Analysis of a stochastic distributed delay epidemic model with relapse and Gamma distribution kernel

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Abstract

In this work, we investigate a stochastic epidemic model with relapse and distributed delay. First, we prove that our model possesses and unique global positive solution. Next, by means of the Lyapunov method, we determine some sufficient criteria for the extinction of the disease and its persistence. In addition, we establish the existence of a unique stationary distribution to our model. Finally, we provide some numerical simulations for the stochastic model to assist and show the applicability and efficiency of our results.

Keywords: Epidemic model, Distributed delay, Extinction, Persistence, Stationary distribution.

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1. Introduction

Statistics reported by the World Health Organisation shows the threats of infectious diseases to global health. It has estimated that tuberculosis caused 1.18 millions of deaths and 2.56 millions of peoples died from pneumonia in 2017. At the end of 2018, a total of 32 millions of peoples were killed by HIV since the beginning of the epidemic. Biological literature has been enriched by

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contribution of mathematicians to understand, predict behaviors and control the spread of detrimental epidemics. Since the basic SIR model introduced in works of Kermack and McKendrick [13, 14, 15], many important extensions have been developed including different characteristics of the epidemics [1, 2, 3, 4, 7, 24, 26, 12, 20, 28, 8, 18]. Since many species should reveal time delay, the introduction of time delays into compartmental epidemic models presents a fascinating improvement as it models sojourn times in a specified state, e.g. the infective state, see [21, 23, 27, 5]. In [19], authors studied the following stochastic SIR epidemic model with distributed time delay

$$\begin{cases} dS(t) &= \left(\mu - \mu S(t) - \beta S(t) \int_{-\infty}^{t} D(t-s)I(s)ds\right) dt + \sigma S dB(t), \\ dI(t) &= \left(\beta S(t) \int_{-\infty}^{t} D(t-s)I(s)ds - (\mu+\lambda+\delta)I(t)\right) dt \\ dR(t) &= \left(\lambda I(t) - \mu R(t)\right) dt, \end{cases} \tag{1}$$

where $S(t), I(t)$ and $R(t)$ represent the densities of susceptible, infected and recovered individuals at time t respectively. The parameter μ is the recruitment rate, β the infection transmission rate, λ the recovery rate and δ the death rate revealed by the disease. The quantity $\int_{-\infty}^{t} D(t-s)I(s)ds$ is the force of infection at time t. The function $D : [0,\infty) \to [0,\infty)$, called delay kernel, is an L^1-function verifying $\int_0^\infty D(s)ds = 1$. The standard Brownian motion $B(t)$ with intensity σ is used to model the environmental noise on susceptible class.
Connected with the above statements, we suggest to study the dynamics of a stochastic SIR epidemic model incorporating relapse, possessing distributed time delay with environmental noise proportional to $S(t), I(t)$ and $R(t)$ as

$$\begin{cases} dS(t) &= \left(\mu - \mu S(t) - \beta S(t) \int_{-\infty}^{t} D(t-s)I(s)ds\right) dt + \sigma_1 S(t) dB_1(t), \\ dI(t) &= \left(\beta S(t) \int_{-\infty}^{t} D(t-s)I(s)ds - (\mu+\lambda+\delta)I(t) + \gamma R(t)\right) dt \\ & \quad +\sigma_2 I(t) dB_2(t) \\ dR(t) &= \left(\lambda I(t) - (\mu+\gamma) R(t)\right) dt + \sigma_3 R(t) dB_3(t), \end{cases} \tag{2}$$

where γ is the rate of reinfection, and $B_i(t), i = 1,2,3$ are independent Brownian motions with intensities $\sigma_i, i = 1,2,3$.
In practice, it is convenient to use the kernel delay D with Gamma distribution

[22], that is

$$D(s) = \frac{s^n \alpha^{n+1} e^{-\alpha s}}{n!}, \quad s > 0,$$

where the positive number α is the rate of decay of effect of past memories. In this paper, we consider the weak kernel (i.e. $n = 1$) and for simplicity we let

$$Z(t) = \int_{-\infty}^{t} \alpha e^{-\alpha(t-s)} I(s) ds,$$

then it follows that

$$dZ(t) = \alpha \left(I(t) - Z(t) \right) dt.$$

We also allow the component $Z(t)$ to be affected by the environmental variability as it depends of the perturbed density of infected individuals. Hence, system (2) can be rewritten as

$$\begin{cases} dS(t) &= \left(\mu - \mu S(t) - \beta S(t) Z(t) \right) dt + \sigma_1 S(t) dB_1(t), \\ dI(t) &= \left(\beta S(t) Z(t) - (\mu + \lambda + \delta) I(t) + \gamma R(t) \right) dt + \sigma_2 I(t) dB_2(t), \\ dR(t) &= \left(\lambda I(t) - (\mu + \gamma) R(t) \right) dt + \sigma_3 R(t) dB_3(t), \\ dZ(t) &= \alpha \left(I(t) - Z(t) \right) dt + \sigma_4 Z(t) dB_4(t), \end{cases} \tag{3}$$

where $B_4(t)$ stands for Brownian motion with intensity σ_4.
Despite the realism presented by epidemic models with distributed delay, their stochastic analysis is still quite difficult and limited. To the best of our knowledge, only sufficient conditions are obtained for the extinction and persistence of the diseases with such kind of delay (see, for instance, (1)). Actually, more general results are proved for models with maturation and bounded delays [9, 25].

The content of this paper is structured as follows. Section 2 is devoted to establish the existence of a unique global positive solution for the stochastic epidemic model (3). In Section 3, by means of Lyapunov functions, we establish a sufficient criterion for the exponential extinction of the disease. The persistence in mean result under some appropriate conditions is proved in Section 4. In Section 5, we analyze the existence of an ergodic stationary distribution to system (3). Finally, in Section 6, some numerical examples are presented to provide a practical exhibition and explanation to our analytical results.

