Research Article

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Hyers-Ulam stability for coupled random fixed point theorems and applications to periodic boundary value random problems

Abstract: In this paper we prove some existence, uniqueness and Hyers-Ulam stability results for the coupled random fixed point of a pair of contractive type random operators on separable complete metric spaces. The approach is based on a new version of the Perov type fixed point theorem for contractions. Some applications to integral equations and to boundary value problems are also given.

Keywords: fixed point, random operator, periodic boundary random value problem

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1 Introduction

In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem known as the Banach Contraction Principle, which is one of the most important results of analysis, and is considered as the main source of metric fixed point theory. It is well known that the classical Banach contraction principle possesses many applications to operational equations, optimization theory and other topics. In 1964, Perov [12] extended the Banach contraction principle for single–valued contraction on spaces endowed with vector-valued metrics.

Random fixed point theorems for contraction mappings were proved by Hanš̃ [4, 5], Š̃pačck [7] and Mukherjee [9, 10]. Recently, Sinacer *et al.* [16] proved a new version of Perov's fixed point theorem (Perov type random fixed point).

The investigation of qualitative properties such as existence, uniqueness and stability for random differential equations has received much attention, as can be seen in [11, 17] and the references therein, but for the study of the existence of fixed points for an operator, it is useful to consider a more general concept, namely coupled fixed points. The concept of coupled fixed point for an operators was introduced in 1987 by D. Guo and V. Lakshmikantham (see [3]).

One of the main topics in the theory of functional equations is Hyers-Ulam stability. The starting point of this topic was the problem of S.M. Ulam [20] and the solution given by Hyers to this problem in the case of the Cauchy functional equation [6]. Generally, we say that a functional equation is stable in Hyers-Ulam sense if for every solution of the perturbed equation there exists a solution of the equation that is close to it. For more details and results on this topic we refer to [2, 6, 15].

For examples and other considerations regarding Hyers-Ulam stability of the coupled system of differential Equations see [1, 8, 13, 14, 18, 19].

On the other hand, to the best of our knowledge, there is no paper which investigates the existence and Hyers-Ulam stability results for the coupled random fixed point of a pair of contractive type random operators on separable complete metric spaces. To fill this gap, in this paper, we study the existence and uniqueness and we investigate the Hyers-Ulam stability results for the coupled random fixed point of a pair of contractive type random operators.

Therefore, in this paper we present some coupled fixed points results for contractive type operators on spaces endowed with vector-valued metrics and, as an application, we discuss the existence, uniqueness and Hyers-Ulam stability of the solution of a periodic boundary value problem related to a system of random differential equations. The approach is based on random Perov-type fixed point theorem for contractions in metric spaces endowed with vector-valued metrics.

The plan of this paper is as follows. In Section 2 we introduce notations, definitions, and preliminary facts that are useful throughout the paper. In Section 3 we present some existence, uniqueness and Hyers-Ulam stability results for the coupled random fixed point of a pair of contractive type operators on separable complete metric spaces. The approach is based on a random Perov type fixed point theorem for contractions. Finally, in Section 4 an example is given to show the applicability of our results on random integral equations and to boundary value problems.

2 Preliminaries

In this section we recall some notations, definitions, and auxiliary results which will be used throughout this paper. If $x, y \in \mathbb{R}^n$ with $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$, we set $|x| = (|x_1|, \cdots, |x_n|)$, $\max(x, y) = (\max(x_1, y_1), \cdots, \max(x_n, y_n))$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \quad x_i > 0\}$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \cdots, n$.

Definition 2.1. *Let X be a nonempty set. By a vector-valued metric on X we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:*
(i) $d(u, v) \geq 0$ for all $u,\ v \in X$.; if $d(u, v) = 0$ then u=v;
(ii) $d(u, v) = d(v, u)$ for all $u,\ v \in X$;
$(iii)$$d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Here, if $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, \cdots, n$. We call the pair (X, d) a generalized metric space with

$$d(x, y) := \begin{pmatrix} d_1(x, y) \\ \vdots \\ d_n(x, y) \end{pmatrix}$$

Notice that d is a generalized metric space on X if and only if $d_i,\ i = 1, 2, \cdots, n$ are metrics on X. Similarly, by a vector-valued norm on a linear space X, we mean a mapping $\|.\| : X \to \mathbb{R}^n_+$ with $\|x\| = 0$ only for $x = 0$; $\|\lambda x\| = |\lambda| \|x\|$ for $x \in X,\ \lambda \in_R$, and $\|x + y\| \leq \|x\| + \|y\|$ for every $x,\ y \in X$. To any vector-valued norm $\|.\|$ one can associate the vector valued metric $d(x, y) := \|x - y\|$, and one says that $(X, \|.\|)$ is a generalized Banach space if X is complete with respect to d.

