TRAJECTORY AND GLOBAL ATTRACTORS FOR GENERALIZED PROCESSES

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Dedicated to Peter Kloeden on occasion of his 70th birthday

Abstract. In this work the theory of generalized processes is used to describe the dynamics of a nonautonomous multivalued problem and, through this approach, some conditions for the existence of trajectory attractors are proved. By projecting the trajectory attractor on the phase space, the uniform attractor for the multivalued process associated to the problem is obtained and some conditions to guarantee the invariance of the uniform attractor are given. Furthermore, the existence of the uniform attractor for a class of p-Laplacian nonautonomous problems with dynamical boundary conditions is established.

1. Introduction

One way to study the asymptotic behaviour of nonautonomous multivalued evolution problems was developed in [9, 10] and lies on looking at the space of all solutions as a phase space (endowed with a suitable topology) and studying the asymptotic behaviour of the system by considering the dynamics of the translation semigroup on this trajectory space. This theory is known as trajectory attractor theory and its results are gathered in the work [11]. When dealing with well posed problems, the projection of the trajectory attractor on the phase space concentrates the forward dynamics of nonautonomous problems.

It is suggested in [12], Remark 4.2, that one could "construct" a global attractor for a nonautonomous evolution problem without uniqueness of solution by using the sections of a uniform trajectory attractor. Nevertheless, as far as we know, it is not done yet. We believe that the main difficulty lies on the choice of an appropriated theoretical framework to describe the dynamics of nonautonomous multivalued problems.

The theory of multivalued processes [5, 29] has been successfully applied in the study of the asymptotic dynamics of problems without uniqueness of solution. Another theoretical framework that can be considered is the so called generalized processes one, where the elementary

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objects are the solution curves, on which we can impose suitable conditions, concatenation for instance, which are impossible to be required in a setting where the elementary objects are the states ([2, 13, 34]).

For well posed problems we can find other theoretical approaches to study the pullback and forward asymptotic dynamics, and a detailed description of them, including relations between the attractors associated to each one, can be found in [4, 8].

As an application of the abstract theory we consider the nonautonomous p-Laplacian problem with dynamic boundary condition:

$$\begin{cases} u_t - \Delta_p u + f_1(t,u) = g_1(t,x), & (t,x) \in (\tau,+\infty)\times\Omega, \\ u_t + |\nabla u|^{p-2}\partial_{\vec{n}} u + f_2(t,u) = g_2(t,x), & (t,x)\in(\tau,+\infty)\times\Gamma, \\ u(\tau) = u_0, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N, N \geq 3$, is a bounded domain with smooth boundary $\Gamma = \partial\Omega$, $\vec{n}$ is the outer normal to Γ, $\tau \in \mathbb{R}$ is an initial time, u_0 is an initial state in a space we will properly define later and Δ_p denotes the p-Laplacian operator, defined by $\Delta_p u = div(|\nabla u|^{p-2}\nabla u), p \geq 2$. The perturbations f_i and the external forces g_i, $i = 1, 2$, satisfy the following assumptions:

(H1) Let $g_1 \in L^{p'}_{loc}(\mathbb{R}; L^{r_1'}(\Omega))$, $g_2 \in L^{p'}_{loc}(\mathbb{R}; L^{r_2'}(\Gamma))$, where s' denotes the conjugate exponent of s, which means, $\frac{1}{s} + \frac{1}{s'} = 1$, with

$$r_1 \in \begin{cases} (p, pN/(N-p)], & \text{if } p \in (2,N), \\ (p,+\infty), & \text{if } p = N, \\ [p,+\infty), & \text{if } p > N \end{cases}$$

and

$$r_2 \in \begin{cases} (2, (N-1)p/(N-p)], & \text{if } p \in (2,N), \\ (2,+\infty), & \text{if } p = N, \\ [2,+\infty), & \text{if } p > N; \end{cases}$$

(H2) For $i = 1, 2$, $f_i \in C(\mathbb{R}^2)$ satisfy the following growth conditions

$$\begin{cases} a_1(t)|s|^{r_1} - k_1 \leq f_1(t,s)s, \\ a_2(t)|s|^{r_2} - k_2 \leq f_2(t,s)s, \end{cases}$$

a.e. for $t \in \mathbb{R}$ and all $s \in \mathbb{R}$, where k_i are positive constants and $a_i \in L^1_{loc}(\mathbb{R})$ are real functions satisfying $a_i(t) \geq a_0 > 0$ for some $a_0 \in \mathbb{R}$.

(H3) There exist constants C_i, $i = 1, 2$, such that $|f_i(t,s)| \leq C_i(|s|^{r_i - 1} + 1)$ a.e. for $t \in \mathbb{R}$ and all $s \in \mathbb{R}$.

Problems with dynamic boundary conditions have not been very much explored yet in comparison with other boundary conditions and have emerged in the literature at least since the early seventies, [14, 19, 20, 26]. This kind of problems has applications in several areas of science such as hydrodynamics, thermoelasticity and heat transfer [15, 18, 24, 27, 30].

For well posed problems, we can refer to the following works. The autonomous version of the problem presented here appears in [16, 17]. When $g_i, i = 1, 2$, depend on t, the authors have established in [39] the existence of a uniform attractor. In the works [25, 38] we can find results on the existence of a $\mathscr{D}$-pullback attractor for nonautonomous p-Laplacian problems with dynamic boundary conditions. Existence of solution and existence of pullback attractors for problems with this type of boundary conditions were also established in [32], where the authors have inserted a nonautonomous term in perturbations $f_i, i = 1, 2$, and in [33], where a problem with infinite delay is considered.

In this work we conjugate both theories, multivalued processes and generalized processes, to describe the forward dynamics of a nonautonomous multivalued problem and we obtain the uniform attractor in multivalued context as the projection of the trajectory attractor on the phase space, following an idea that has arisen in [12], where the authors mentioned that it was possible to obtain this result but they omitted the proof. After that we collect some results about multivalued processes to prove the invariance of the uniform attractor under appropriate conditions. Finally, we ensure the existence of the trajectory attractor for Problem (P) which implies the existence of the uniform attractor.

We organize this paper as follows. In the next section, we recall some notations, definitions and properties of suitable spaces for the study of Problem (P). In Section 3 we adapt the theory of trajectory attractors given in [11] for the framework of generalized processes, and show that the existence of trajectory attractors ensures, under certain hypotheses, the existence of the invariant compact uniform global attractor. In Section 4 we prove the existence of the trajectory attractor for our Problem (P), which guarantees the existence of the uniform global attractor for the same problem.

2. Preliminaries

In this section we define the appropriate spaces to study Problem (P), following [15].

Consider the Lebesgue space

$$L^r(\Gamma) = \{v : \|v\|_{L^r(\Gamma)} < \infty\},$$

where $\|v\|_{L^p(\Gamma)} = \left(\int_\Gamma |v|^p dS\right)^{1/p}$, for $p \in [1, \infty)$, dS is the surface measure on Γ induced by dx and $\|v\|_{L^\infty(\Gamma)} = \inf\{C;\ |v(x)| \leq C \text{ a.e. in } \Gamma\}$.

The phase space to be considered is given by

$$\mathbb{X}^p := L^p(\Omega, dx) \times L^p(\Gamma, dS) = \{F = (f, g); f \in L^p(\Omega) \text{ and } g \in L^p(\Gamma)\},$$

with the norm

$$\|F\|_{\mathbb{X}^p} = \left(\int_\Omega |f|^p dx + \int_\Gamma |g|^p dS\right)^{\frac{1}{p}},$$

for $1 \leq p < \infty$, and

$$\|F\|_{\mathbb{X}^\infty} := \max\left\{\|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Gamma)}\right\},$$

for $p = +\infty$. This space can be identified with the space $L^p(\overline{\Omega}, d\mu)$ where $d\mu = dx \oplus dS$, i.e., if $A \subset \overline{\Omega}$ is $\mu-$measurable, then $\mu(A) = |A \cap \Omega| + S(A \cap \Gamma)$.

Note that the space $\mathbb{X}^2$, with the following inner product

$$\langle\cdot,\cdot\rangle_{\mathbb{X}^2} := \langle\cdot,\cdot\rangle_{L^2(\Omega)} + \langle\cdot,\cdot\rangle_{L^2(\Gamma)},$$

is a separable Hilbert space.

For $p \in (1, \infty)$ we define the fractional order Sobolev space

$$W^{1-\frac{1}{p},p}(\Gamma) = \left\{u \in L^p(\Gamma) : \int_\Gamma\int_\Gamma \frac{|u(x) - u(y)|^p}{|x - y|^{p+N-2}} dS_x dS_y < \infty\right\}.$$

Consider the vector subspace of $W^{1,p}(\Omega) \times W^{1-\frac{1}{p},p}(\Gamma)$, given by

$$\mathbb{V}^p = \{U = (u, v); u \in W^{1,p}(\Omega) \text{ and } v = \gamma(u)\},$$

where $\gamma : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\Gamma)$ is the continuous trace operator. In $\mathbb{V}^p$, we can consider the usual norm $\|U\|_{\mathbb{V}^p} = \|u\|_{W^{1,p}(\Omega)} + \|\gamma(u)\|_{W^{1-\frac{1}{p},p}(\Gamma)}$. The $\mathbb{V}^p$ space is densely and compactly contained in the Hilbert space $\mathbb{X}^2$ for $2 \leq p < +\infty$, as it can be seen in [16].

Note that we can identify $u \in W^{1,p}(\Omega)$ with a couple $U = (u, \gamma(u)) \in \mathbb{V}^p$, the continuity of γ ensures the equivalence between the norms of $W^{1,p}(\Omega)$ and $\mathbb{V}^p$. We can show that $\mathbb{V}^p$ is a reflexive and separable space for $1 < p < \infty$. Furthermore,

$$\mathbb{V}^p \subset\subset \mathbb{X}^2 \subset (\mathbb{V}^p)^* \text{ for } 2 \leq p < +\infty.$$

3. Abstract Results

Now we define a generalized process ([2, 32, 34]) as an extension of the generalized semiflows for nonautonomous problems ([1, 35]). Then we define the uniform trajectory attractor for a generalized process and we establish conditions to guarantee the existence of such attractors for nonautonomous multivalued cases.

3.1. **Generalized Process.** Let (X, ρ) be a complete metric space. For $x \in X$, $A, B \subset X$ and $\varepsilon > 0$ we define

$$\rho(x, A) := \inf_{a \in A}\{\rho(x, a)\};$$
$$dist(A, B) := \sup_{a \in A} \inf_{b \in B}\{\rho(a, b)\};$$
$$\mathcal{O}_\varepsilon(A) := \{z \in X; \rho(z, A) < \varepsilon\}.$$

Denote by $\mathcal{P}(X)$, $\mathcal{B}(X)$ and $\mathcal{K}(X)$ the nonempty, nonempty and bounded and nonempty and compact subsets of X, respectively.