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying usual conditions. Define the space $\mathbb{R}^n_+$ as follows

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0, i = 1, \cdots, n\}.$$

Let X_t be a regular homogeneous Markov process in $\mathbb{R}^n$ described by the SDE

$$dX_t = f(X)dt + \sum_{r=1}^{k} g_r(X)dB_r(t), \quad X(0) = X_0, \tag{4}$$

with the diffusion matrix defined by

$$A(x) = (a_{ij}(x)), \quad a_{ij}(x) = \sum_{r=1}^{k} g_r^i(x) g_r^j(x).$$

We introduce the differential operator $\mathcal{L}$ associated to (4) and acts on any twice continuously differentiable function V as follows

$$\mathcal{L}V(x) = \sum_{i=1}^{n} f_i(x)\frac{\partial V(x)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 V(x)}{\partial x_i \partial x_j}.$$

The following lemma is a worthwhile criterion for the existence of ergodic stationary distributions [16].

Lemma 1. *Assume the following assumptions hold,*

1. *There exists a positive number κ such that $\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \kappa|\xi|^2, \xi \in D$, where $D \subset \mathbb{R}^n$ is a bounded open set with compact closure.*

2. *There exists a nonnegative $\mathcal{C}^2-$function $V : D^c \to \mathbb{R}$ such that $\mathcal{L}V$ is negative for any $x \in D^c = \mathbb{R}^n \setminus D$.*

Then, the Markov process $X(t)$ has a unique ergodic stationary distribution $\mu(\cdot)$ with density in $\mathbb{R}^n$ such that

$$\mathbb{P}\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T h(X(t))dt = \int_{\mathbb{R}^n} h(x)\mu(dx)\right\} = 1,$$

for any $x \in \mathbb{R}^n$ and $h(\cdot)$ is an integrable function with respect to the measure μ.

2. Existence and uniqueness of global positive solution

In the sequel, we show that system (3) admits a unique global nonnegative solution for any initial condition on $\mathbb{R}^4_+$.

Theorem 2.1. *For any given initial value $(S(0), I(0), R(0), Z(0)) \in \mathbb{R}^4_+$, there exists a unique local solution $(S(t), I(t), R(t), Z(t))$ to system (3) on $t \geq 0$ and the solution remains in $\mathbb{R}^4_+$ almost surely.*

Proof. Let $(S(0), I(0), R(0), Z(0)) \in \mathbb{R}^4_+$. Since the considered system has a locally Lipschitz coefficients of linear growth, there exists a unique solution $(S(t), I(t), R(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. Now, we shall prove that $\tau_e = \infty$ *a.s.* To this end, we need to construct a $\mathcal{C}^2-$function V : $\mathbb{R}^4_+ \mapsto \mathbb{R}_+ \cup \{0\}$ such that

$$\liminf_{k\to\infty,(S,I,R,Z)\in\mathbb{R}^4_+\setminus D_k} V(S, I, R, Z) = \infty \quad \text{and} \quad \mathcal{L}V(S, I, R, Z) \leq C,$$

where $D_k = \left(\frac{1}{k}, k\right)^4$ for $k \geq k_0 \geq 1$, k_0 is a sufficiently large integer such that $(S(0), I(0), R(0), Z(0)) \in D_{k_0}$ and C is a positive constant.
For any $k \geq k_0$, we define $\tau_k = \inf\{t \in [0, \tau_e) : (S, I, R, Z) \notin D_k\}$. Obviously, $(\tau_k)_{k\geq k_0}$ is an increasing sequence with limit τ_∞ verifying $\tau_\infty \leq \tau_e$ *a.s.* It is sufficient to prove that $\tau_\infty = \infty$ *a.s.* If this is false, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$. Hence, there is an integer $k_1 \geq k_0$ such that $\mathbb{P}\{\tau_k \leq T\} \geq \varepsilon$ for all $k \geq k_1$.
Let a and b be two positive constants and define

$$V(S, I, R, Z) = \left(S - a - a\ln\frac{S}{a}\right) + (I - 1 - \ln I) + (R - 1 - \ln R) + b\left(Z - b - b\ln\frac{Z}{b}\right).$$

Since $-\ln x \xrightarrow[x\to 0^+]{} \infty$ and $-x - m - m\ln\frac{x}{m} \xrightarrow[x\to\infty]{} \infty$ for any $m > 0$, one can easily obtain that

$$\liminf_{k\to\infty,(S,I,R,Z)\in\mathbb{R}^4_+\setminus D_k} V(S, I, R, Z) = \infty.$$

Now, using Itô's formula on V, we obtain that

$$\begin{aligned}dV(S,I,R,Z) =& \mathcal{L}V(S,I,R,Z)dt + \left(1-\frac{a}{S}\right)\sigma_1 S dB_1 + \left(1-\frac{1}{I}\right)\sigma_2 I dB_2 \\ &+ \left(1-\frac{1}{R}\right)\sigma_3 R dB_3 + \left(b-\frac{b^2}{Z}\right)\sigma_4 Z dB_4,\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}V(S,I,R,Z) =& \left(1-\frac{a}{S}\right)(\mu-\mu S-\beta SZ) + \left(1-\frac{1}{I}\right)(\beta SZ-(\mu+\lambda+\delta)I+\gamma R) \\ &+ \left(1-\frac{1}{R}\right)(\lambda I-(\mu+\gamma)R) + \left(b-\frac{b^2}{Z}\right)(\alpha I-\alpha Z) + \frac{a\sigma_1^2}{2} \\ &+ \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{b\sigma_4^2}{2} \\ =& 3\mu + a\mu + \lambda + \delta + \gamma + \frac{a\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} + \frac{b\sigma_4^2}{2} + (b\alpha - (\mu+\delta))\, I \\ &+ (a\beta - b\alpha)Z - \left(\mu\left(S+R+\frac{a}{S}\right) + \frac{\beta SZ}{I} + \frac{\gamma R}{I} + \frac{\lambda I}{R} + \frac{b^2\alpha I}{Z}\right).\end{aligned}$$

Choosing $a = \frac{\mu+\delta}{\beta}$ and $b = \frac{\mu+\delta}{\alpha}$ leads to

$$\mathcal{L}V(S,I,R,Z) \le C, \tag{5}$$

where $C = 3\mu + a\mu + \lambda + \delta + \gamma + \dfrac{a\sigma_1^2}{2} + \dfrac{\sigma_2^2}{2} + \dfrac{\sigma_3^2}{2} + \dfrac{b\sigma_4^2}{2}$.
The remaining part of the proof is similar to Theorem 1.2 of [6]. However, for the convenience of the reader, we will include it here.
By (5), we have

$$\begin{aligned}dV(S,I,R,Z) \le& Cdt + \left(1-\frac{a}{S}\right)\sigma_1 S dB_1 + \left(1-\frac{1}{I}\right)\sigma_2 I dB_2 \\ &+ \left(1-\frac{1}{R}\right)\sigma_3 R dB_3 + \left(b-\frac{b^2}{Z}\right)\sigma_4 Z dB_4.\end{aligned}$$