Definition 2.2. *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$.*

Theorem 2.1. *[21] Let $M \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$, the following assertions are equivalent:*
(a) M is convergent towards zero;
(b) $M^k \to 0$ as $k \to \infty$;

(*c*) *The matrix* $(I - M)$ *is nonsingular and*

$$(I - M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots ,$$

(*d*) *The matrix* $(I - M)$ *is nonsingular and* $(I - M)^{-1}$ *has nonnegative elements.*

Let (Ω, Σ) be a measurable space and X is a metric space, we shall use $B(X)$ to denote the Borel σ-algebra on $\Omega \times X$. Then $\Sigma \otimes B(X)$ denotes the smallest σ-algebra on $\Omega \times X$ which contains all the sets $A \times S$, where $Q \in \Sigma$ and $S \in B(X)$.

Definition 2.3. *Recall that a mapping* $f : \Omega \times X \longrightarrow X$ *is said to be a random operator if, for any* $x \in X$, $f(\cdot, x)$ *is measurable.*

Definition 2.4. *A random fixed point of* f *is a measurable function* $y : \Omega \to X$ *such that* $y(\omega) = f(\omega, y(\omega))$ *for all* $\omega \in \Omega$.

We recall now Perov's fixed point theorem (see [16]).

Theorem 2.2. *Let* $(\Omega, \mathcal{F})$ *be a measurable space, let* X *be a real separable generalized Banach space and let* $F : \Omega \times X \to X$ *be a continuous random operator. Let* $M(\omega) \in M_n(\mathbb{R}_+)$ *be a random variable matrix such that for every* $\omega \in \Omega$ *the matrix* $M(\omega)$ *converges to* 0 *and*

$$d(F(\omega, x_1), F(\omega, x_2)) \leq M(\omega) d(x_1, x_2) \text{ for each } x_1, x_2 \in X,\ \omega \in \Omega.$$

Then
(i) *there exists a random variable* $x^* : \Omega \to X$ *which is the unique random fixed point of* F.
(ii) *one has the following estimation*

$$d(x_n(\omega), x^*(\omega)) \leq M^n(\omega)(I - M(\omega))^{-1} d(x_0(\omega), x_1(\omega)); \tag{1}$$

3 Main results

Let X be a separable generalized metric space. We will focus our attention to the following system of random equations:

$$\begin{cases} x(\omega) = T_1(\omega, x, y) \\ y(\omega) = T_2(\omega, x, y) \end{cases} \tag{2}$$

where $T_1, T_2 : \Omega \times X \times X \to X$

By definition, a solution $(x, y) \in X \times X$ of the above system is called a coupled random fixed point for the pair (T_1, T_2).

For the proof of our main theorem we need the following notions and results.

Definition 3.1. *Let* X *be a separable generalized metric space and* $F : \Omega \times X \to X$ *be a random operator. Then, the fixed point equation*

$$x(\omega) = F(\omega, x) \tag{3}$$

is said to be generalized Hyers-Ulam stable if there exists an increasing function $\psi : \Omega \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$, *continuous in* 0 *with* $\psi(\omega, 0) = 0$, *such that, for any* $\varepsilon := \varepsilon_1, \cdots, \varepsilon_n$ *with* $\varepsilon_i > 0$ *for* $i \in \{1, \cdots, n\}$ *and any solution* $y^* \in X$ *of the inequation*

$$d(y(\omega), F(\omega, y)) \leq \varepsilon \tag{4}$$

there exists a solution x^* *of (3) such that*

$$d(x^*(\omega), y^*(\omega)) \leq \psi(\varepsilon, \omega) \tag{5}$$

In particular, if $\psi(t,\omega) = C(\omega)\cdot t$, $t \in \mathbb{R}^n_+$ (where $C(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$), then the fixed point equation (3) is called Hyers-Ulam stable.

We recall now a direct consequence of random versions of the Perov fixed point theorem.

Theorem 3.1. *Let $(\Omega, \mathcal{F})$ be a measurable space, X be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a continuous random operator, and let $M(\omega) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matric such that for every $\omega \in \Omega$ the matrix, $M(\omega)$ converge to 0 and*

$$d(F(\omega, x), F(\omega, y)) \leq M(\omega)d(x, y) \text{ for each } x, y \in X,\ \omega \in \Omega.$$

then the fixed $x(\omega) = F(x, \omega)$, $x \in X$ is Hyers-Ulam stable.