Consider the family of translation operators $\{T(t)\}_{t\in\mathbb{R}}$ acting on any function whose domain is the real line, i.e., given a function $f : \mathbb{R} \to A$, where $A \neq \emptyset$ is an arbitrary set we define $T(t)f(s) := f(t + s)$, $\forall s \in \mathbb{R}$.

Let Ψ be a Banach Space and $\Xi = \{\sigma; \sigma : \mathbb{R} \to \Psi\}$ endowed with some topology. The family of operators $\{T(t)\}_{t\in\mathbb{R}}$ acting on Ξ is a group.

Let now $\Sigma \subset \Xi$ be an invariant subset under translation, which means $T(t)\Sigma = \Sigma$, and suppose that Σ is a complete metric space with the topology inherited from Ξ.

Definition 3.1 (Generalized Process)**.** *A* ***generalized process*** $\mathscr{G} = \{\mathscr{G}_\sigma(\tau)\}_{\sigma\in\Sigma,\tau\in\mathbb{R}}$ *in* X *is a family of sets* $\mathscr{G}_\sigma(\tau)$ *consisting of functions* $\varphi : [\tau, \infty) \to X$*, called solutions, satisfying the following conditions:*

(C1) *For each* $\sigma \in \Sigma$*,* $\tau \in \mathbb{R}$ *and* $z \in X$ *there exists at least one* $\varphi \in \mathscr{G}_\sigma(\tau)$ *with* $\varphi(\tau) = z$*;*
(C2) *If* $\varphi \in \mathscr{G}_\sigma(\tau)$ *and* $s \geq 0$*, then* $\varphi^{+s} \in \mathscr{G}_\sigma(\tau + s)$*, where* $\varphi^{+s} := \varphi_{|[\tau+s,\infty)}$*;*
(C3) *If* $\varphi \in \mathscr{G}_\sigma(\tau)$ *and* $s \in \mathbb{R}$*, then* $T(s)\varphi \in \mathscr{G}_{T(s)\sigma}(\tau - s)$*;*
(C4) *If* $\{\varphi_j\}_{j\in\mathbb{N}} \subset \mathscr{G}_\sigma(\tau)$ *and* $\varphi_j(\tau) \to z$*, there is a subsequence* $\{\varphi_{j_k}\}_{k\in\mathbb{N}}$ *of* $\{\varphi_j\}_{j\in\mathbb{N}}$ *and* $\varphi \in \mathscr{G}_\sigma(\tau)$ *with* $\varphi(\tau) = z$ *and such that* $\varphi_{j_k}(t) \to \varphi(t)$ *when* $k \to \infty$*, for each* $t \geq \tau$*.*

Remark 3.2. *Note that, if* $\mathscr{G}$ *is a generalized process, for* $\varphi \in \mathscr{G}_\sigma(\tau)$ *we have, if* $s \geq 0$*,* $T(s)(\varphi^{+s}) \in \mathscr{G}_{T(s)\sigma}(\tau)$ *and* $T(s)(\varphi^{+s}) = (T(s)\varphi)^{+s}$*.*

Indeed, on the one hand, for $s \geq 0$ *and* $t \geq \tau$ *we have*

$$(T(s)(\varphi^{+s}))(t) = (\varphi^{+s})(t + s) = \varphi_{|[\tau+s,+\infty)}(t + s) = \varphi(t + s),$$

and, on the other hand,

$$(T(s)\varphi)^{+s}(t) = (T(s)\varphi)_{|[\tau-s+s,+\infty)}(t) = \varphi_{|[\tau,+\infty)}(t+s) = \varphi(t+s).$$

Definition 3.3. *A generalized process $\mathscr{G} = \{\mathscr{G}_\sigma(\tau)\}_{\sigma\in\Sigma,\tau\in\mathbb{R}}$ is said to be* ***locally uniformly upper semicontinuous (LUUS)*** *if it satisfies the following condition:*

(C4') *If $\{\varphi_j\}_{j\in\mathbb{N}}$ is a sequence such that $\varphi_j \in \mathscr{G}_{\sigma_j}(\tau)$ and $\varphi_j(\tau) \to z$, then there are $\sigma \in \Sigma$ and $\varphi \in \mathscr{G}_\sigma(\tau)$ with $\varphi(\tau) = z$ and subsequences $\{\sigma_{j_k}\}_{k\in\mathbb{N}}$ and $\{\varphi_{j_k}\}_{k\in\mathbb{N}}$ such that $\sigma_{j_k} \to \sigma$ and $\varphi_{j_k} \to \varphi$ uniformly on compact subsets of $[\tau, +\infty)$ when $k \to \infty$.*

We say that a generalized process $\mathscr{G} = \{\mathscr{G}_\sigma(\tau)\}_{\sigma\in\Sigma,\tau\in\mathbb{R}}$ is ***exact*** *if it satisfies the following condition:*

(C5) *(Concatenation) let $\varphi \in \mathscr{G}_\sigma(\tau)$ and $\psi \in \mathscr{G}_\sigma(r)$ such that $\varphi(s) = \psi(s)$ for some $s \geq r \geq \tau$. If θ is defined by*

$$\theta(t) := \begin{cases} \varphi(t), & t \in [\tau, s], \\ \psi(t), & t \in (s, \infty), \end{cases}$$

then $\theta \in \mathscr{G}_\sigma(\tau)$.

From now on $\mathscr{G}$ will always denote a generalized process (which does not need to be exact or LUUS).

Definition 3.4. *We define the family of operators $\{H(t); t \geq 0\}$, where for each $t \geq 0$, $\sigma \in \Sigma$ and $\varphi \in \mathscr{G}_\sigma(\tau)$ it is given by*

$$H(t)\varphi := (T(t)\varphi)^{+t} \in \mathscr{G}_{T(t)\sigma}(\tau).$$

Definition 3.5. *Given $\mathscr{G}$ and $\tau \in \mathbb{R}$ we define the* ***trajectory space*** *(associated to $\mathscr{G}$) by*

$$\mathscr{G}_\Sigma(\tau) := \bigcup_{\sigma\in\Sigma} \mathscr{G}_\sigma(\tau).$$

In [9, 10, 11] and [12] the authors define the trajectory set, denoted by $\mathcal{K}_\Sigma^+$, which we identify here with $\mathscr{G}_\Sigma(0)$, i.e.,

$$\mathcal{K}_\Sigma^+ = \mathscr{G}_\Sigma(0).$$

In these previous works, despite dealing with nonautonomous problems, all trajectories are considered starting at zero, which means that there is an implicit restriction on the domain of each curve after being translated. In this work we prefer to make explicit both procedures, translation and restriction. By considering the operators H instead of the pure translation T (as it is usual in the study of trajectory attractors), we allow each curve to start at any $\tau \in \mathbb{R}$.

At the end of this section, we will briefly discuss the dependence of the initial instant on this theory, for the time being we will keep that term in our studies.

We clearly have the following:

Lemma 3.6. *Given $\tau \in \mathbb{R}$ we have*

$$H(t)\mathscr{G}_\Sigma(\tau) \subset \mathscr{G}_\Sigma(\tau).$$

Lemma 3.7. *The family of operators $\{H(t)\}_{t\geq 0}$ defines a semigroup acting on $\mathscr{G}_\Sigma(\tau)$.*

Proof: Indeed, given $\varphi \in \mathscr{G}_\Sigma(\tau)$, there exists $\sigma \in \Sigma$ such that $\varphi \in \mathscr{G}_\sigma(\tau)$ and

- $H(0)\varphi = \varphi$, then $H(0) = Id$;
- given $t_1, t_2 \geq 0$ we have

$$\begin{aligned} H(t_1)H(t_2)\varphi(t) &= H(t_1)(T(t_2)\varphi)^{+t_2}(t) = H(t_1)(T(t_2)\varphi)_{|[\tau-t_2+t_2,+\infty)}(t) \\ &= H(t_1)\varphi_{|[\tau-t_2+t_2,+\infty)}(t_2+t) = (T(t_1)\varphi_{|[\tau-t_2-t_1+t_2,+\infty)})^{+t_1}(t_2+t) \\ &= (T(t_1)\varphi_{|[\tau-t_2-t_1+t_2+t_1,+\infty)})(t_2+t) = \varphi_{|[\tau-t_2-t_1+t_2+t_1,+\infty)}(t_1+t_2+t) \\ &= T(t_1+t_2)\varphi_{|[\tau-t_2-t_1+t_2+t_1,+\infty)}(t) = (T(t_1+t_2)\varphi)^{t_1+t_2}(t) = H(t_1+t_2)\varphi(t), \end{aligned}$$

for all $t \geq \tau$.

■

Let $\mathcal{F}_\tau$ be a metric space and Θ_τ a Hausdorff and Fréchet-Urysohn space, such that (as a set) $\mathcal{F}_\tau \cap \Theta_\tau \neq \emptyset$. Suppose that

$$\mathscr{G}_\Sigma(\tau) \subset \mathcal{F}_\tau \cap \Theta_\tau.$$

We want to define an attractor in $\mathscr{G}_\Sigma(\tau) \cap \Theta_\tau$ for bounded trajectories in $\mathscr{G}_\Sigma(\tau) \cap \mathcal{F}_\tau$.

Definition 3.8. *Given $\tau \in \mathbb{R}$, a set $P \subset \Theta_\tau$ is a uniform attractor (w.r.t. $\sigma \in \Sigma$) of the trajectory space $\mathscr{G}_\Sigma(\tau)$ in the topology of Θ_τ if for all bounded $\mathbf{B} \subset \mathcal{F}_\tau$ such that $\mathbf{B} \subset \mathscr{G}_\Sigma(\tau)$, the set P attracts $\mathbf{B}$ in the topology of Θ_τ under $H(t)$ when $t \to +\infty$, i.e., for all neighborhoods $\mathcal{O}(P)$ in Θ_τ there exists $t_1 \geq 0$ such that $H(t)\mathbf{B} \subset \mathcal{O}(P)$ if $t \geq t_1$.*

Definition 3.9 (Uniform Trajectory Attractor). *A compact set $\mathcal{U}_\Sigma \subset \Theta_\tau$ is the uniform trajectory attractor (w.r.t. $\sigma \in \Sigma$) of semigroup $\{H(t)\}_{t\geq 0}$ in $\mathscr{G}_\Sigma(\tau)$ on the topology of Θ_τ if:*

i) *$\mathcal{U}_\Sigma$ is a uniform attractor;*
ii) *$H(t)\mathcal{U}_\Sigma = \mathcal{U}_\Sigma$ for all $t \geq 0$;*
iii) *$\mathcal{U}_\Sigma$ is the minimal closed set that is a uniform attractor.*

Definition 3.10. *We say that the trajectory space $\mathscr{G}_\Sigma(\tau)$ is (Θ_τ, Σ)-**closed** if the set $\bigcup_{\sigma\in\Sigma} \mathscr{G}_\sigma(\tau) \times \{\sigma\}$ is closed in the space $\Theta_\tau \times \Sigma$ with the usual product topology.*

Proposition 3.11 (Proposition 2.1, p. 262, [11]). *Let Σ a compact metric space and $\{\mathscr{G}_\sigma(\tau)\}_{\sigma\in\Sigma}$ is (Θ_τ, Σ)-closed. Then the set $\mathscr{G}_\Sigma(\tau)$ is closed in Θ_τ.*

Proposition 3.11 can be proved by using sequences, which justifies the assumption that Θ_τ is Fréchet-Urysohn. This assumption also appears in [10] and [11], where the authors ensure the continuity of the semigroup $\{H(t)\}_{t\geq 0}$ by using sequences.