Integrating both sides of the last inequality over $[0, \tau_k \wedge T]$ yields

$$\begin{aligned}&\mathbb{E}V\left(S(\tau_k\wedge T), I(\tau_k\wedge T), R(\tau_k\wedge T), Z(\tau_k\wedge T)\right) \\ &\quad\le V(S(0), I(0), R(0), Z(0)) + CT,\end{aligned}$$

which gives that

$$
\begin{aligned}
&V(S(0),I(0),R(0),Z(0))+CT \\
&\qquad\qquad \geq \mathbb{E}\left[1_{\{}\tau_k \leq T\}V\left(S(\tau_k\wedge T),I(\tau_k\wedge T),R(\tau_k\wedge T),Z(\tau_k\wedge T)\right)\right] \\
&\qquad\qquad \geq \varepsilon\theta_k,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_k = &\left(k-a-a\ln\frac{k}{a}\right)\wedge\left(\frac{1}{k}-a+a\ln(ka)\right)\wedge(k-1-\ln k) \\
&\wedge\left(\frac{1}{k}-1+\ln k\right)\wedge b\left(k-b-b\ln\frac{k}{b}\right)\wedge b\left(\frac{1}{k}-b+b\ln(kb)\right).
\end{aligned}
$$

Letting $k\to\infty$ leads to the contradiction $\infty > V(S(0),I(0),R(0),Z(0))+CT = \infty$. Hence $\tau_\infty = \infty$ *a.s.* This concludes the proof. □

3. Extinction of the disease

In this section, we establish sufficient conditions for the extinction of the disease in our model (3). We set the parameter

$$
\mathcal{R}_0 = \frac{\beta}{\mu+\lambda+\delta} + \frac{\lambda\gamma}{(\mu+\lambda+\delta)(\mu+\gamma)}.
$$

Then, we have the following theorem.

Theorem 3.1. *Let $(S(t),I(t),R(t),Z(t))$ be the solution to system (3) for an initial value $(S(0),I(0),R(0),Z(0))$ in $\mathbb{R}^4_+$. Assume that*

$$
\mathcal{R}_0 < 1, \quad \sigma_1^2 \leq 2\mu \quad \text{and}
$$

$$
\min\{\mu+\lambda+\delta,\mu+\gamma,\alpha\}(\sqrt{\mathcal{R}_0}-1) + \frac{\beta\alpha\sigma_1\sqrt{\mathcal{R}_0}}{(\mu+\lambda+\delta)\sqrt{2\mu-\sigma_1^2}} < 0.
$$

Then, the disease dies out exponentially with probability one. Moreover, the distribution of $S(t)$ converges weakly to the measure which has the density

$$
p(x) = Qx^{-\frac{2\mu}{\sigma_1^2}-2}e^{-\frac{2\mu}{\sigma_1^2 x}}, \quad x>0,
$$

where the normalisation constant $Q = \left(\frac{2\mu}{\sigma_1^2}\right)^{\frac{2\mu}{\sigma_1^2}+1}\left(\Gamma\left(\frac{2\mu}{\sigma_1^2}+1\right)\right)^{-1}$.

Proof. First, we consider X_t solution to the following SDE

$$dX_t = (\mu - \mu X)\, dt + \sigma_1 X dB_1(t), \quad X(0) = S(0) > 0. \tag{6}$$

Let $q(x) = \exp\left(-2\int_1^x \frac{\mu - \mu u}{(\sigma_1 u)^2} du\right)$. We have

$$q(x) = \exp\left(-2\frac{\mu}{\sigma_1^2}\left(1 - \ln x - \frac{1}{x}\right)\right) = e^{-\frac{2\mu}{\sigma_1^2}} x^{\frac{2\mu}{\sigma_1^2}} e^{\frac{2\mu}{\sigma_1^2 x}}.$$

Hence,

$$\int_1^\infty q(x)dx = \infty, \int_0^1 q(x)dx = \infty \quad \text{and} \quad \int_0^\infty \frac{1}{\sigma_1^2} x^{-2} q(x)^{-1} dx < \infty.$$

Therefore, by Theorem 1.16 in [17], (6) has the ergodic property and the density of its invariant law is given by

$$p(x) = Q x^{-\frac{2\mu}{\sigma_1^2} - 2} e^{-\frac{2\mu}{\sigma_1^2 x}},$$

where $Q = \left(\frac{2\mu}{\sigma_1^2}\right)^{\frac{2\mu}{\sigma_1^2}+1} \left(\Gamma\left(\frac{2\mu}{\sigma_1^2} + 1\right)\right)^{-1}$. By the comparison theorem, we have $S(t) \leq X(t),\ t \geq 0\ a.s.$ By direct computations, one can obtain

$$\begin{aligned}
\int_0^\infty x p(x) dx &= Q \int_0^\infty x^{-\frac{2\mu}{\sigma_1^2} - 1} e^{-\frac{2\mu}{\sigma_1^2 x}} dx \\
&= Q \left(\frac{2\mu}{\sigma_1^2}\right)^{-\frac{2\mu}{\sigma_1^2}} \Gamma\left(\frac{2\mu}{\sigma_1^2}\right) \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty x^2 p(x) dx &= Q \int_0^\infty x^{-\frac{2\mu}{\sigma_1^2}} e^{-\frac{2\mu}{\sigma_1^2 x}} dx \\
&= Q \left(\frac{2\mu}{\sigma_1^2}\right)^{-\frac{2\mu}{\sigma_1^2}+1} \Gamma\left(\frac{2\mu}{\sigma_1^2} - 1\right) \\
&= \frac{2\mu}{2\mu - \sigma_1^2}.
\end{aligned}$$

Then,

$$\int_0^\infty (x-1)^2 p(x) dx = \int_0^\infty (x^2 - 2x + 1) p(x) dx = \frac{\sigma_1^2}{2\mu - \sigma_1^2}.$$