Proof. From random versions of the Perov's fixed point theorem we deduce that

$Fix(F) = x^*(\omega)$. Let $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_n)$, with $\varepsilon_i > 0$ for each $i \in \{1, \cdots, n\}$ and let $y^*(\omega)$ be a solution of the inequation

$$d(y(\omega), F(y, \omega) \leq \varepsilon.$$

Then, we succesively have that

$$\begin{aligned} d(x^*(\omega), y^*(\omega)) &= d(F(x^*, \omega), y^*(\omega)) \\ &\leq d(F(x^*, \omega), F(y^*, \omega)) + d(F(y^*, \omega), y^*(\omega)) \\ &\leq M(\omega)d(x^*(\omega), y^*(\omega)) + \varepsilon \end{aligned}$$

Thus, using Theorem 2.2,

$$d(x^*(\omega), y^*(\omega)) \leq (I - M(\omega))^{-1}\varepsilon$$

□

Definition 3.2. *Let X be a separable metric space and let $T_1, T_2 : \Omega \times X \times X \longrightarrow X$ be two operators. Then the system of random equations*

$$\begin{cases} x(\omega) = T_1(x, y, \omega) \\ y(\omega) = T_2(x, y, \omega), \end{cases} \tag{6}$$

is said to be Hyers-Ulam stable if there exist random variables $c_1, c_2, c_3, c_4 > 0$ such that for each $\varepsilon_1, \varepsilon_2 > 0$ and each solution-pair $(u^*(\omega), v^*(\omega)) \in X \times X$ of the inequations

$$\begin{aligned} d(u^*(\omega), T_1(u^*, v^*, \omega)) \leq \varepsilon_1 \\ d(v^*(\omega), T_2(u^*, v^*, \omega)) \leq \varepsilon_2 \end{aligned} \tag{7}$$

there exists a solution $(x^*, y^*) \in X \times X$ of (6) such that

$$\begin{aligned} d(u^*(\omega), x^*(\omega)) \leq c_1(\omega)\varepsilon_1 + c_2(\omega)\varepsilon_2 \\ d(v^*(\omega), y^*(\omega)) \leq c_3(\omega)\varepsilon_1 + c_4(\omega)\varepsilon_2 \end{aligned} \tag{8}$$

Now, we present our main result which is the following existence, uniqueness and Hyers-Ulam stability theorem for the coupled fixed point of a pair of singlevalued random operators (T_1, T_2).

Theorem 3.2. *Let $(\Omega, \mathcal{F})$ be a measurable space, X be a complete separable metric space and $T_1, T_2 : \Omega \times X \times X \to X$ be two random operators such that,*

$$\begin{aligned} d(T_1(x, y, \omega), T_1(u, v, \omega)) \leq k_1(\omega)d(x, u) + k_2(\omega)d(y, v) \\ d(T_2(x, y, \omega), T_2(u, v, \omega)) \leq k_3(\omega)d(x, u) + k_4(\omega)d(y, v) \end{aligned} \tag{9}$$

for all $(x, y), (u, v) \in X \times X$. We suppose that $M(\omega) := \begin{pmatrix} k_1(\omega) & k_2(\omega) \\ k_3(\omega) & k_4(\omega) \end{pmatrix}$ converges to zero. Then:

(i) *There exists a random variable* $(x^*, y^*) : \Omega \times \Omega \to X \times X$ *which is the unique random fixed point of* (T_1, T_2). *such that*

$$\begin{cases} x^*(\omega) = T_1(x^*, y^*, \omega) \\ y^*(\omega) = T_2(x^*, y^*, \omega), \end{cases} \tag{10}$$

(ii) *the sequence* $(T_1^n(x, y, \omega), T_2^n(x, y, \omega))_{n\in\mathbb{N}}$ *converges to* (x^*, y^*) *as* $n \to \infty$, *where*

$$\begin{array}{l} T_1^{n+1}(x, y, \omega) = T_1^n(T_1(x, y, \omega), T_2(x, y, \omega)) \\ T_2^{n+1}(x, y, \omega) = T_2^n(T_1(x, y, \omega), T_2(x, y, \omega)) \end{array} \tag{11}$$

(iii) *the random system*

$$\begin{cases} x(\omega) = T_1(x, y, \omega) \\ y(\omega) = T_2(x, y, \omega), \end{cases} \tag{12}$$

is Hyers-Ulam stable.