Theorem 3.12. *Suppose that Σ is compact, $T(t) : \Sigma \to \Sigma$ is continuous for all $t \in \mathbb{R}$, the trajectory space $\mathscr{G}_\Sigma(\tau)$ is (Θ_τ, Σ)-closed and $H(t) : \Theta_\tau \to \Theta_\tau$ is continuous. If there is a uniform attractor P of $\mathscr{G}_\Sigma(\tau)$ in Θ_τ such that P is compact in Θ_τ and bounded in $\mathcal{F}_\tau$, then the semigroup $\{H(t), t \geq 0\}$ acting on $\mathscr{G}_\Sigma(\tau)$ possesses the uniform trajectory attractor $\mathcal{U}_\Sigma \subset \mathscr{G}_\Sigma(\tau) \cap P$. The set $\mathcal{U}_\Sigma$ is bounded in $\mathcal{F}_\tau$.*

The proof of the above theorem can be found in [10] and [11], where it is supposed that $\mathcal{F}_\tau$ is a Banach space and $\mathcal{F}_\tau \subset \Theta_\tau$. Nevertheless, the proof follows from similar arguments.

3.2. **On the global attractor.** Let X be a Banach space and consider, for each $\tau \in \mathbb{R}$, the set $\mathbb{R}_\tau = [\tau, +\infty)$ and, for each $\sigma \in \Sigma$, the set

$$\mathscr{G}_\sigma(\tau) \subset C(\mathbb{R}_\tau; X) \cap L^\infty(\mathbb{R}_\tau; X).$$

In this section we consider $\Theta_\tau = C_{loc}(\mathbb{R}_\tau; X)$ ($C_{loc}(\mathbb{R}_\tau; X)$ is the set $C(\mathbb{R}_\tau; X)$ with the local uniform convergence topology) and $\mathcal{F}_\tau = L^\infty(\mathbb{R}_\tau; X)$, which means we are looking for a set $\mathcal{U}_\Sigma \subset \mathscr{G}_\Sigma(\tau)$ bounded in $L^\infty(\mathbb{R}_\tau; X)$, compact in $C_{loc}(\mathbb{R}_\tau; X)$, invariant with respect to the semigroup $\{H(t)\}_{t\geq 0}$ and such that, for all $\mathbf{B} \subset \mathscr{G}_\Sigma(\tau)$, $\mathbf{B}$ bounded in $L^\infty(\mathbb{R}_\tau; X)$, and for all $M \geq \tau$, we have

$$dist_{C([\tau,M];X)}\left(\Pi_{[\tau,M]}H(t)\mathbf{B}, \Pi_{[\tau,M]}\mathcal{U}_\Sigma\right) \to 0, \quad \text{when } t \to +\infty,$$

where $\Pi_{[\tau,M]}f := f_{|[\tau,M]}$ for $f \in C_{loc}(\mathbb{R}_\tau; X)$.

Note that, given $M \geq \tau$ and $0 \leq t \leq M - \tau$, if $f \in C([\tau, M]; X)$ then

$$H(t)f(s) = (T(t))^{+t}f(s) = T(t)f_{|[\tau+t,M]}(s) = f_{|[\tau+t,M]}(t+s),$$

therefore, we can consider the continuous mappings

$$H(t) : C([\tau, M]; X) \to C([\tau, M - t]; X), \quad \forall\ 0 \leq t \leq M - \tau.$$

Proposition 3.13 (Proposition 1.3, p. 222, [11])**.** *The semigroup $\{H(t)\}_{t\geq 0}$ is continuous in $C_{loc}(\mathbb{R}_\tau; X)$.*

From the above proposition and Theorem 3.12 we have the following:

Theorem 3.14. *Suppose that $\Sigma \subset \Xi$ is compact, $T(t) : \Sigma \to \Sigma$ continuous for all $t \in \mathbb{R}$, and the trajectory space $\mathscr{G}_\Sigma(\tau)$ is $(C_{loc}(\mathbb{R}_\tau; X), \Sigma)$-closed. If there exists a uniform attractor P for $\mathscr{G}_\Sigma(\tau)$ in $C_{loc}(\mathbb{R}_\tau; X)$ such that P is compact in $C_{loc}(\mathbb{R}_\tau; X)$ and bounded in $L^\infty(\mathbb{R}_\tau, X)$, then the semigroup $\{H(t); t \geq 0\}$ acting in $\mathscr{G}_\Sigma(\tau)$ possesses the uniform trajectory attractor.*

Proposition 3.15. *If the generalized process $\mathscr{G}$ is LUUS, then $\mathscr{G}_\Sigma(\tau)$ is $(C_{loc}(\mathbb{R}_\tau; X), \Sigma)$-closed.*

Proof: Let $\{\sigma_n\}_{n\in\mathbb{N}} \subset \Sigma$, $\varphi_n \in \mathscr{G}_{\sigma_n}(\tau)$, $\sigma \in \Sigma$ and $\varphi \in \mathscr{G}_\Sigma(\tau)$ such that

$$\varphi_n \to \varphi \text{ in } C_{loc}(\mathbb{R}_\tau; X) \text{ and } \sigma_n \to \sigma \text{ in } \Sigma.$$

In particular $\varphi_n(\tau) \to \varphi(\tau)$ in X. Since $\mathscr{G}$ is LUUS there are $\tilde{\sigma} \in \Sigma$ and $\tilde{\varphi} \in \mathscr{G}_{\tilde{\sigma}}(\tau)$ and subsequences, which we do not relabel, such that $\tilde{\varphi}(\tau) = \varphi(\tau)$, $\varphi_n \to \tilde{\varphi}$ in compact subsets of $[\tau, +\infty)$ and $\sigma_n \to \tilde{\sigma}$.

Therefore, $\tilde{\sigma} = \sigma$ and $\tilde{\varphi} = \varphi$ which ensures that $\varphi \in \mathscr{G}_\sigma(\tau)$.

■

Proposition 3.15 and Theorem 3.14 guarantee the next result.

Theorem 3.16. *Let Σ be compact and $T(t) : \Sigma \to \Sigma$ continuous for all $t \in \mathbb{R}$. Suppose that the generalized process $\mathscr{G}$ is LUUS. If there is a uniform attractor P for $\mathscr{G}_\Sigma(\tau)$ in $C_{loc}(\mathbb{R}_\tau; X)$ such that P is compact in $C_{loc}(\mathbb{R}_\tau; X)$ and bounded in $L^\infty(\mathbb{R}_\tau, X)$, then the semigroup $\{H(t); t \geq 0\}$ acting in $\mathscr{G}_\Sigma(\tau)$ possesses the uniform trajectory attractor $\mathcal{U}_\Sigma$.*

Next we will show how to obtain a set containing the asymptotic concentrations of the system in the phase space X by using a trajectory attractor, as in [10, 11] and [12].

For all $\mathbf{B} \subset \mathscr{G}_\Sigma(\tau)$ we define the sections

$$\mathbf{B}(t) := \{\varphi(t); \varphi \in \mathbf{B}\},\ \forall\ t \geq \tau$$

and

$$\mathcal{U}_\Sigma(t) := \{\varphi(t); \varphi \in \mathcal{U}_\Sigma\},\ \forall\ t \geq \tau.$$

Definition 3.17. *A set $\mathcal{A}_\Sigma \subset X$ is a uniform global attractor (w.r.t. $\sigma \in \Sigma$) in X for the generalized process $\mathscr{G}$ if*

i) *the set $\mathcal{A}_\Sigma$ is compact in X;*
ii) *for all $\mathbf{B} \subset \mathscr{G}_\Sigma(\tau)$ bounded in $L^\infty(\mathbb{R}_\tau, X)$ we have*

$$dist_X(\mathbf{B}(t), \mathcal{A}_\Sigma) \to 0 \ (t \to +\infty);$$

iii) *$\mathcal{A}_\Sigma$ is the minimal closed set satisfying ii).*

Corollary 3.18. *Under the hypotheses of Theorem 3.14, the set*

$$\mathcal{A}_\Sigma = \mathcal{U}_\Sigma(\tau)$$

is the uniform global attractor of $\mathscr{G}$ in X.

Proof: As $\mathcal{U}_\Sigma$ is compact in $C_{loc}(\mathbb{R}_\tau, X)$, we have $\mathcal{U}_\Sigma(\tau)$ is compact in X.

Clearly $\mathcal{U}_\Sigma(\tau)$ attracts bounded subsets of $\mathscr{G}_\Sigma(\tau)$.

In order to show the minimality, suppose that there exists U_1 closed satisfying ii) in Definition 3.17. As $\mathcal{U}_\Sigma \subset \mathscr{G}_\Sigma(\tau)$ and it is bounded in $L^\infty(\mathbb{R}_\tau, X)$ we have

$$dist_X(\mathcal{U}_\Sigma(t), U_1) \to 0 \ (t \to +\infty).$$

But, if $t \geq \tau$, $\mathcal{U}_\Sigma(t) = \mathcal{U}_\Sigma(\tau)$, since $\mathcal{U}_\Sigma$ is invariant by $H(t)$. Indeed, if $t \geq \tau$, let $s \geq 0$ such that $\tau + s = t$, we have

$$\mathcal{U}_\Sigma(t) = \mathcal{U}_\Sigma(\tau + s) = \{\varphi(\tau + s); \varphi \in \mathcal{U}_\Sigma\} = (H(s)\mathcal{U}_\Sigma)(\tau) = \mathcal{U}_\Sigma(\tau). \tag{3.1}$$

Therefore, $\mathcal{U}_\Sigma(\tau) \subset U_1$.