Next, we consider the nonnegative $\mathcal{C}^2-$function $V_1(I,R,Z)=\alpha_1 I+\alpha_2 R+\alpha_3 Z$, where $\alpha_1=\frac{\omega_1}{\mu+\lambda+\delta}$, $\alpha_2=\frac{\omega_2}{\mu+\gamma}$, and $\alpha_3=\frac{\omega_3}{\alpha}$. $\omega_i, i=1,\cdots,3$ are positive constants to be determined. Using Itô's formula for $\ln V_1$, we have

$$\begin{aligned}
d\ln V_1 =&\frac{1}{V_1}\left\{\alpha_1(\beta SZ-(\mu+\lambda+\delta)I+\gamma R)+\alpha_2(\lambda I-(\mu+\gamma)R)+\alpha_3\alpha(I-Z)\right\}dt\\
&-\frac{1}{2V_1^2}\left((\alpha_1\sigma_2 I)^2+(\alpha_2\sigma_3 R)^2+(\alpha_3\sigma_4 Z)^2\right)dt\\
&+\frac{1}{V_1}\left(\sigma_2 I dB_2+\sigma_3 R dB_3+\sigma_4 Z dB_4\right)\\
=&\mathcal{L}\ln V_1 dt+\frac{1}{V}\left(\sigma_2 I dB_2+\sigma_3 R dB_3+\sigma_4 Z dB_4\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}\ln V_1 =&\frac{1}{V_1}\left\{\alpha_1(\beta Z-(\mu+\lambda+\delta)I+\gamma R)+\alpha_2(\lambda I-(\mu+\gamma)R)+\alpha_3\alpha(I-Z)\right\}\\
&+\frac{\alpha_1\beta}{V_1}(S-1)Z-\frac{1}{2V^2}\left((\alpha_1\sigma_2 I)^2+(\alpha_2\sigma_3 R)^2+(\alpha_3\sigma_4 Z)^2\right)\\
\le&\frac{\alpha_1\beta}{\alpha_3}|X-1|\\
&+\frac{1}{V_1}\left\{\alpha_1(\beta Z-(\mu+\lambda+\delta)I+\gamma R)+\alpha_2(\lambda I-(\mu+\gamma)R)+\alpha_3\alpha(I-Z)\right\}\\
\le&\frac{\alpha_1\beta}{\alpha_3}|X-1|+\frac{1}{V_1}\left\{\frac{\omega_1}{\mu+\lambda+\delta}(\beta Z-(\mu+\lambda+\delta)I+\gamma R)\right.\\
&\left.+\frac{\omega_2}{\mu+\gamma}(\lambda I-(\mu+\gamma)R)+\omega_3\alpha(I-Z)\right\}.
\end{aligned}$$

Let M_0 be the matrix defined by

$$M_0=\begin{pmatrix} 0 & \frac{\gamma}{\mu+\lambda+\delta} & \frac{\beta}{\mu+\lambda+\delta}\\ \frac{\lambda}{\mu+\gamma} & 0 & 0\\ 1 & 0 & 0\end{pmatrix},$$

with eigenvalue $\sqrt{\mathcal{R}_0}=\sqrt{\frac{\beta}{\mu+\lambda+\delta}+\frac{\lambda\gamma}{(\mu+\lambda+\delta)(\mu+\gamma)}}$. Choosing the vector $(\omega_1,\omega_2,\omega_3)=\left(\sqrt{\mathcal{R}_0},\frac{\lambda}{\mu+\gamma},1\right)$ leads to

$$(\omega_1,\omega_2,\omega_3)\,M_0=\sqrt{\mathcal{R}_0}\,(\omega_1,\omega_2,\omega_3)\,.$$

Therefore,

$$\begin{aligned}
\mathcal{L}\ln V_1 \leq & \frac{\alpha_1\beta}{\alpha_3}|X-1| + \frac{1}{V_1}\left(\omega_1\ \omega_2\ \omega_3\right)\left(M_0(I,R,Z)^T - (I,R,Z)^T\right) \\
= & \frac{\alpha_1\beta}{\alpha_3}|X-1| + \frac{1}{V_1}\left(\sqrt{\mathcal{R}_0}-1\right)(\omega_1 I + \omega_2 R + \omega_3 Z) \\
\leq & \frac{\alpha_1\beta}{\alpha_3}|X-1| + \min\{\mu+\lambda+\delta, \mu+\gamma, \alpha\}\left(\sqrt{\mathcal{R}_0}-1\right).
\end{aligned}$$

Hence, we obtain the inequality

$$\begin{aligned}
d\ln V_1 \quad \leq \quad & \frac{\alpha_1\beta}{\alpha_3}|X-1| + \min\{\mu+\lambda+\delta, \mu+\gamma, \alpha\}\left(\sqrt{\mathcal{R}_0}-1\right) \\
& + \frac{1}{V}\left(\sigma_2 I dB_2 + \sigma_3 R dB_3 + +\sigma_4 Z dB_4\right).
\end{aligned}$$

Integrating both sides and dividing by t gives

$$\begin{aligned}
\frac{\ln V_1(I(t),R(t),Z(t))}{t} \leq & \frac{\ln V_1(I(0),R(0),Z(0))}{t} \\
& + \min\{\mu+\lambda+\delta, \mu+\gamma, \alpha\}\left(\sqrt{\mathcal{R}_0}-1\right) \\
& + \frac{\alpha_1\beta}{\alpha_3}\frac{1}{t}\int_0^t |X(s)-1|ds + \frac{N_t}{t},
\end{aligned}$$

where $N_t = \int_0^t \frac{\sigma_2 I}{V_1}dB_2 + \int_0^t \frac{\sigma_3 R}{V_1}dB_3 + \int_0^t \frac{\sigma_4 Z}{V_1}dB_4$ is a local martingale with finite quadratic variation. Hence, $\lim\limits_{t\to\infty}\frac{N_t}{t} = 0$ $a.s.$ by the strong law of large numbers for local martingales.