Proof. **(i)** Let us define $T : \Omega \times X \times X \longrightarrow X \times X$ by

$$T(x, y, \omega) = \begin{pmatrix} T_1(x, y, \omega) \\ T_2(x, y, \omega) \end{pmatrix} = (T_1(x, y, \omega), T_2(x, y, \omega)).$$

Denote $Z := X \times X$ and consider $\tilde{d} : Z \times Z \longrightarrow \mathbb{R}^2_+$,

$$\tilde{d}((x, y), (u, v)) = \begin{pmatrix} d(x, u) \\ d(y, v) \end{pmatrix}. \tag{13}$$

Then we have

$$\begin{aligned} \tilde{d}(T(x, y, \omega), T(u, v, \omega)) &= \tilde{d}\left(\begin{pmatrix} T_1(x, y, \omega) \\ T_2(x, y, \omega) \end{pmatrix} \begin{pmatrix} T_1(u, v, \omega) \\ T_2(u, v, \omega) \end{pmatrix}\right) \\ &= \begin{pmatrix} d(T_1(x, y, \omega), T_1(u, v, \omega)) \\ d(T_2(x, y, \omega), T_2(u, v, \omega)) \end{pmatrix} \\ &\leq \begin{pmatrix} k_1(\omega)d(x, u) + k_2(\omega)d(y, v) \\ k_3(\omega)d(x, u) + k_4(\omega)d(y, v) \end{pmatrix} \\ &= \begin{pmatrix} k_1(\omega), k_2(\omega) \\ k_3(\omega), k_4(\omega) \end{pmatrix} \begin{pmatrix} d(x, u) \\ d(y, v) \end{pmatrix} = M(w)\tilde{d}((x, y), (u, v)). \end{aligned}$$

Applying the random Perov fixed point Theorem 2.2 (i), there exists a unique element $(x^*(\omega), y^*(\omega)) \in X \times X$ such that

$$(x^*(\omega), y^*(\omega)) = T(x^*, y^*, \omega),$$

which is equivalent to

$$\begin{cases} x^*(\omega) = T_1(x^*, y^*, \omega) \\ y^*(\omega) = T_2(x^*, y^*, \omega), \end{cases} \tag{14}$$

(ii) Moreover, for each $(x, y) \in X \times X$, we have that $T^n(x, y, \omega) \to (x^*(\omega), y^*(\omega))$ as $n \to \infty$, where $T^0(x, y, \omega) := (x(\omega), y(\omega))$, $T^1(x, y, \omega) = T(x, y, \omega) = (T_1(x, y, \omega), T_2(x, y, \omega))$
$T^2(x, y, \omega) = T(T_1(x, y, \omega), T_2(x, y, \omega)) = (T_1(x, y, \omega), T_2(x, y, \omega))$
and, in general,

$$\begin{array}{l} T_1^{n+1}(x, y, \omega) = T_1^n(T_1(x, y, \omega), T_2(x, y, \omega)) \\ T_2^{n+1}(x, y, \omega) = T_2^n(T_1(x, y, \omega), T_2(x, y, \omega)). \end{array}$$

We obtain that $T^n(x, y, \omega) = (T_1^n(x, y, \omega), T_2^n(x, y, \omega)) \longrightarrow (x^*(\omega), y^*(\omega))$ as $n \longrightarrow \infty$, for all $(x, y) \in X \times X$.
Thus, for all $(x, y) \in X \times X$, we have that $T_1^n(x, y, \omega) \longrightarrow x^*(\omega)$ as $n \to \infty$
$T_2^n(x, y, \omega) \longrightarrow y^*(\omega)$ as $n \to \infty$.

(iii) By (i) and (ii), for each ω there exists a unique element $(x^*(\omega), y^*(\omega)) \in X \times X$ such that (x^*, y^*) is a solution for (14) and the sequence $(T_1^n(x, y, \omega)T_2^n(x, y, \omega))$ converges to $(x^*(\omega), y^*(\omega))$ as $n \longrightarrow \infty$. Let $\varepsilon_1, \varepsilon_2 > 0$ and $(u^*(\omega), v^*(\omega)) \in X \times X$ such that

$$\begin{aligned} d(u^*(\omega), T_1(u^*, v^*, \omega)) \leq \varepsilon_1 \\ d(v^*(\omega), T_2(u^*, v^*, \omega)) \leq \varepsilon_2. \end{aligned} \tag{15}$$

