2, we have that $L^1(\phi_{\Lambda_\varphi}(t)\,dt) \hookrightarrow [S, \Lambda_\varphi] \hookrightarrow L^1(\theta_\varphi(t)\,dt)$. If (8) holds, then θ_φ is equivalent to $\theta_\varphi(t) + \varphi(t)/t$, which is equivalent (by Remark 4.3-(b)) to ϕ_{Λ_φ}. So, $L^1(\theta_\varphi(t)\,dt) = L^1(\phi_{\Lambda_\varphi}(t)\,dt) = [S, \Lambda_\varphi]$. □

We consider now the special case of the Lorentz spaces $L^{p,q}$. We show that for $q = 1$, the domain coincides with an L^1-space with respect to an absolutely continuous measure, but this result does not hold if $1 < q \le \infty$:

Proposition 4.5 *(a) For $1 < p < \infty$,*

$$[S, L^{p,1}] = L^1(t^{-1/p'} dt) . \tag{9}$$

(b) If $1 < p < \infty$ and $1 \le q \le \infty$, then $L^1(t^{-1/p'} dt) \subset [S, L^{p,q}]$.

(c) For every $1 < q \le \infty$, there does not exist a nonnegative function $v \in L^1_{\text{loc}}(\mathbb{R}^+)$ for which $[S, L^{p,q}] = L^1(v(t)\, dt)$.

Proof. To prove (a), we observe that the function $\varphi(t) = t^{1/p}$ satisfies (8):

$$\theta_\varphi(t) = \frac{1}{p-1} t^{-(1-1/p)} = \frac{1}{p-1} \frac{\varphi(t)}{t}.$$

The result follows from Theorem 4.4, since $\Lambda_\varphi = L^{p,1}$

(b) is a consequence of (a) and the fact that $L^{p,1} \subset L^{p,q}$.

Suppose now that $[S, L^{p,q}] = L^1(v(t)\, dt)$. Then, using a small modification of the result in [7, p. 316], it follows that, since $L^1(v(t)\, dt) \subset [S, L^{p,q}]$, there exists a constant $C > 0$ such that $C \le t^{1/p'} v(t)$, and hence $L^1(v(t)\, dt) \subset [S, L^{p,1}]$. Therefore, $[S, L^{p,q}] = [S, L^{p,1}]$. But, taking a decreasing function $f \in L^{p,q} \setminus L^{p,1}$, we find that $f \in L^{p,q} \subset [S, L^{p,q}]$, and $f \le Sf \in L^{p,1}$, which is a contradiction. □

Remark 4.6 Proposition 4.5 shows that $L^1(t^{-1/p'} dt)$ is the largest L^1-space contained in $[S, L^{p,\infty}]$. If we consider the converse embedding $[S, L^{p,\infty}] \subset L^1(v(t)\, dt)$, then a necessary condition is that

$$\int_0^\infty \frac{v(t)}{t^{1/p}}\, dt < \infty. \tag{10}$$

On the other hand, if (10) holds, then any decreasing function in $[S, L^{p,\infty}]$ belongs also to $L^1(v(t)\, dt)$.

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