As the solution to system (6) is ergodic and $\int_0^\infty xp(x)dx < \infty$, then

$$\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\int_0^t |X(s)-1|ds = \int_0^\infty |x-1|p(x)dx \leq & \left(\int_0^\infty (x-1)^2 p(x)dx\right)^{\frac{1}{2}} \\
= & \frac{\sigma_1}{\sqrt{2\mu-\sigma_1^2}}.
\end{aligned}$$

Consequently

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln V_1(I(t),R(t),Z(t))}{t} &\leq \min\{\mu+\lambda+\delta,\mu+\gamma,\alpha\}\left(\sqrt{\mathcal{R}_0}-1\right)\\ &\quad+\frac{\alpha_1\beta}{\alpha_3}\frac{\sigma_1}{\sqrt{2\mu-\sigma_1^2}}\\ &=\min\{\mu+\lambda+\delta,\mu+\gamma,\alpha\}\left(\sqrt{\mathcal{R}_0}-1\right)\\ &\quad+\frac{\beta\alpha\sigma_1\sqrt{\mathcal{R}_0}}{(\mu+\lambda+\delta)\sqrt{2\mu-\sigma_1^2}}\\ &<0 \quad a.s.,\end{aligned}$$

which is the desired result. Moreover,

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}<0, \quad \limsup_{t\to\infty}\frac{\ln Z(t)}{t}<0 \quad a.s.$$

This implies that $I(t)$ goes to 0 exponentially with probability 1. The proof is then complete. □

4. Persistence in mean of the Disease

The study of the persistence of diseases is an important approach to know more on epidemics behaviors, since it provides the conditions under which the diseases are prevalent. For reasons of simplification, we define the quantity

$$\mathcal{R}_s^0=\frac{\beta K}{\mu+\lambda+\delta+\frac{\sigma_2^2}{2}+\beta K\frac{\sigma_4^2}{2\alpha}-\frac{\lambda\gamma}{\mu+\gamma+\frac{\sigma_3^2}{2}}},$$

where $K=\dfrac{\mu}{\mu+\frac{\sigma_1^2}{2}}$. We then have the following result.

Theorem 4.1. *If* $\mathcal{R}_s^0>1$, *then for any given initial value* $(S(0),I(0),R(0),Z(0))\in\mathbb{R}_+^4$, *the corresponding solution to* (3) *verifies*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(s)ds\geq\Lambda\left(1-\frac{1}{\mathcal{R}_s^0}\right) \quad a.s., \tag{7}$$

where $\Lambda=\dfrac{\mu\alpha}{\beta\left(\alpha+\dfrac{\sigma_4^2}{2}\right)}$. *In other words, the epidemic will be permanent.*

Proof. Let $(S(0), I(0), R(0), Z(0)) \in \mathbb{R}^4_+$ and consider the function

$$V_2(S, I, R, Z) = -\beta_1 \ln S - \ln I - \beta_2 \ln R - \beta_3 \ln Z + \beta_4 Z, \tag{8}$$

where $\beta_i, i = 1, \cdots, 4$ are positive constants to be specified later. The function V_2 is continuous and goes to ∞ as (S, I, R, Z) goes to the boundary of $\mathbb{R}^4_+$. Thus it must has a lower bound M. Then we define the nonnegative function $\tilde{V}_2 = V_2(S, I, R, Z) - M$. Applying Itô's formula to $\tilde{V}_2$, we obtain

$$\begin{aligned}
d\tilde{V}_2 =& -\frac{\beta_1}{S}(\mu - \mu S - \beta SZ) + \beta_1 \frac{\sigma_1^2}{2} - \frac{1}{I}(\beta SZ - (\mu + \lambda + \delta)I + \gamma R) + \frac{\sigma_2^2}{2} \\
& - \frac{\beta_2}{R}(\lambda I - (\mu + \gamma)R) + \beta_2 \frac{\sigma_3^2}{2} - \beta_3 \frac{\alpha}{Z}(I - Z) + \beta_3 \frac{\sigma_4^2}{2} + \beta_4 \alpha (I - Z) \\
& - (\beta_1 \sigma_1 dB_1 + \sigma_2 dB_2 + \beta_2 \sigma_3 dB_3 + \sigma_4 (\beta_3 - \beta_4 Z) dB_4) \\
=& \mathcal{L}\tilde{V}_2 dt - (\beta_1 \sigma_1 dB_1 + \sigma_2 dB_2 + \beta_2 \sigma_3 dB_3),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}\tilde{V}_2 =& -\frac{\beta SZ}{I} - \frac{\beta_1 \mu}{S} - \frac{\beta_3 \alpha I}{Z} - \frac{\gamma R}{I} - \frac{\beta_2 \lambda I}{R} + (\beta_1 \beta - \beta_4 \alpha) Z + \beta_1 \left(\mu + \frac{\sigma_1^2}{2}\right) \\
& + \mu + \lambda + \delta + \frac{\sigma_2^2}{2} + \beta_2 \left(\mu + \gamma + \frac{\sigma_3^2}{2}\right) + \beta_3 \left(\alpha + \frac{\sigma_4^2}{2}\right) + \beta_4 \alpha I,
\end{aligned}$$

Choosing $\beta_4 = \beta_1 \beta / \alpha$ and using the facts that

$$-\frac{\beta SZ}{I} - \frac{\beta_1 \mu}{S} - \frac{\beta_3 \alpha I}{Z} \leq -3\sqrt[3]{\beta \mu \alpha \beta_1 \beta_3} \quad \text{and} \quad -\frac{\gamma R}{I} - \frac{\beta_2 \lambda I}{R} \leq -2\sqrt{\lambda \gamma \beta_2},$$

we obtain

$$\begin{aligned}
\mathcal{L}\tilde{V}_2 \leq& -3\sqrt[3]{\beta \mu \alpha \beta_1 \beta_3} - 2\sqrt{\lambda \gamma \beta_2} + \beta_1 \left(\mu + \frac{\sigma_1^2}{2}\right) + \mu + \lambda + \delta + \frac{\sigma_2^2}{2} \\
& + \beta_2 \left(\mu + \gamma + \frac{\sigma_3^2}{2}\right) + \beta_3 \left(\alpha + \frac{\sigma_4^2}{2}\right) + \beta_4 \alpha I.
\end{aligned}$$

For $\beta_2 = \lambda\gamma / \left(\mu + \gamma + \frac{\sigma_3^2}{2}\right)^2$, $\beta_1 = \beta K^2 \left(\alpha + \frac{\sigma_4^2}{2}\right) / (\mu\alpha)$ and $\beta_3 = \beta K / \left(\alpha + \frac{\sigma_4^2}{2}\right)$, we get

$$\begin{aligned}
\mathcal{L}\tilde{V}_2 \leq& -\beta K + \mu + \lambda + \delta + \frac{\sigma_2^2}{2} + \beta K \frac{\sigma_4^2}{2\alpha} - \frac{\lambda\gamma}{\mu + \gamma + \frac{\sigma_3^2}{2}} + \beta_1 \beta I \\
=& -\beta K \left(1 - \frac{1}{\mathcal{R}_s^0}\right) + \beta_1 \beta I.
\end{aligned}$$