Then,

$$\begin{aligned} \tilde{d}((u^*, v^*), (x^*, y^*)) \leq & \tilde{d}((u^*, v^*), (T_1(u^*, v^*, \omega), T_2(u^*, v^*, \omega))) \\ & + \tilde{d}((T_1(u^*, v^*, \omega), T_2(u^*, v^*, \omega)), (x^*, y^*)) \\ = & \tilde{d}((u^*, v^*), (T_1(u^*, v^*, \omega), T_2(u^*, v^*, \omega))) \\ & + \tilde{d}((T_1(u^*, v^*, \omega), T_2(u^*, v^*, \omega)), (T_1(x^*, y^*, \omega), T_2(x^*, y^*, \omega))) \\ = & \begin{pmatrix} d(u^*(\omega), T_1(u^*, v^*, \omega)) \\ d(v^*(\omega), T_2(u^*, v^*, \omega)) \end{pmatrix} + \begin{pmatrix} d(T_1(u^*, v^*, \omega), T_1(x^*, y^*, \omega)) \\ d(T_2(u^*, v^*, \omega), T_2(x^*, y^*, \omega)) \end{pmatrix} \\ \leq & \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} + \tilde{d}(T(u^*, v^*, \omega), T(x^*, y^*, \omega)) \\ \leq & \varepsilon + M(\omega)\tilde{d}((u^*, v^*), (x^*, y^*)). \end{aligned}$$

Since $(I - M(\omega))$ is invertible and $(I - M(\omega))^{-1}$ has positive elements, we immediately obtain

$$\tilde{d}((u^*, v^*), (x^*, y^*)) = \begin{pmatrix} d(x^*, y^*) \\ d(u^*, v^*) \end{pmatrix} \leq (I - M(\omega))^{-1}\varepsilon.$$

If we denote $(I - M(\omega))^{-1} = \begin{pmatrix} c_1(\omega) & c_2(\omega) \\ c_3(\omega) & c_4(\omega) \end{pmatrix}$, we then obtain

$$\begin{aligned} d(u^*(\omega), x^*(\omega)) \leq c_1(\omega)\varepsilon_1 + c_2(\omega)\varepsilon_2 \\ d(v^*(\omega), y^*(\omega)) \leq c_3(\omega)\varepsilon_1 + c_4(\omega)\varepsilon_2 \end{aligned} \tag{16}$$

proving that the system (14) is Hyers-Ulam stable.

□

4 An application to random differential equations with periodic boundary conditions

In this section we study the existence, uniqueness and Hyers-Ulam stability of a solution to a periodic boundary value problem as an application of the coupled random fixed point Theorem 3.2 presented in Section 2.

We consider the periodic boundary value problem

$$\begin{cases} x^{'}(t, \omega) = f(t, x(t, \omega), \omega) + g(t, y(t, \omega), \omega) \\ y^{'}(t, \omega) = f(t, y(t, \omega), \omega) + g(t, x(t, \omega), \omega) \\ x(0, \omega) = x(T, \omega) \\ y(0, \omega) = y(T, \omega) \end{cases} \tag{17}$$

where $f, g : [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R}$, $(\Omega, \mathcal{A})$ is a measurable space and $x_0, y_0 : \Omega \to \mathbb{R}$ are random variables.

Definition 4.1. *A function $f : [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is called random Carathèodory if the following conditions are satisfied:*

(i) *the map $(t, \omega) \to f(t, x, \omega)$ is jointly measurable for all $x \in \mathbb{R}$,*

(ii) *the map* $x \to f(t,x,\omega)$ *is continuous for all* $t \in [0,T]$ *and* $\omega \in \Omega$.

Let f, g be two Carathory functions satisfying the following conditions.

(H1) There exist $\lambda_1 > 0$, $\lambda_2 > 0$ and there exist random variables $P_1, P_2 : \Omega \to \mathbb{R}_+$, such that for all $x(\omega), y(\omega) \in \mathbb{R}$, $x(\omega) \le y(\omega)$

$$0 \le f(t,x(t,\omega),\omega) + \lambda_1 x(t,\omega) - (f(t,y(t,\omega),\omega) + \lambda_1 y(t,\omega) \le P_1(\omega)(x-y) \tag{18}$$

$$-P_2(\omega)(x-y) \le g(t,x(t,\omega),\omega) - \lambda_2 x(t,\omega) - (f(t,y(t,\omega),\omega) - \lambda_2 y(t,\omega) \le 0, \tag{19}$$

where $M(\omega) = \begin{pmatrix} \frac{P_1(\omega)}{\lambda_1+\lambda_2} & \frac{P_2(\omega)}{\lambda_1+\lambda_2} \\ \frac{P_2(\omega)}{\lambda_1+\lambda_2} & \frac{P_1(\omega)}{\lambda_1+\lambda_2} \end{pmatrix}$ is a random variable matrix convergent to zero.