■

Although the proof of Corollary 3.18 can be found in [11], we reproduce it here to emphasize property (3.1), which will be used later.

Remark 3.19. *Note that Corollary 3.18 remains valid if, instead of asking $\mathscr{G}_\Sigma(\tau)$ to be $(C_{loc}(\mathbb{R}_\tau; X), \Sigma)$-closed, we impose that $\mathscr{G}$ is an LUUS generalized process.*

3.3. **Uniform Attractor for Multivalued Semiprocess.** In [29], in order to deal with nonautonoumous evolution problems without uniqueness of solution, the authors have developed a theory involving a family of set-valued operators which they called multivalued process. Next we briefly present the main elements of multivalued process theory and then we describe how to connect multivalued processes and generalized processes in such way that, by joining both theories, we can effectively "construct" a global attractor for a nonautonomous evolution problem without uniqueness of solution by using the sections of a uniform trajectory attractor.

Definition 3.20. *Given $\tau \in \mathbb{R}$ and $t \geq \tau$, a family of multivalued operators $U(t,\tau) : X \to P(X)$ is a* ***multivalued process*** *if:*

(1) *$U(\tau, \tau) = Id_X$ for all $\tau \in \mathbb{R}$;*
(2) *$U(t,\tau)x \subset U(t,s)U(s,\tau)x$, $\forall\ t \geq s \geq \tau$ and $x \in X$.*

We say that a multivalued process is ***exact*** *if:*

(2') *$U(t,\tau)x = U(t,s)U(s,\tau)x$, $\forall\ t \geq s \geq \tau$ and $x \in X$.*

Now consider a family of multivalued processes $\{U_\sigma; \sigma \in \Sigma\}$. Note that by setting

$$\mathbb{U}_\Sigma(t,\tau)x := \bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)x, \tag{3.2}$$

the family $\{\mathbb{U}_\Sigma(t,\tau)\}_{t\geq\tau,\tau\in\mathbb{R}}$ is itself a multivalued process. Following [29], we define a uniform attractor in X for $\{U_\sigma\}_{\sigma\in\Sigma}$:

Definition 3.21. *The set $\mathbb{O}_\Sigma$ is the uniform global attractor for the family of multivalued processes $\{U_\sigma\}_{\sigma\in\Sigma}$ in X if:*

(1) *for all B bounded in X and $\tau \in \mathbb{R}$*

$$\lim_{t\to+\infty} dist_X\left(\mathbb{U}_\Sigma(t,\tau)B, \mathbb{O}_\Sigma\right) = 0;$$

(2) *$\mathbb{O}_\Sigma \subset \mathbb{U}_\Sigma(t,\tau)\mathbb{O}_\Sigma$, $\forall\, t \geq \tau$ (negative invariance);*

(3) *for all closed Y satisfying item (1), $\mathbb{O}_\Sigma \subset Y$.*

Remark 3.22. *Let $\mathscr{G}$ be a generalized process, for each $\sigma \in \Sigma$ fixed, we have a multivalued process associated with $\mathscr{G}_\sigma = \cup_{\tau\in\mathbb{R}}\mathscr{G}_\sigma(\tau)$ which is given by*

$$U_\sigma(t,\tau)D := \{\varphi(t); \varphi \in \mathscr{G}_\sigma(\tau),\ \textit{with}\ \varphi(\tau) \in D\},\ \tau \in \mathbb{R},\ t \geq \tau,$$

for each $D \subset X$. The family $\{U_\sigma(t,\tau)\}_{t\geq\tau,\tau\in\mathbb{R}}$ defines a multivalued process according to the Definition 3.20 (see [32, 34]).

According to Remark 3.22, a generalized process $\mathscr{G}$ generates a family of multivalued processes $\{U_\sigma\}_{\sigma\in\Sigma}$, which in turn generates another multivalued process $\mathbb{U}_\Sigma$ as in (3.2). Next result guarantees the existence of a uniform global attractor for a family of multivalued processes associated to a generalized process.

Theorem 3.23. *Under the hypotheses of Theorem 3.14 if, for any bounded $B \subset X$, the set*

$$\mathbb{B} = \{\varphi \in \mathscr{G}_\Sigma(\tau); \varphi(\tau) \in B\}$$

is bounded in $L^\infty(\mathbb{R}_\tau; X)$, then there exists the uniform global attractor $\mathbb{O}_\Sigma$ in X and $\mathbb{O}_\Sigma = \mathscr{U}_\Sigma(\tau)$. Furthermore, $\mathbb{O}_\Sigma$ is compact in X.

Proof: Note that,

$$\mathbb{B}(t) = \mathbb{U}_\Sigma(t,\tau)B,$$

therefore, from Corollary 3.18, the set $\mathscr{U}_\Sigma(\tau)$ is the minimal closed set that attracts bounded sets of X.

It remains to prove the negative invariance. Note that, for all $s \geq 0$, we have

$$H(s)\mathscr{U}_\Sigma = \left\{\varphi_{|[\tau,+\infty)}(s+\cdot); \varphi \in \mathscr{U}_\Sigma\right\}.$$

Therefore, for all $t \geq \tau$, take $s \geq 0$ such that $s + \tau = t$,

$$\begin{aligned}\mathscr{U}_\Sigma(\tau) = \left(H(s)\mathscr{U}_\Sigma\right)(\tau) &= \left\{\varphi_{|[\tau,+\infty)}(s+\tau); \varphi \in \mathscr{U}_\Sigma\right\} = \left\{\varphi_{|[\tau,+\infty)}(t); \varphi \in \mathscr{U}_\Sigma\right\} \\ &\subset \{\varphi(t); \varphi \in \mathscr{G}_\Sigma(\tau) \text{ and } \varphi(\tau) \in \mathscr{U}_\Sigma(\tau)\} = \mathbb{U}_\Sigma(t,\tau)\mathscr{U}_\Sigma(\tau).\end{aligned} \tag{3.3}$$

As $t \geq \tau$ is arbitrary, the theorem is proved.

■

In [21] the authors have proved this result in autonomous case, in [10] and [11] there are similar results by assuming uniqueness of solutions. In [12] the authors mention the above result but they omit the proof.

Remark 3.24. *Note that in the above theorem we can also impose that $\mathscr{G}$ is an LUUS generalized process instead of asking $\mathscr{G}_\Sigma(\tau)$ to be $(C_{loc}(\mathbb{R}_\tau; X), \Sigma)$-closed. See Remark 3.19.*

3.4. **On the Invariance of the Uniform Global Attractor.** In what follows we roughly describe some results of [29] in order to ensure that the uniform global attractor for the family of multivalued process $\mathbb{U}_\Sigma$ associated with an exact generalized process $\mathscr{G}$ is invariant. We carry this out by joining the multivalued process and the generalized process theory.

In this section we are assuming that $\{U_\sigma\}_{\sigma\in\Sigma}$ is a family of multivalued processes (not necessarily generated by a generalized process) with the following property

$$U_\sigma(t,\tau)x = U_{T(h)\sigma}(t-h,\tau-h)x, \forall\ h \in \mathbb{R} \text{ and } x \in X, \tag{3.4}$$

for each $\sigma \in \Sigma$.

Definition 3.25. *The family $\{U_\sigma\}_{\sigma\in\Sigma}$ is **uniformly asymptotically upper semicompact (UAUS)** if given $\tau \in \mathbb{R}$ and a bounded set $B \subset X$ for which there exists $T = T(B,\tau)$ such that the set*

$$\bigcup_{t\geq T} \mathbb{U}_\Sigma(t,\tau)B$$

is bounded in X, any sequence $\{\xi_n\}_{n\in\mathbb{N}}$ with $\xi_n \in U_{\sigma_n}(t_n,\tau)B$, $\sigma_n \in \Sigma$ and $t_n \to +\infty$ is a precompact sequence in X.

Definition 3.26. *A family of multivalued processes $\{U_\sigma\}_{\sigma\in\Sigma}$ is **pointwise dissipative** if there exists B_0 bounded in X such that*

$$dist_X\left(\mathbb{U}_\Sigma(t,\tau)x, B_0\right) \to 0 \text{ when } t \to +\infty.$$

Theorem 3.27. *(See Theorems 1 and 2, [29]) Let Σ be compact and $T(h) : \Sigma \to \Sigma$ continuous, for all $h \in \mathbb{R}_+$. Suppose that for all $B \subset X$ bounded and $\tau \in \mathbb{R}$ there exists $T = T(B,\tau)$ such that the set*

$$\bigcup_{t\geq T} \mathbb{U}_\Sigma(t,\tau)B$$

is bounded in X. If

- (i) *the family $\{U_\sigma\}_{\sigma\in\Sigma}$ is UAUS;*
- (ii) *for all $t \geq \tau$ the application $X \times \Sigma \ni (x,\sigma) \mapsto U_\sigma(t,\tau)x$ has closed values in X and closed graph,*

then the family $\{U_\sigma\}$ possesses the uniform global attractor $\mathbb{O}_\Sigma \neq X$.

Furthermore, if instead of assuming (i) and (ii) we suppose

- (i') *the family $\{U_\sigma\}_{\sigma\in\Sigma}$ is UAUS and pointwise dissipative;*
- (ii') *for all $t \geq \tau$ the mapping $X \times \Sigma \ni (x,\sigma) \mapsto U_\sigma(t,\tau)x$ has closed values in X and is upper semicontinuous,*

then the family $\{U_\sigma\}$ possesses the uniform global attractor $\mathbb{O}_\Sigma \neq X$ and $\mathbb{O}_\Sigma$ is compact.

In the proofs of the previous results in [29], the authors define the multivalued operator $G : \mathbb{R}_+ \times X \times \Sigma \to \mathcal{P}(X \times \Sigma)$ by

$$G(t,(x,\sigma)) = (U_\sigma(t,0)x, T(t)\sigma)\,.$$

They ensure that the operator G is a multivalued semiflow (in the sense of [28]) which possesses a global attractor $\mathcal{R}$, such that

- $\mathcal{R} \neq X \times \Sigma$;
- $\mathcal{R} \subset G(t, \mathcal{R})$, $\forall\, t \in \mathbb{R}_+$;
- for all $C \subset X \times \Sigma$ bounded, we have
$$dist_{X\times\Sigma}(G(t, C), \mathcal{R}) \to 0,$$
when $t \to +\infty$;
- for all P closed satisfying the above property we have that $\mathcal{R} \subset P$.

It is a consequence of the proof that
$$\mathcal{R} = \mathbb{O}_\Sigma \times \omega(\Sigma),$$
where $\mathbb{O}_\Sigma$ is the uniform global attractor and $\omega(\Sigma)$ is the ω-limit of Σ under $\{T(h)\}$.