We further obtain

$$
\begin{aligned}
d\tilde{V}_2 \quad \leq \quad & \left(-\beta K\left(1-\frac{1}{\mathcal{R}_s^0}\right)+\beta_1\beta I\right)dt \\
& -\left(\beta_1\sigma_1 dB_1+\sigma_2 dB_2+\beta_2\sigma_3 dB_3+\sigma_4(\beta_3-\beta_4 Z)dB_4\right).
\end{aligned}
$$

Integrating both sides of the last inequality and dividing by t, leads to

$$
\begin{aligned}
0 \leq & \frac{\tilde{V}_2\left(S(t),I(t),R(t),Z(t)\right)}{t} \\
\leq & \frac{\tilde{V}_2\left(S(0),I(0),R(0),Z(0)\right)}{t}-\beta K\left(1-\frac{1}{\mathcal{R}_s^0}\right) \\
& +\beta_1\beta\frac{1}{t}\int_0^t I(s)ds+M_t,
\end{aligned}
$$

where $M_t=-\dfrac{1}{t}(\beta_1\sigma_1 B_1(t)+\sigma_2 B_2(t)+\beta_2\sigma_3 B_3(t)+\sigma_4\displaystyle\int_0^t(\beta_3-\beta_4 Z)dB_4)$ is a continuous local martingale. Using the large numbers theorem for martingales, we obtain $\lim\limits_{t\to\infty} M_t=0$. Therefore

$$
\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(s)ds \geq \frac{K}{\beta_1}\left(1-\frac{1}{\mathcal{R}_s^0}\right) \quad a.s.
$$

This completes the proof. □

5. Stationary distribution and positive recurrence

In the previous sections, we have investigated both the extinction and the permanence of the disease under some conditions. In this section, we will show that the solution to system (3) possesses a unique stationary distribution by means of the method given in [16] and used in [29].

Theorem 5.1. *For any* $(S(0),I(0),R(0),Z(0))\in\mathbb{R}_+^4$, *the corresponding solution to system (3) is positive recurrent and has a unique ergodic stationary distribution if the condition* $\mathcal{R}_s^0>1$ *holds.*

Proof. It is easy to verify the first condition of Lemma 1. We only have to prove that the second condition is also verified. Let us consider the $\mathcal{C}^2$-function V_6 defined by

$$
V_5(S,I,R,Z)=MV_2+V_3+V_4,
$$

where the function V_2 is given in the previous section, $V_3 = -\ln S - \frac{1}{\alpha}\ln Z$ and $V_4 = \frac{1}{m+1}\left(S + I + R + \frac{\mu+\delta}{2\alpha}Z\right)^{m+1} + cR$. M, m and c are positive constants satisfying some conditions as we shall see later.

Since V_6 is continuous and tends to ∞ as (S, I, R, Z) tends to the boundary of $\mathbb{R}^4_+$, it admits a lower bound M^* and achieves this lower bound at a point in $\mathbb{R}^4_+$. Therefore, we construct the nonnegative function $\tilde{V}_5$ by

$$\tilde{V}_5(S, I, R, Z) = V_5(S, I, R, Z) - M^*.$$

By simple computations, we can get that

$$\mathcal{L}V_2 \leq -\Theta + \beta\beta_1 I,$$

where

$$\Theta = \frac{K}{\beta_1}\left(1 - \frac{1}{\mathcal{R}^0_s}\right) > 0.$$

Moreover,

$$\mathcal{L}V_3 = -\frac{\mu}{S} - \frac{I}{Z} + \beta Z + 1 + \mu + \frac{\sigma_1^2}{2} + \frac{\sigma_4^2}{2\alpha}.$$

Similarly, we can deduce that

$$\begin{aligned}
\mathcal{L}V_4 = & \left(S + I + R + \frac{\mu+\delta}{2\alpha}Z\right)^m \left(\mu - \mu S - \frac{\mu+\delta}{2}I - \mu R - \frac{\mu+\delta}{2}Z\right) \\
& + \frac{m}{2}\left(S + I + R + \frac{\mu+\delta}{2\alpha}Z\right)^{m-1} \left(\sigma_1^2 S^2 + \sigma_2^2 I^2 + \sigma_3^2 R^2 + \sigma_4^2 Z^2\right) \\
& + c\lambda I - c(\mu+\gamma)R.
\end{aligned}$$

We choose m sufficiently small so that the following inequality holds,

$$m\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right) < \min\{\mu, \frac{\mu+\delta}{2}, \frac{(\mu+\delta)^2}{4\alpha}\}.$$

Then, we have

$$\begin{aligned}
\mathcal{L}V_4 \quad \leq \quad & -\mu m S^{m+1} - \frac{\mu+\delta}{4}I^{m+1} - \frac{\mu}{2}R^{m+1} - c(\mu+\gamma)R \\
& -\frac{1}{2\alpha^m}\left(\frac{\mu+\delta}{2}\right)^{m+1} Z^{m+1} + \Upsilon,
\end{aligned}$$

where

$$\Upsilon = \sup_{(S,I,R,Z)\in\mathbb{R}^4_+} \Bigg\{ -\mu(1-m)S^{m+1} - \frac{\mu+\delta}{4} I^{m+1} - \frac{\mu}{2} R^{m+1}$$
$$- \frac{1}{2\alpha^m}\left(\frac{\mu+\delta}{2}\right)^{m+1} Z^{m+1} + c\lambda I$$
$$+ \frac{m}{2}\left(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2\right)\left(S+I+R+\frac{\mu+\delta}{2\alpha}Z\right)^{m-1}\left(S^2+I^2+R^2\right)$$
$$+\mu\left(S+I+R+\frac{\mu+\delta}{2\alpha}Z\right)^{m}\Bigg\}.$$