We study the existence of a solution of the following periodic system:

$$x'(t,\omega) + \lambda_1 x(t,\omega) - \lambda_2 y(t,\omega) = f(t,x(t,\omega),\omega) + g(t,y(t,\omega),\omega) + \lambda_1 x(t,\omega) - \lambda_2 y(t,\omega) \tag{20}$$

$$y'(t,\omega) + \lambda_1 y(t,\omega) - \lambda_2 x(t,\omega) = f(t,y(t,\omega),\omega) + g(t,x(t,\omega),\omega) + \lambda_1 y(t,\omega) - \lambda_2 x(t,\omega) \tag{21}$$

together with the periodic conditions,

$$x(0,\omega) = x(T,\omega) \text{ and } y(0,\omega) = y(T,\omega). \tag{22}$$

This problem is equivalent to the integral equations:

$$\begin{aligned} x(t,\omega) = \int_0^T & G_1(t,s)[f(s,x(s,\omega),\omega) + g(s,y(s,\omega),\omega) + \lambda_1 x(s,\omega) - \lambda_2 y(s,\omega)] \\ & + G_2(t,s)[f(s,y(s,\omega),\omega) + g(s,x(s,\omega),\omega) + \lambda_1 y(s,\omega) - \lambda_2 x(s,\omega)]ds \end{aligned}$$

$$\begin{aligned} y(t,\omega) = \int_0^T & G_1(t,s)[f(s,y(s,\omega),\omega) + g(s,x(s,\omega),\omega) + \lambda_1 y(s,\omega) - \lambda_2 x(s,\omega)] \\ & + G_2(t,s)[f(s,x(s,\omega),\omega) + g(s,y(s,\omega),\omega) + \lambda_1 x(s,\omega) - \lambda_2 y(s,\omega)]ds \end{aligned}$$

where

$$G_1(t,s) = \begin{cases} \frac{1}{2}\left[\frac{e^{\sigma_1(t-s)}}{1-e^{\sigma_1 T}} + \frac{e^{\sigma_2(t-s)}}{1-e^{\sigma_2 T}}\right] & 0 \le s < t \le T \\ \frac{1}{2}\left[\frac{e^{\sigma_1(t+T-s)}}{1-e^{\sigma_1 T}} + \frac{e^{\sigma_2(t+T-s)}}{1-e^{\sigma_2 T}}\right] & 0 \le t < s \le T, \end{cases}$$

$$G_2(t,s) = \begin{cases} \frac{1}{2}\left[\frac{e^{\sigma_2(t-s)}}{1-e^{\sigma_2 T}} - \frac{e^{\sigma_1(t-s)}}{1-e^{\sigma_1 T}}\right] & 0 \le s < t \le T \\ \frac{1}{2}\left[\frac{e^{\sigma_2(t+T-s)}}{1-e^{\sigma_2 T}} - \frac{e^{\sigma_1(t+T-s)}}{1-e^{\sigma_1 T}}\right] & 0 \le t < s \le T, \end{cases}$$

Here $\sigma_1 = -(\lambda_1 + \lambda_2)$ and $\sigma_2 = (\lambda_2 - \lambda_1)$

It was shown in [14] (Lemma 3.2) that, if

$$ln\left(\frac{2e-1}{e}\right) \le (\lambda_2 - \lambda_1)T \text{ and } (\lambda_1 + \lambda_2)T \le 1, \tag{23}$$

then $G_1(t,s) \ge 0$ and $G_2(t,s) \le 0$, $0 \le t, s \le T$.

Consider the operator $A : C([0,T],\mathbb{R}) \times C([0,T],\mathbb{R}) \times \Omega \to C([0,T],\mathbb{R})$ where

$$\begin{aligned} A(t,x(t,\omega),y(t,\omega),\omega) = \int_0^T & G_1(t,s)[f(s,x(s,\omega),\omega) + g(s,y(s,\omega),\omega) + \lambda_1 x(s,\omega) - \lambda_2 y(s,\omega)] \\ & + G_2(t,s)[f(s,y(s,\omega),\omega) + g(s,x(s,\omega),\omega) + \lambda_1 y(s,\omega) - \lambda_2 x(s,\omega)]ds \end{aligned}$$

Thus, (x,y) is a random solution of (20)-(22).

For the proof of our main result we need the following notion.