In this work we make minor changes on the above approach, allowing $\tau \in \mathbb{R}$ and $G : \mathbb{R}_+ \times X \times \Sigma \to \mathcal{P}(X)$ given by
$$G(t,(x,\sigma)) = (U_\sigma(t+\tau,\tau)x, T(t)\sigma)\,. \tag{3.5}$$
By slightly adapting the proof we can ensure Theorem 3.27.

Remark that the invariance of $\mathcal{R}$ is a consequence of the exactness of the multivalued semiflow G, which means the following property
$$G(t_1 + t_2, \xi) = G(t_1, G(t_2, \xi)),\ \forall\ t_1, t_2 \in \mathbb{R}_+,$$
and the compactness of $\mathcal{R}$ guarantees that
$$G(t, \mathcal{R}) = \mathcal{R},\ \forall\ t \in \mathbb{R}_+. \tag{3.6}$$
(see [28], Remark 8.)

Lemma 3.28. *Let $\{U_\sigma\}_{\sigma\in\Sigma}$ be a family of multivalued processes such that, for each $\sigma \in \Sigma$, the multivalued process U_σ is exact. The operator G defined in (3.5) is an exact semiflow, in other words,*

(1) $G(0, \cdot) = Id_X$;
(2) $G(t_1 + t_2, \cdot) = G(t_1, G(t_2, \cdot)),\ \forall\ t_1, t_2 \geq 0.$

Proof: Item (1) was proved in [29].

In order to show item (2), let $t_1, t_2 \geq 0$ and $(x, \sigma) \in X \times \Sigma$, and then we have
$$\begin{aligned} G(t_1+t_2,(x,\sigma)) &= (U_\sigma(t_1+t_2+\tau,\tau)x, T(t_1+t_2)\sigma) \\ &= (U_\sigma(t_1+t_2+\tau,t_2+\tau)U_\sigma(t_2+\tau,\tau)x, T(t_1)T(t_2)\sigma), \\ &= (U_{T(t_2)\sigma}(t_1+\tau,\tau)U_\sigma(t_2+\tau,\tau)x, T(t_1)T(t_2)\sigma) \\ &= G(t_1,(U_\sigma(t_2+\tau,\tau)x, T(t_2)\sigma)) \\ &= G(t_1, G(t_2,(x,\sigma))) \end{aligned}$$
from property (3.4).

■

We refer the reader to Proposition A2.1 of [37] for a proof of Lemma 3.28, similar to this one, under the hypothesis of uniqueness of solution.

Theorem 3.29. *Let $\tau \in \mathbb{R}$ and suppose that $\{U_\sigma\}_{\sigma\in\Sigma}$ is a family of exact multivalued processes and for all $B \subset X$ bounded and $\tau \in \mathbb{R}$ there exists $T = T(B,\tau)$ such that the set*

$$\bigcup_{t\geq T} \mathbb{U}_\Sigma(t,\tau)B$$

is bounded in X and $\{U_\sigma\}_{\sigma\in\Sigma}$ satisfies Hypotheses (i') and (ii') of Theorem 3.27. Assume further that Σ is compact, $T(h) : \Sigma \to \Sigma$ is continuous for all $h \in \mathbb{R}$. Then there is the uniform global attractor $\mathbb{O}_\Sigma$ which is compact and invariant, i.e.

$$\mathbb{U}_\Sigma(t,\tau)\mathbb{O}_\Sigma = \mathbb{O}_\Sigma, \ \forall t \geq \tau.$$

Proof: Theorem 3.27 ensures the existence of the compact uniform global attractor $\mathbb{O}_\Sigma$. Furthermore, the global attractor of the generalized semiflow G is characterized by

$$\mathcal{R} = \mathbb{O}_\Sigma \times \omega(\Sigma).$$

Note that, as $T(h)\Sigma = \Sigma$ for all $h \in \mathbb{R}$, we have

$$\omega(\Sigma) = \Sigma.$$

Since U_σ is an exact multivalued process for each $\sigma \in \Sigma$, it follows from Lemma 3.28 and (3.6) that

$$G(t,\mathcal{R}) = \mathcal{R} \ \ \forall t \geq 0.$$

We also have that

$$G(t,\mathbb{O}_\Sigma \times \Sigma) = \bigcup_{\sigma\in\Sigma} \left(U_\sigma(t+\tau,\tau)\mathbb{O}_\Sigma, T(t)\sigma\right),$$

therefore,

$$\mathbb{O}_\Sigma = \Pi_1\mathcal{R} = \Pi_1 G(t,\mathcal{R}) = \Pi_1 G(t,\mathbb{O}_\Sigma \times \Sigma) = \bigcup_{\sigma\in\Sigma} U_\sigma(t+\tau,\tau)\mathbb{O}_\Sigma = \mathbb{U}_\Sigma(t,\tau)\mathbb{O}_\Sigma.$$

■

Analogous results can be found in [22] and [23], where the approach is based on the characterization of the uniform global attractor as a union of ω-limit sets.

Our aim is to prove the existence of an invariant attractor by using a trajectory attractor generated by an exact generalized process. Thus, we have defined different dynamical systems which are somehow connected. It is worth organizing them now.

1. The generalized process $\mathscr{G} = \{\mathscr{G}_\sigma(\tau)\}_{\sigma\in\Sigma,\tau\in\mathbb{R}}$ is a family of sets of trajectories in X (Definition 3.1).
   - (a) Each set $\mathscr{G}_\sigma(\tau)$ contains the trajectories starting at $\tau \in \mathbb{R}$ which corresponds to a specific index σ.
   - (b) $\mathscr{G}_\Sigma(\tau) = \cup_{\sigma\in\Sigma}\mathscr{G}_\sigma(\tau)$ (Definition 3.5).
   - (c) We also use the notation $\mathscr{G}_\sigma = \cup_{\tau\in\mathbb{R}}\mathscr{G}_\sigma(\tau)$.
2. $\{H(\cdot)\}$ is a semigroup acting on $\mathscr{G}_\Sigma(\tau)$ and its attractor, a trajectory attractor, is denoted by $\mathcal{U}_\Sigma$ (Definitions 3.4 and 3.9.)
3. The projection $\mathcal{U}_\Sigma(\tau)$ of $\mathcal{U}_\Sigma$ on the phase space X is a uniform attractor for $\mathscr{G}$ and we denote it by $\mathcal{A}_\Sigma$. Then we have $\mathcal{A}_\Sigma = \mathcal{U}_\Sigma(\tau)$ (Definition 3.17 and Corollary 3.18).
4. For each $\sigma \in \Sigma$ we associate a multivalued process $\{U_\sigma(t,\tau)\}$ with $\mathscr{G}_\sigma$,
5. and a "larger" multivalued process is associated to the whole set $\mathscr{G}$ by setting $\mathbb{U}_\Sigma(t,\tau) = \cup_{\sigma\in\Sigma}U_\sigma(t,\tau)$ (Remark 3.22 and (3.2)).

(6) We denote by $\mathbb{O}_\Sigma$ the uniform global attractor for $\mathbb{U}_\Sigma(t,\tau)$ (Definition 3.21).

Now, note that also Σ has its own dynamics determined by translations operators $\{T(h)\}$, then we can define:

(7) $G(t,(x,\sigma)) = (U_\sigma(t+\tau,\tau)x, T(t)\sigma)$ which is a multivalued semiflow (3.5).
(8) Under appropriate conditions, there is a global attractor $\mathcal{R} \subset X \times \Sigma$ for $G(t,(x,\sigma))$ and $\mathcal{R} = \mathbb{O}_\Sigma \times \omega(\Sigma)$, where $\omega(\Sigma)$ is the ω-limit set of Σ under $T(\cdot)$.
(9) We are going to prove that $\mathbb{O}_\Sigma = \mathcal{A}_\Sigma$ and the invariance of this attractor.

Firstly we prove that the family $\{U_\sigma;\sigma\in\Sigma\}$ associated with $\mathscr{G}$ satisfies (3.4).

Lemma 3.30. *The family of multivalued process $\{U_\sigma;\sigma\in\Sigma\}$ associated with a generalized process $\mathscr{G}$ satisfies*

$$U_\sigma(t,\tau)x = U_{T(h)\sigma}(t-h,\tau-h)x, \forall\ h\in\mathbb{R} \text{ and } x\in X$$

Proof: Indeed, we have that

$$U_{T(h)\sigma}(t-h,\tau-h)x \subset U_\sigma(t,\tau)x,$$

for all $h\in\mathbb{R}$ and $x\in X$, see [5]. In other hand, using the inclusion above, we have

$$U_\sigma(t,\tau)x = U_{T(h-h)\sigma}(t-(h-h),\tau-(h-h))x \subset U_{T(h)\sigma}(t-h,\tau-h)x.$$

for all $h\in\mathbb{R}$ and $x\in X$.

■

In the previous result, as throughout all this work, we are considering generalized processes rather than multivalued processes, in such way we take into account the properties of each trajectory, and by imposing suitable conditions on such trajectories we can conclude the desired properties for the generalized process, and also for both the multivalued process and the multivalued semiflow associated with these trajectories. For instance, the exactness of the generalized process $\mathscr{G}$ implies the exactness of each multivalued process U_σ associated with $\mathscr{G}$, see [32] and [34], which ensures the exactness of multivalued process $\mathbb{U}_\Sigma$ defined in (3.2) and the exactness of multivalued semiflow $G(t,(x,\sigma))$ associated with $\mathscr{G}$, from Lemma 3.28. Another example is that, for an LUUS generalized process, the trajectory space $\mathscr{G}_\Sigma(\tau)$ is closed, see Propositions 3.15 and 3.11.

One of the advantages of dealing with generalized processes is that we can obtain such exactness through the concatenation property of the curves, see [34]. These facts will be used in our next result, where we prove that if $\mathscr{G}$ is an exact and LUUS generalized process, then the family of multivalued process $\{U_\sigma\}_{\sigma\in\Sigma}$ associated with $\mathscr{G}$ possesses the compact invariant uniform global attractor.

Theorem 3.31. *Let X be an infinite dimensional Banach space, Σ a compact metric space and $T(h):\Sigma\to\Sigma$ continuous for all $h\in\mathbb{R}$. Suppose that $\mathscr{G}$ is an exact and LUUS generalized process. Suppose also that, for all $B\in X$ bounded we have that the set*

$$\mathbb{B} = \{\varphi;\varphi\in\mathscr{G}_\Sigma(\tau) \text{ and } \varphi(\tau)\in B\}$$

is a bounded set in $L^\infty(\mathbb{R}_\tau;X)$. If there is a uniform trajectory attractor P of $\mathscr{G}_\Sigma(\tau)$ in $C_{loc}(\mathbb{R}_\tau;X)$ such that P is compact in $C_{loc}(\mathbb{R}_\tau;X)$ and bounded in $L^\infty(\mathbb{R}_\tau;X)$, then there is the compact uniform global attractor $\mathcal{A}_\Sigma$ which in addition is invariant, i.e.

$$\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)\mathcal{A}_\Sigma = \mathcal{A}_\Sigma\ \forall\ t\geq\tau.$$

Proof: It follows from Corollary 3.18 that if there is an attractor P as described above, then there exists the compact uniform global attractor $\mathcal{A}_\Sigma = \mathcal{U}_\Sigma(\tau)$.