Hence,

$$\mathcal{L}\tilde{V}_5 \leq -M\Theta + \beta\beta_1 I - c(\mu+\gamma)R - \frac{\mu}{S} - \frac{I}{Z} - \mu m S^{m+1}$$
$$- \frac{\mu+\delta}{4} I^{m+1} - \frac{\mu}{2} R^{m+1} - \frac{1}{4\alpha^m}\left(\frac{\mu+\delta}{2}\right)^{m+1} Z^{m+1} + \Psi,$$

where

$$\Psi = \sup_{Z>0}\left\{ -\frac{1}{4\alpha^m}\left(\frac{\mu+\delta}{2}\right)^{m+1} Z^{m+1} + \beta Z + \Upsilon + 1 + \mu + \frac{\sigma_1^2}{2} + \frac{\sigma_4^2}{2\alpha}\right\}.$$

Let $\varepsilon_i, i = 1, \cdots, 3$ be positive reals and define the closed set

$$D = \left\{(S,I,R,Z)\in\mathbb{R}^4_+; \varepsilon_1 \leq S \leq \frac{1}{\varepsilon_1}, \varepsilon_2 \leq I \leq \frac{1}{\varepsilon_2}, \varepsilon_2^2 \leq S \leq \frac{1}{\varepsilon_2^2}, \text{ and } \varepsilon_3 \leq Z \leq \frac{1}{\varepsilon_3}\right\}.$$

We divide the domaine $\mathbb{R}^4_+ \setminus D$ into eight domains

$$D_1 = \left\{(S,I,R,Z)\in\mathbb{R}^4_+;\ S<\varepsilon_1\right\},\quad D_2 = \left\{(S,I,R,Z)\in\mathbb{R}^4_+;\ S>\frac{1}{\varepsilon_1}\right\},$$
$$D_3 = \left\{(S,I,R,Z)\in\mathbb{R}^4_+;\ Z<\varepsilon_3\right\},\quad D_4 = \left\{(S,I,R,Z)\in\mathbb{R}^4_+;\ Z>\frac{1}{\varepsilon_3}\right\},$$
$$D_5 = \left\{(S,I,R,Z)\in\mathbb{R}^4_+; S\geq\varepsilon_1, I<\varepsilon_2, Z\geq\varepsilon_3\right\},$$
$$D_6 = \left\{(S,I,R,Z)\in\mathbb{R}^4_+; R>\frac{1}{\varepsilon_2^2}\right\},$$
$$D_7 = \left\{(S,I,R,Z)\in\mathbb{R}^4_+; I>\varepsilon_2, R<\varepsilon_2^2\right\},\ D_8 = \left\{(S,I,R,Z)\in\mathbb{R}^4_+; I>\frac{1}{\varepsilon_2}\right\}.$$

Let us show that $\mathcal{L}\tilde{V}_5$ is negative at any vector in $\mathbb{R}^4_+ \setminus D$.

Case 1. If $(S,I,R,Z)\in D_1$, we use the fact that $-\frac{\mu}{S} < -\frac{\mu}{\varepsilon_1}$ and choose ε_1

sufficiently small such that

$$\begin{aligned}\mathcal{L}\tilde{V}_5 \leq & -M\Theta + \beta\beta_1 I - \frac{\mu}{S} - \frac{\mu+\delta}{4} I^{m+1} + \Psi \\ \leq & -M\Theta - \frac{\mu}{\varepsilon_1} + \Psi_1 \\ < & 0,\end{aligned}$$

where $\Psi_1 = \sup_{Z>0} \left\{ \beta\beta_1 I - \frac{\mu+\delta}{4} I^{m+1} + \Psi \right\}$.

Case 2. Let $(S, I, R, Z) \in D_2$. Since $-\mu m S^{m+1} < -\frac{\mu m}{\varepsilon_1^{m+1}}$, we choose ε_1 sufficiently small such that

$$\begin{aligned}\mathcal{L}\tilde{V}_5 \leq & -M\Theta - \mu m S^{m+1} - mu\frac{m}{2} S^{m+1} + \Psi_1 \\ \leq & -M\Theta - \frac{\mu m}{\varepsilon_1^{m+1}} + \Psi_1 \\ < & 0.\end{aligned}$$

Case 3. If $(S, I, R, Z) \in D_3$, then, there exists a sufficiently large M such that

$$\mathcal{L}\tilde{V}_5 \leq -M\Theta + \Psi_1 < 0.$$

Case 4. If $(S, I, R, Z) \in D_4$. With the fact that $-Z^{m+1} < -\frac{1}{\varepsilon_3^{m+1}}$, we choose ε_3 sufficiently small such that

$$\begin{aligned}\mathcal{L}\tilde{V}_5 \leq & -M\Theta - \frac{1}{4\alpha^m} \left(\frac{\mu+\delta}{2} \right)^{m+1} Z^{m+1} + \Psi_1 \\ \leq & -M\Theta - \frac{1}{4\alpha^m} \left(\frac{\mu+\delta}{2\varepsilon_3} \right)^{m+1} + \Psi_1 \\ < & 0.\end{aligned}$$

Case 5. If $(S, I, R, Z) \in D_5$, there is a sufficiently large number M such that

$$\mathcal{L}\tilde{V}_5 \leq -M\Theta + \beta\beta_1\varepsilon_2 + \Psi_1 < 0.$$

Case 6. Similarly to the third case, if $(S, I, R, Z) \in D_5$, with a large value of M one can get

$$\mathcal{L}\tilde{V}_5 \leq -M\Theta + \Psi_1 < 0.$$

Case 7. If $(S, I, R, Z) \in D_7 \cup D_8$, there is a sufficiently large number M such that

$$\mathcal{L}\tilde{V}_5 \leq -M\Theta + \Psi_1 < 0.$$

With the above choices of M, ε_1 and ε_3 and for a given $\varepsilon_2 < 1$, we get

$$\mathcal{L}\tilde{V}_5(S, I, R, Z) < 0, \quad \text{for any} \quad (S, I, R, Z) \in \mathbb{R}^4_+ \setminus D.$$

This concludes the proof. □

6. Numerical simulations and conclusion

Adopting Milstein's scheme for SDE discretization [10], we simulated system (3) for many parameter sets with the aim of illustrating the results exhibited in this paper. We choose the initial value $(S(0), I(0), R(0)) = (0.5, 0.15, 0.1)$ and use R software to perform simulations. Theorem 3.1 presents conditions under

Table 1: Parameters values used in Fig. 1.

μ	β	λ	δ	γ	α	σ_1	σ_2	σ_3	σ_4
0.12	0.27	0.48	0.1	0.113	0.11	0.2	0.23	0.15	0.07

Table 2: Parameters values used in Fig. 2 and Fig. 3.