Definition 4.2. *The system*

$$\begin{cases} x(t,\omega) = & \int\limits_0^T G_1(t,s)[f(s,x(s,\omega),\omega)+g(s,y(s,\omega),\omega)+\lambda_1 x(s,\omega)-\lambda_2 y(s,\omega)] \\ & +G_2(t,s)[f(s,y(s,\omega),\omega)+g(s,x(s,\omega),\omega)+\lambda_1 y(s,\omega)-\lambda_2 x(s,\omega)]ds \\ y(t,\omega) = & \int\limits_0^T G_1(t,s)[f(s,y(s,\omega),\omega)+g(s,x(s,\omega),\omega)+\lambda_1 y(s,\omega)-\lambda_2 x(s,\omega)] \\ & +G_2(t,s)[f(s,x(s,\omega),\omega)+g(s,y(s,\omega),\omega)+\lambda_1 x(s,\omega)-\lambda_2 y(s,\omega)]ds \end{cases} \tag{24}$$

is said to be Hyers-Ulam stable if there exist $c_1(\omega), c_2(\omega) > 0$ such that for each ε_1,$\varepsilon_2 > 0$ and each solution (u^*, v^*) of the following inequation system

$$\begin{cases} |u^*(t,\omega) - \int\limits_0^T G_1(t,s)[f(s,u^*(s,\omega),\omega)+g(s,v^*(s,\omega),\omega)+\lambda_1 u^*(s,\omega)-\lambda_2 v^*(s,\omega)] \\ +G_2(t,s)[f(s,v^*(s,\omega),\omega)+g(s,u^*(s,\omega),\omega)+\lambda_1 v^*(s,\omega)-\lambda_2 u^*(s,\omega)]ds| \le \varepsilon_1 \\ |v^*(t,\omega) - \int\limits_0^T G_1(t,s)[f(s,v^*(s,\omega),\omega)+g(s,u^*(s,\omega),\omega)+\lambda_1 v^*(s,\omega)-\lambda_2 u^*(s,\omega)] \\ +G_2(t,s)[f(s,u^*(s,\omega),\omega)+g(s,v^*(s,\omega),\omega)+\lambda_1 u^*(s,\omega)-\lambda_2 v^*(s,\omega)]ds| \le \varepsilon_2, \end{cases} \tag{25}$$

there exists a solution (x^*, y^*) of (24) such that

$$|x^*(t,\omega), u^*(t,\omega))| \le c_1(\omega)\varepsilon_1 + c_2(\omega)\varepsilon_2$$
$$|y^*(t,\omega), v^*(t,\omega))| \le c_3(\omega)\varepsilon_1 + c_4(\omega)\varepsilon_2$$

Our main result is the following existence, uniqueness and Hyers-Ulam stability of a random solution to a periodic boundary value problem.

Theorem 4.1. *Consider the problem (17) with f, g be two Carathory functions and suppose that Assumption $(H1)$ is satisfied. If (23) is fulfilled, then:*
(i) *there exists a unique random solution x^*, y^* of the periodic boundary value problem (17).*
(ii) *the system (24) is Hyers-Ulam stable.*