We first show that the family $\{U_\sigma\}_{\sigma\in\Sigma}$ of multivalued processes associated with the generalized process $\mathscr{G}$ has the following property: for any bounded $B \subset X$, there is $T = T(B,\tau)$, such that the set

$$\bigcup_{\sigma\in\Sigma}\bigcup_{t\geq T} U_\sigma(t,\tau)B$$

is bounded in X and also prove that $\{U_\sigma\}_{\sigma\in\Sigma}$ is a UAUS family.

Let B a bounded set of X, the set $\mathbb{B}$ associated with B is bounded in $L^\infty(\mathbb{R}_\tau; X)$, and then there exists $R > 0$ such that for each $\varphi \in \mathbb{B}$ we have that $\|\varphi(t)\|_X \leq R$ a.e. in $\mathbb{R}_\tau$. However, $\mathbb{B} \subset \mathscr{G}_\Sigma(\tau)$, which is composed by continuous functions, thus $\|\varphi(t)\|_X \leq R$ for all $t \in \mathbb{R}_\tau$. Therefore, the set $\mathbb{B}(t)$ is bounded in X uniformly for $t \in \mathbb{R}_\tau$. Then, for any $T > \tau$, the set

$$\bigcup_{\sigma\in\Sigma}\bigcup_{t\geq T} U_\sigma(t,\tau)B = \bigcup_{t\geq T}\mathbb{B}(t)$$

is bounded in X.

Now, let $\{\xi_n\}_{n\in\mathbb{N}} \subset X$ be a sequence such that, for each $n \in \mathbb{N}$, $\xi_n \in U_{\sigma_n}(t_n,\tau)B$, $\sigma_n \in \Sigma$ and $t_n \to +\infty$.

It follows from the compactness of the set $\mathcal{A}_\Sigma$ that $\{\xi_n\}_{n\in\mathbb{N}}$ possesses a convergent subsequence.

Note that the set $U_\sigma(t,\tau)x$ is always closed for all $x \in X$, see Proposition 12.2 of [34], and as the generalized process $\mathscr{G}$ is LUUS, the mapping $X \times \Sigma \ni (x,\sigma) \to U_\sigma(t,\tau)x$ is upper semicontinuous. The pointwise dissipativity follows from the existence of the compact attractor $\mathcal{A}_\Sigma$ in X.

These properties show that the hypotheses of Theorem 3.29 are satisfied, what ensures that there exists the compact uniform global attractor $\mathbb{O}_\Sigma$ which is invariant. From minimality of attractors we have

$$\mathcal{A}_\Sigma = \mathbb{O}_\Sigma$$

and then the result follows. ■

3.5. **Independence of the Attractors of Initial Times.** According to [11] the uniform trajectory attractor $\mathcal{U}_\Sigma$ is characterized by

$$\mathcal{U}_\Sigma = \Pi_\tau\mathcal{K}_\Sigma = \left\{\varphi_{|[\tau,+\infty)};\ \varphi \in \mathcal{K}_\Sigma\right\}, \tag{3.7}$$

where $\mathcal{K}_\Sigma$ is the set of all complete trajectories bounded in $\mathcal{F}$, in other words, this set is composed by functions $\varphi : \mathbb{R} \to X$ such that for each $\tau \in \mathbb{R}$ we have $\varphi_{|[\tau,+\infty)} \in \mathscr{G}_\Sigma(\tau) \subset \Theta_\tau \cap \mathcal{F}_\tau$, and the spaces Θ_τ and $\mathcal{F}_\tau$ contain only the restrictions to $[\tau,+\infty)$ of curves in more general spaces Θ and $\mathcal{F}$, then $\mathcal{K}_\Sigma \subset \Theta \cap \mathcal{F}$.

Note that,

$$\mathcal{U}_\Sigma(\tau) = \mathcal{K}_\Sigma(\tau).$$

With the hypotheses of Theorem 3.31 we can rewrite expression (3.3) as

$$\mathcal{U}_\Sigma(\tau) = \mathcal{A}_\Sigma = \mathbb{U}_\Sigma(t,\tau)\mathcal{A}_\Sigma = \mathbb{U}_\Sigma(t,\tau)\mathcal{U}_\Sigma(\tau).$$

The characterization (3.7) and the information above show that, for all $\tau \in \mathbb{R}$ and $t \geq \tau$,

$$\left\{\varphi_{|[\tau,+\infty)}(\tau); \varphi \in \mathcal{K}_\Sigma\right\} = \mathcal{U}_\Sigma(\tau) = \mathbb{U}_\Sigma(t,\tau)\mathcal{U}_\Sigma(\tau) = \{\phi(t); \phi \in \mathcal{K}_\Sigma \text{ and } \phi(\tau) \in \mathcal{U}_\Sigma(\tau)\},$$

and then

$$\mathcal{K}_\Sigma(\tau) = \{\varphi(\tau); \varphi \in \mathcal{K}_\Sigma\} = \{\phi(t); \phi \in \mathcal{K}_\Sigma \text{ and } \phi(\tau) \in \mathcal{U}_\Sigma(\tau)\} = \mathcal{K}_\Sigma(t).$$

Therefore, although in (3.1) the trajectory attractor was considered starting at a specific time, now we have an independence of the initial time for the uniform trajectory attractor, which means that the uniform trajectory attractor $\mathcal{U}_\Sigma$ is the same for all restrictions of the curves in $\Theta \cap \mathcal{F}$ in the initial time τ.

Also, according to [11, Section IV.6], there is a uniformity with respect to initial time in the global attractors, such property which is also attributed to the trajectory attractor, since the global attractor can be seen as a section of the trajectory attractor.

Another interesting fact that motivates this independence was found in [29] where the authors proved that, for each $B \subset X$ bounded, we have

$$\omega_{\tau,\Sigma}(B) \subset \omega_{0,\Sigma}(B),\ \ \tau \geq 0,$$

where the ω-limit set $\omega_{\tau,\Sigma}$ was defined by, for $B \subset X$ and $\tau \in \mathbb{R}$,

$$\omega_{\tau,\Sigma}(B) = \bigcap_{t\geq\tau} \overline{\bigcup_{\sigma\in\Sigma}\bigcup_{s\geq t} U_\sigma(s,\tau)B} = \bigcap_{t\geq\tau} \overline{\bigcup_{s\geq t} \mathbb{U}_\Sigma(s,\tau)B}.$$

And then, they proved that the uniform global attractor is characterized by

$$\Theta_\Sigma = \bigcup_{\{B\in\mathcal{B}(X)\}} \omega_{0,\Sigma}(B),$$

Notice the independence of the attractor of the initial time. Despite they work with positive initial time, this independence for attractor makes us think about the dependence of trajectory attractors of initial times in our more general context.

4. Trajectory and Global Attractors for The Problem (P)

In this section we will prove the existence of the trajectory attractor for Problem (P) and guarantee the existence of the invariant uniform global attractor. The following definition shows how we have to understand the idea of weak solution to Problem (P).

Definition 4.1 (Weak Solution to Problem (P))**.** *Given $U_\tau = (u_\tau, v_\tau) \in \mathbb{X}^2$, $\tau \in \mathbb{R}$, the couple $U(t) = (u(t), v(t))$ is a weak solution to Problem (P) if $v(t) = \gamma(u(t))$ a.e. in (τ, T) for each $T > \tau$, and U satisfies*

(i)
$$\begin{cases} U \in C([\tau,+\infty); \mathbb{X}^2) \cap L^\infty(\tau,+\infty; \mathbb{X}^2); \\ U \in L^p_{loc}(\tau,+\infty; \mathbb{V}^p); \end{cases}$$

(ii)
$$\partial_t U \in L^s_{loc}(\tau,+\infty; (\mathbb{V}^p)^*),\ \ s = min(r_1', r_2', p');$$

(iii) *for all $\Psi = (\varphi, \gamma(\varphi)) \in \mathbb{V}^p$,*
$$\begin{aligned} \langle \partial_t U, \Psi\rangle_{\mathbb{X}^2} + \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle_{L^2(\Omega)} + \langle f_1(t,u), \varphi\rangle_{L^2(\Omega)} + \langle f_2(t,v), \gamma(\varphi)\rangle_{L^2(\Gamma)} \\ = \langle g_1, \varphi\rangle_{L^2(\Omega)} + \langle g_2, \gamma(\varphi)\rangle_{L^2(\Gamma)} \end{aligned} \tag{4.1}$$

a.e. in (τ, T), for each $T > \tau$;

(iv) *$U(\tau) = (u_\tau, v_\tau)$ in $\mathbb{X}^2$, which means, $u(\tau) = u_\tau$ a.e. in Ω and $v(\tau) = v_\tau$ a.e. in Γ.*

Theorem 4.2 (Weak Solution Existence, Theorem 4.11, [32])**.** *Suppose that hypotheses (H1)-(H3) hold. Given $\tau \in \mathbb{R}$ and $U_0 \in \mathbb{X}^2$, there exists a weak solution of Problem (P) with initial condition U_0.*

Next we intend to reformulate expression (4.1) in order to obtain a simpler functional formulation of the problem, which enables the use of the Faedo-Galerkin method to prove the existence of at least one weak solution (see [16] for the same scheme).

As shown in [32], for each $U = (u, \gamma(u)) \in \mathbb{V}^p$, the mapping

$$\beta_p(U, V) = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + \int_\Omega |u|^{p-2} uv dx$$

generates an operator $\beta_p U \in (\mathbb{V}^p)^*$. The other operators to be considered can be interpreted as follows. An element $B \in (\mathbb{V}^p)^* \subset \left(W^{1,p}(\Omega) \times W^{1-\frac{1}{p},p}(\Gamma) \right)^*$ can be considered as a couple

$$B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

and it acts on an element $W = (w_1, w_2) \in \mathbb{V}^p$ in the following sense

$$\langle B, W \rangle_{\mathbb{X}^2} = \langle b_1, w_1 \rangle_{L^2(\Omega)} + \langle b_2, w_2 \rangle_{L^2(\Gamma)}.$$

Then, consider

$$G(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} \in L^{r_1'}(\Omega) \times L^{r_2'}(\Gamma) \subset (\mathbb{V}^p)^*;$$

$$U_t = \begin{pmatrix} u_t \\ \gamma(u)_t \end{pmatrix}, \text{ with } U = \begin{pmatrix} u \\ \gamma(u) \end{pmatrix} \in \mathbb{V}^p;$$

$$\mathcal{F}(t, U) = \begin{pmatrix} f_1(t, u) - |u|^{p-2} u \\ f_2(t, \gamma(u)) \end{pmatrix} = \begin{pmatrix} \tilde{f}_1(t, u) \\ f_2(t, \gamma(u)) \end{pmatrix}.$$

This formulation will be used at appropriate times.