μ	β	λ	δ	γ	α	σ_1	σ_2	σ_3	σ_4
0.09	0.37	0.328	0.1	0.11	0.07	0.05	0.04	0.035	0.04

which the disease dies out exponentially. This is illustrated by Fig. 1 where sufficient conditions are satisfied.

$$\mathcal{R}_0 = 0.7183 < 1, \quad \sigma_1^2 = 0.04 \leq 2\mu = 0.24 \quad \text{and}$$

$$\min\{\mu + \lambda + \delta, \mu + \gamma, \alpha\}(\sqrt{\mathcal{R}_0} - 1) + \frac{\beta\alpha\sigma_1\sqrt{\mathcal{R}_0}}{(\mu + \lambda + \delta)\sqrt{2\mu - \sigma_1^2}} = -7 \cdot 10^{-4} < 0.$$

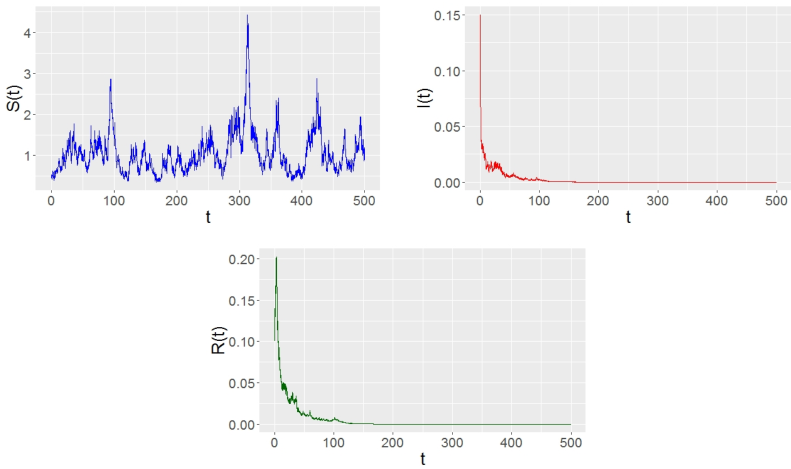

Figure 1: Trajectories of $S(t), I(t)$ and $R(t)$ components of $(S(t), I(t), R(t), Z(t))$ the solution of system (3) with parameters given in Table 1.

When $\mathcal{R}_s^0$ exceeds 1, the disease should be prevalent with probability one. This confirms the behaviour of the epidemic in Fig. 2 with parameters given in table 2 and a value of $\mathcal{R}_s^0$ equals to 1.0636.

By sampling 10000 stochastic paths of the solution to system (3) and keeping the same parameter set presented in table 2, the stationary probability density functions of infected and removed individuals at different times are shown in Fig. 3 respectively. We see that they are close to each other at $t = 2000$ and $t = 3000$, which is confirmed by theorem 5.1.

To conclude, a stochastic SIRI epidemic model with distributed delay is analyzed in this paper. The existence and uniqueness of a global positive solution is the first result we established. Some sufficient conditions are obtained to promise the extinction of the disease. $\mathcal{R}_s^0 > 0$ is an adequate condition to the permanence of the disease as well as the existence of unique stationary distribution to our model. As an important question, it will be noteworthy to study the behaviour of the considered model in the case where $\mathcal{R}_0 \leq 1 \leq \mathcal{R}_s^0$. We leave the response to this question to further works.

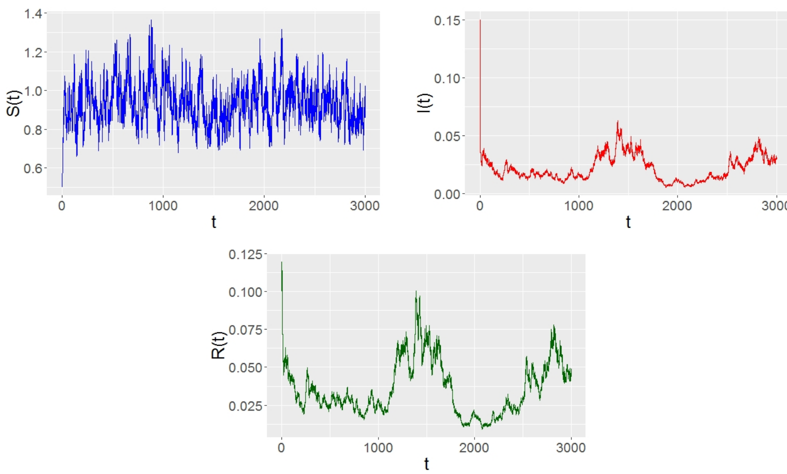

Figure 2: Simulation of paths $S(t), I(t)$ and $R(t)$ where $(S(t), I(t), R(t), Z(t))$ is the solution to system (3) using data of Table 2.

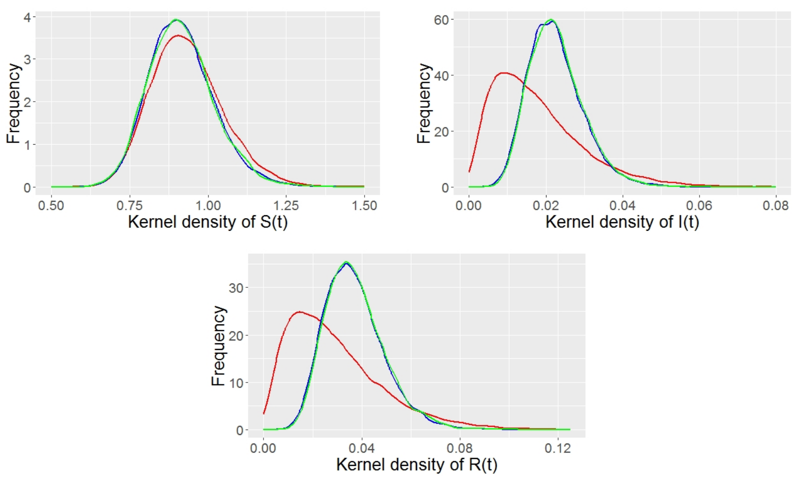

Figure 3: The kernel density functions of susceptible, infected and removed compartments of (3) at $t = 1000$ (red), $t = 2000$ (blue) and $t = 3000$ (green).

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