Proof. **(i)** Observe that

$$
\begin{aligned}
&d(A(x,y,\omega), A(u,v,\omega))\\
&\quad= \sup_{t\in I} |A(x,y,\omega), A(u,v,\omega)|\\
&\quad= \sup_{t\in I} \int_0^T G_1(t,s)[f(s,x(s,\omega),\omega) + g(s,y(s,\omega),\omega) + \lambda_1 x(s,\omega) - \lambda_2 y(s,\omega)]\\
&\qquad + G_2(t,s)[f(s,y(s,\omega),\omega) + g(s,x(s,\omega),\omega) + \lambda_1 y(s,\omega) - \lambda_2 x(s,\omega)]ds\\
&\qquad - \int_0^T G_1(t,s)[f(s,u(s,\omega),\omega) + g(s,v(s,\omega),\omega) + \lambda_1 u(s,\omega) - \lambda_2 v(s,\omega)]\\
&\qquad + G_2(t,s)[f(s,v(s,\omega),\omega) + g(s,u(s,\omega),\omega) + \lambda_1 v(s,\omega) - \lambda_2 u(s,\omega)]ds\\
&\quad= \sup_{t\in I} \int_0^T G_1(t,s)\,([f(s,x(s,\omega),\omega) + g(s,y(s,\omega),\omega) + \lambda_1 x(s,\omega) - \lambda_2 y(s,\omega)]\\
&\qquad - [f(s,u(s,\omega),\omega) + g(s,v(s,\omega),\omega) + \lambda_1 u(s,\omega) - \lambda_2 v(s,\omega)])\\
&\qquad + G_2(t,s)([f(s,y(s,\omega),\omega) + g(s,x(s,\omega),\omega) + \lambda_1 y(s,\omega) - \lambda_2 x(s,\omega)]\\
&\qquad - [f(s,v(s,\omega),\omega) + g(s,u(s,\omega),\omega) + \lambda_1 v(s,\omega) - \lambda_2 u(s,\omega)])ds\\
&\quad= \sup_{t\in I} \int_0^T G_1(t,s)\,([f(s,x(s,\omega),\omega) + g(s,y(s,\omega),\omega) + \lambda_1 x(s,\omega) - \lambda_2 y(s,\omega)]\\
&\qquad - [f(s,u(s,\omega),\omega) + g(s,v(s,\omega),\omega) + \lambda_1 u(s,\omega) - \lambda_2 v(s,\omega)])\\
&\qquad - G_2(t,s)([f(s,v(s,\omega),\omega) + g(s,u(s,\omega),\omega) + \lambda_1 v(s,\omega) - \lambda_2 u(s,\omega)]\\
&\qquad - [f(s,y(s,\omega),\omega) + g(s,x(s,\omega),\omega) + \lambda_1 y(s,\omega) - \lambda_2 x(s,\omega)])ds\\
&\quad\le \sup_{t\in I} \int_0^T G_1(t,s)[P_1(\omega)(x-u) + P_2(\omega)(v-y)]\\
&\qquad - G_2(t,s)[P_1(\omega)(v-y) + P_2(\omega)(x-u)]ds\\
&\quad= [P_1(\omega)d(x,u) + P_2(\omega)d(y,v)] \sup_{t\in I} \left| \int_0^T [G_1(t-s) - G_2(t,s)]ds \right|\\
&\quad= [P_1(\omega)d(x,u) + P_2(\omega)d(y,v)] \cdot \sup_{t\in I} \left| \int_0^t \frac{e^{\sigma_1(t-s)}}{1-e^{\sigma_1 T}} ds \int_t^T \frac{e^{\sigma_1(t+T-s)}}{1-e^{\sigma_1 T}} ds \right|\\
&\quad= \frac{P_1(\omega)}{\lambda_1+\lambda_2} d(x,u) + \frac{P_2(\omega)}{\lambda_1+\lambda_2} d(y,v).
\end{aligned}
$$

In a similar way we deduce that

$$
d(A(y,x,\omega), A(v,u,\omega)) \le \frac{P_1(\omega)}{\lambda_1+\lambda_2} d(y,v) + \frac{P_2(\omega)}{\lambda_1+\lambda_2} d(x,u).
$$

If we denote $k_1(\omega) := \frac{P_1(\omega)}{\lambda_1+\lambda_2}$ and $k_2(\omega) := \frac{P_2(\omega)}{\lambda_1+\lambda_2}$, then

$$
d(A(x,y,\omega), A(u,v,\omega)) \le k_1(\omega)d(x,u) + k_2(\omega)d(y,v)
$$

and

$$
d(A(y,x,\omega), A(v,u,\omega)) \le k_2(\omega)d(x,u) + k_1(\omega)d(y,v).
$$

Thus,

$$\begin{pmatrix} d(A(x,y,\omega),A(u,v,\omega)) \\ d(A(y,x,\omega),A(v,u,\omega)) \end{pmatrix} \leq \begin{pmatrix} k_1(\omega)d(x,u)+k_2(\omega)d(y,v) \\ k_2(\omega)d(y,v)+k_1(\omega)d(x,u) \end{pmatrix}$$
$$= \begin{pmatrix} k_1(\omega),k_2(\omega) \\ k_2(\omega),k_1(\omega) \end{pmatrix} \begin{pmatrix} d(x,u) \\ d(y,v). \end{pmatrix} = M(w)\tilde{d}((x,y),(u,v))$$

where $M(\omega)$ is a matrix convergent to zero.
From Theorem 2.2 there exists a unique random solution of problem (20-22).

(ii) By the first part of our proof and by Theorem 3.2 (iii) we deduce that there exists a unique element $(x^*(\omega),y^*(\omega))\in C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$ such that (x^*,y^*) is a solution for (14).
Let $\varepsilon_1,\varepsilon_2>0$ and $(u^*(\omega),v^*(\omega))\in C([0,T],\mathbb{R})\times C([0,T],\mathbb{R})$ such that

$$\begin{aligned} d(u^*(\omega),A(u^*,v^*,\omega)) &\leq \varepsilon_1 \\ d(v^*(\omega),A(v^*,u^*,\omega)) &\leq \varepsilon_2. \end{aligned} \tag{26}$$

Then

$$\begin{pmatrix} d(x^*,y^*) \\ d(u^*,v^*) \end{pmatrix} \leq (I-M(\omega))^{-1}\varepsilon$$

and the system (24) is Hyers-Ulam stable.

□

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