Suppose the following additional hypotheses for Problem (P):

(H4) Functions g_1 and g_2 satisfy

$$\|G\|_a^2 := \sup_{t \in \mathbb{R}} \int_t^{t+1} \left(\|g_1(l)\|_{L^2(\Omega)}^2 + \|g_2(l)\|_{L^2(\Gamma)}^2 \right) dl < \infty;$$

(H5) For $R > 0$, functions $f_i, i = 1, 2$, are bounded on the cylinder $Q(R) = \{(v, s); |v| < R, \ s \in \mathbb{R}\}$ and there are functions $\alpha_i(l, R)$, such that $\alpha_i(l, R) \to 0$ when $l \to 0^+$ and satisfy

$$|f_i(t_1, s_1) - f_i(t_2, s_2)| \leq \alpha_i(|t_1 - t_2| + |s_1 - s_2|, R), \ \forall \ (t_1, s_1), (t_2, s_2) \in Q(R).$$

4.1. **Estimates.** The following estimates are useful to ensure the existence of trajectory attractors.

Lemma 4.3. *Given $\tau \in \mathbb{R}$, $t > \tau$ and $U_\tau \in \mathbb{X}^2$, if U is a weak solution to Problem (P) with $U(\tau) = U_\tau$, we have the following estimates:*

$$\|U(t)\|_{\mathbb{X}^2}^2 \leq \|U_\tau\|_{\mathbb{X}^2}^2 e^{\theta(\tau - t)} + M\|G\|_a^2 + \frac{C}{\theta}, \tag{4.2}$$

$$\eta \int_h^{h+1} \|u\|^p_{W^{1,p}(\Omega)} ds + \frac{a_0}{2}\int_h^{h+1} \|u\|^{r_1}_{L^{r_1}(\Omega)} ds + a_0 \int_h^{h+1} \|\gamma(u)\|^{r_2}_{L^{r_2}(\Gamma)} ds \\ \leq \|U_\tau\|^2_{\mathbb{X}^2} e^{\theta(\tau-h)} + 2M\|G\|^2_a + \frac{C}{\theta}, \quad h \geq \tau, \tag{4.3}$$

with θ, η, M and C positive constants independent of τ, t and h.

Proof: First of all, note that, from Lemma 2.1 in [32], there exist positive constants κ_1, κ_2 and κ_3 such that

(i) $\|u\|^2_{L^2(\Omega)} \leq \kappa_1 \|u\|^{r_1}_{L^{r_1}(\Omega)} + C_{\kappa_1}$;
(ii) $\|\gamma(u)\|^2_{L^2(\Gamma)} \leq \kappa_2 \|\gamma(u)\|^{r_2}_{L^{r_2}(\Gamma)} + C_{\kappa_2}$;
(iii) $\|u\|^p_{L^p(\Omega)} \leq \kappa_3 \|u\|^{r_1}_{L^{r_1}(\Omega)} + C_{\kappa_3}$,

with C_{κ_i} positive constants, $i = 1, 2, 3$.

Taking $\Psi = U$ in (4.1) and using Hölder's and Young's inequalities, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|U\|^2_{\mathbb{X}^2} &+ \|\nabla u\|^p_{L^p(\Omega)} + \int_\Omega (a_1(t)|u|^{r_1} - k_1)\, dx + \int_\Gamma (a_2(t)|\gamma(u)|^{r_2} - k_2)\, dS \\ &\leq \langle g_1(t), u\rangle_{L^2(\Omega)} + \langle g_2(t), \gamma(u)\rangle_{L^2(\Gamma)} \leq \|g_1(t)\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|g_2(t)\|_{L^2(\Gamma)}\|\gamma(u)\|_{L^2(\Gamma)} \\ &\leq M_\varepsilon \|g_1(t)\|^2_{L^2(\Omega)} + \varepsilon\|u\|^2_{L^2(\Omega)} + M_\varepsilon\|g_2(t)\|^2_{L^2(\Gamma)} + \varepsilon\|\gamma(u)\|^2_{L^2(\Gamma)},\end{aligned}$$

where $\varepsilon > 0$ will be chosen appropriately. Since $a_1(t), a_2(t) \geq a_0$, we obtain

$$\begin{aligned}\frac{d}{dt}\|U\|^2_{\mathbb{X}^2} &+ 2\|\nabla u\|^p_{L^p(\Omega)} + 2a_0\|u\|^{r_1}_{L^{r_1}(\Omega)} + 2a_0\|\gamma(u)\|^{r_2}_{L^{r_2}(\Gamma)} \\ &\leq 2M_\varepsilon\|g_1(t)\|^2_{L^2(\Omega)} + 2\varepsilon\|u\|^2_{L^2(\Omega)} + 2M_\varepsilon\|g_2(t)\|^2_{L^2(\Gamma)} + 2\varepsilon\|\gamma(u)\|^2_{L^2(\Gamma)} + 2k_1|\Omega| + 2k_2 S(\Gamma).\end{aligned}$$

Using (i) and (ii) from the beginning of the proof, we have

$$\begin{aligned}\frac{d}{dt}\|U\|^2_{\mathbb{X}^2} &+ 2\|\nabla u\|^p_{L^p(\Omega)} + \left(\frac{a_0}{\kappa_1} - 2\varepsilon\right)\|u\|^2_{L^2(\Omega)} + \left(\frac{a_0}{\kappa_2} - 2\varepsilon\right)\|\gamma(u)\|^2_{L^2(\Gamma)} \\ &\qquad + a_0\|u\|^{r_1}_{L^{r_1}(\Omega)} + a_0\|\gamma(u)\|^{r_2}_{L^{r_2}(\Gamma)} \\ &\leq 2M_\varepsilon\|g_1(t)\|^2_{L^2(\Omega)} + 2M_\varepsilon\|g_2(t)\|^2_{L^2(\Gamma)} + 2k_1|\Omega| + 2k_2S(\Gamma) + \frac{a_0C_{\kappa_1}}{\kappa_1} + \frac{a_0C_{\kappa_2}}{\kappa_2}.\end{aligned}$$

Taking $\varepsilon > 0$ such that $\left(\frac{a_0}{\kappa_i} - 2\varepsilon\right) > 0$, for $i = 1, 2$, and take $\theta = \min\left\{\frac{a_0}{\kappa_1} - 2\varepsilon, \frac{a_0}{\kappa_2} - 2\varepsilon\right\}$,

$$\begin{aligned}\frac{d}{dt}\|U\|^2_{\mathbb{X}^2} &+ \theta\|U\|^2_{\mathbb{X}^2} + 2\|\nabla u\|^p_{L^p(\Omega)} + a_0\|u\|^{r_1}_{L^{r_1}(\Omega)} + a_0\|\gamma(u)\|^{r_2}_{L^{r_2}(\Gamma)} \\ &\leq 2M_\varepsilon\|g_1(t)\|^2_{L^2(\Omega)} + 2M_\varepsilon\|g_2(t)\|^2_{L^2(\Gamma)} + 2k_1|\Omega| + 2k_2S(\Gamma) + \frac{a_0C_{\kappa_1}}{\kappa_1} + \frac{a_0C_{\kappa_2}}{\kappa_2}.\end{aligned}$$

From (iii)

$$\begin{aligned}\frac{d}{dt}\|U\|^2_{\mathbb{X}^2} &+ \theta\|U\|^2_{\mathbb{X}^2} + 2\|\nabla u\|^p_{L^p(\Omega)} + \frac{a_0}{2\kappa_3}\|u\|^p_{L^p(\Omega)} + \frac{a_0}{2}\|u\|^{r_1}_{L^{r_1}(\Omega)} + a_0\|\gamma(u)\|^{r_2}_{L^{r_2}(\Gamma)} \\ &\leq 2M_\varepsilon\|g_1(t)\|^2_{L^2(\Omega)} + 2M_\varepsilon\|g_2(t)\|^2_{L^2(\Gamma)} + 2k_1|\Omega| + 2k_2S(\Gamma) + \frac{a_0C_{\kappa_1}}{\kappa_1} + \frac{a_0C_{\kappa_2}}{\kappa_2} + \frac{a_0C_{\kappa_3}}{2\kappa_3}.\end{aligned}$$

Taking $\eta = \min\left\{2, \frac{a_0}{2\kappa_3}\right\}$, $M = 2M_\varepsilon$ and $C = \frac{a_0 C_{\kappa_1}}{\kappa_1} + \frac{a_0 C_{\kappa_2}}{\kappa_2} + \frac{a_0 C_{\kappa_3}}{2\kappa_3} + 2|\Omega|k_1 + 2S(\Gamma)k_2$

$$\begin{aligned}\frac{d}{dt}\|U\|^2_{\mathbb{X}^2} + \theta\|U\|^2_{\mathbb{X}^2} + \eta\|u\|^p_{W^{1,p}(\Omega)} + \frac{a_0}{2}\|u\|^{r_1}_{L^{r_1}(\Omega)} + a_0\|\gamma(u)\|^{r_2}_{L^{r_2}(\Gamma)} \\ \leq M\left(\|g_1(t)\|^2_{L^2(\Omega)} + \|g_2(t)\|^2_{L^2(\Gamma)}\right) + C.\end{aligned} \tag{4.4}$$

In particular,

$$\frac{d}{dt}\|U\|^2_{\mathbb{X}^2} + \theta\|U\|^2_{\mathbb{X}^2} \leq M\left(\|g_1(t)\|^2_{L^2(\Omega)} + \|g_2(t)\|^2_{L^2(\Gamma)}\right) + C.$$

Multiplying this inequality by $e^{\theta t}$,

$$\frac{d}{dt}\left(\|U\|^2_{\mathbb{X}^2}e^{\theta t}\right) \leq M\left(\|g_1(t)\|^2_{L^2(\Omega)} + \|g_2(t)\|^2_{L^2(\Gamma)}\right)e^{\theta t} + Ce^{\theta t}.$$

and integrating from τ to t,

$$\|U(t)\|^2_{\mathbb{X}^2}e^{\theta t} - \|U(\tau)\|^2_{\mathbb{X}^2}e^{\theta\tau} \leq M\int_\tau^t \left(\|g_1(l)\|^2_{L^2(\Omega)} + \|g_2(l)\|^2_{L^2(\Gamma)}\right)e^{\theta l}dl + \frac{C}{\theta}\left(e^{\theta t} - e^{\theta\tau}\right).$$

Note that, assuming $0 \leq t-\tau \leq k$, where $1 \leq k < \infty$, we have (see [9])

$$\begin{aligned}&\int_\tau^t \left(\|g_1(l)\|^2_{L^2(\Omega)} + \|g_2(l)\|^2_{L^2(\Gamma)}\right)e^{\theta l}dl \\ &\quad\leq e^{\theta t}\int_{t-1}^t \left(\|g_1(l)\|^2_{L^2(\Omega)} + \|g_2(l)\|^2_{L^2(\Gamma)}\right)dl \\ &\qquad + e^{\theta(t-1)}\int_{t-2}^{t-1} \left(\|g_1(l)\|^2_{L^2(\Omega)} + \|g_2(l)\|^2_{L^2(\Gamma)}\right)dl \\ &\qquad\quad + ... + e^{\theta(t-(k-1))}\int_\tau^{t-(k-1)} \left(\|g_1(l)\|^2_{L^2(\Omega)} + \|g_2(l)\|^2_{L^2(\Gamma)}\right)dl \\ &\quad\leq \|G\|^2_a e^{\theta t}\left(1 + e^{-\theta} + e^{-2\theta} + ... + e^{-(k-1)\theta}\right) \\ &\quad\leq \|G\|^2_a e^{\theta t}\left(1 - e^{-\theta}\right)^{-1}.\end{aligned}$$

Therefore,

$$\|U(t)\|^2_{\mathbb{X}^2} \leq \|U_\tau\|^2_{\mathbb{X}^2}e^{\theta(\tau-t)} + M\|G\|^2_a\left(1 - e^{-\theta}\right)^{-1} + \frac{C}{\theta},$$

proving (4.2).

In order to prove (4.3), we integrate (4.4) from h to $h+1$, $h \geq \tau$, use (4.2) and incorporate the constants, obtaining

$$\begin{aligned}\eta\int_h^{h+1}\|u\|^p_{W^{1,p}(\Omega)}ds + \frac{a_0}{2}\int_h^{h+1}\|u\|^{r_1}_{L^{r_1}(\Omega)}ds + a_0\int_h^{h+1}\|\gamma(u)\|^{r_2}_{L^{r_2}(\Gamma)}ds \\ \leq \|U_\tau\|^2_{\mathbb{X}^2}e^{\theta(\tau-h)} + 2M\|G\|^2_a + \frac{C}{\theta}.\end{aligned}$$

■

Remark 4.4. *Note that, if $1 \leq s \leq 2$, then $0 \leq s-1 \leq 1$ and then $1 \leq 2^{s-1} \leq 2$. Therefore, from Lemma 15.2 in [3], we have*

$$(a+b+c)^s \leq 2^{s-1}a^s + 2^{s-1}(b+c)^s \leq 2^{s-1}(a^s + 2^{s-1}(b^s+c^s)) \leq 4(a^s+b^s+c^s).$$

Lemma 4.5. *Let $\tau \in \mathbb{R}$, $h > \tau$ and $U_\tau \in \mathbb{X}^2$. If U is a weak solution to Problem (P) with $U(\tau) = U_\tau$, then*

$$\|\partial_t U\|^s_{L^s(h,h+1;(\mathbb{V}^p)^*)} \leq \tilde{M}\|U_\tau\|^2_{\mathbb{X}^2} e^{\theta(\tau-h)} + \tilde{M}\|G\|^2_a + \tilde{C}, \tag{4.5}$$

where $s = \min\{p', r_1', r_2'\}$ and θ, $\tilde{M}$ and $\tilde{C}$ are positive constants independent of τ and h.

Proof: As in [32] we have

$$\partial_t U = -\beta_p U - \mathcal{F}(t,U) + G(t) \quad \text{in} \quad L^s(h,h+1;(\mathbb{V}^p)^*).$$

Thus,

$$\|\partial_t U\|_{L^s(h,h+1;(\mathbb{V}^p)^*)} = \|-\beta_p U - \mathcal{F}(t,U) + G(t)\|_{L^s(h,h+1;(\mathbb{V}^p)^*)},$$

and then,

$$\begin{aligned}\int_h^{h+1} \|\partial_t U\|^s_{(\mathbb{V}^p)^*} dt &= \int_h^{h+1} \|-\beta_p U - \mathcal{F}(t,U) + G(t)\|^s_{(\mathbb{V}^p)^*} dt \\ &\leq \int_h^{h+1} \left(\|\beta_p U\|_{(\mathbb{V}^p)^*} + \|\mathcal{F}(t,U)\|_{(\mathbb{V}^p)^*} + \|G(t)\|_{(\mathbb{V}^p)^*}\right)^s dt \\ &\leq 4\int_h^{h+1} \left(\|\beta_p U\|^s_{(\mathbb{V}^p)^*} + \|\mathcal{F}(t,U)\|^s_{(\mathbb{V}^p)^*} + \|G(t)\|^s_{(\mathbb{V}^p)^*}\right) dt.\end{aligned} \tag{4.6}$$

We will estimate each term of the last integral in the expression (4.6) separately. Note that

$$\|\beta_p U\|_{(\mathbb{V}^p)^*} \leq \|u\|^{p-1}_{W^{1,p}(\Omega)},$$

(see [32] for details), and then

$$\|\beta_p U\|^s_{(\mathbb{V}^p)^*} \leq \|u\|^{(p-1)s}_{W^{1,p}(\Omega)} \leq \|u\|^p_{W^{1,p}(\Omega)} + 1,$$

since $(p-1)s \leq (p-1)p' = p$.

Therefore,

$$\int_h^{h+1} \|\beta_p U\|^s_{(\mathbb{V}^p)^*} dt \leq \int_h^{h+1} \|u\|^p_{W^{1,p}(\Omega)} dt + 1.$$

Now, we know that

$$\|G(t)\|_{(\mathbb{V}^p)^*} \leq \|g_1(t)\|_{L^{r_1'}(\Omega)} + \|g_2(t)\|_{L^{r_2'}(\Gamma)},$$

thus

$$\begin{aligned}\|G(t)\|^s_{(\mathbb{V}^p)^*} &\leq 2^{s-1}\left(\|g_1(t)\|^s_{L^{r_1'}(\Omega)} + \|g_2(t)\|^s_{L^{r_2'}(\Gamma)}\right) \\ &\leq 2^{s-1}\left(\|g_1(t)\|^{r_1'}_{L^{r_1'}(\Omega)} + 1 + \|g_2(t)\|^{r_2'}_{L^{r_2'}(\Gamma)} + 1\right).\end{aligned} \tag{4.7}$$

Note that, since $r_i' \leq 2$ for $i = 1, 2$, from Lemma 1.2 of [32], there are positive constants κ_i and C_{κ_i} such that

(1) $\|g_1\|^{r_1'}_{L^{r_1'}(\Omega)} \le \kappa_1 \|g_1\|^2_{L^2(\Omega)} + C_{\kappa_1}$;
(2) $\|g_2\|^{r_2'}_{L^{r_2'}(\Gamma)} \le \kappa_2 \|g_2\|^2_{L^2(\Gamma)} + C_{\kappa_2}$.

Therefore, taking $\kappa = \max\{\kappa_1, \kappa_2\}$, we have

$$\begin{aligned}\int_h^{h+1} \|G(t)\|^s_{(\mathbb{V}^p)^*} dt &\le 2^{s-1} \left(\int_h^{h+1} \|g_1(t)\|^{r_1'}_{L^{r_1'}(\Omega)} + \|g_2(t)\|^{r_2'}_{L^{r_2'}(\Gamma)} dt + 2 \right) \\ &\le 2 \left(\int_h^{h+1} \kappa_1 \|g_1(t)\|^2_{L^2(\Omega)} + \kappa_2 \|g_2(t)\|^2_{L^2(\Gamma)} dt + (2 + C_{\kappa_1} + C_{\kappa_2}) \right) \\ &\le 2 \left(\kappa \|G\|^2_a + (2 + C_{\kappa_1} + C_{\kappa_2}) \right)\end{aligned}$$

Finally,

$$\|\mathcal{F}(t,U)\|_{(\mathbb{V}^p)^*} \le \|\tilde{f}_1(t,u)\|_{L^{r_1'}(\Omega)} + \|f_2(t,\gamma(u))\|_{L^{r_2'}(\Gamma)},$$

and from (H3),

$$\begin{aligned}\|\mathcal{F}(t,U)\|^s_{(\mathbb{V}^p)^*} &\le 2^{s-1} \left[\|\tilde{f}_1(t,u)\|^s_{L^{r_1'}(\Omega)} + \|f_2(t,\gamma(u))\|^s_{L^{r_2'}(\Gamma)} \right] \\ &\le 2^{s-1} \left[\|\tilde{f}_1(t,u)\|^{r_1'}_{L^{r_1'}(\Omega)} + 1 + \|f_2(t,\gamma(u))\|^{r_2'}_{L^{r_2'}(\Gamma)} + 1 \right] \\ &\le 2^{s-1} \Bigg[2^{r_1'-1} \left(2^{r_1'-1} C_1^{r_1'} \left(\|u\|^{r_1}_{L^{r_1}(\Omega)} + |\Omega| \right) + \|u\|^{r_1}_{L^{r_1}(\Omega)} + |\Omega| \right) + 1 \\ &\qquad + 2^{r_2'-1} C_2^{r_2'} \left(\|\gamma(u)\|^{r_2}_{L^{r_2}(\Gamma)} + S(\Gamma) \right) + 1 \Bigg].\end{aligned}$$

See the proof of Proposition 4.5 in [32] for details. Then,

$$\begin{aligned}\int_h^{h+1} \|\mathcal{F}(t,U)\|^s_{(\mathbb{V}^p)^*} dt \le 8 \Bigg[& C_1^{r_1'} \left(\int_h^{h+1} \|u\|^{r_1}_{L^{r_1}(\Omega)} dt \right) \\ & + \int_h^{h+1} \|u\|^{r_1}_{L^{r_1}(\Omega)} dt + C_2^{r_2'} \left(\int_h^{h+1} \|\gamma(u)\|^{r_2}_{L^{r_2}(\Gamma)} dt \right) \Bigg] + \bar{M},\end{aligned} \tag{4.8}$$

where $\bar{M}$ depends on $|\Omega|$ and $S(\Gamma)$.

Therefore,

$$\begin{aligned}\int_h^{h+1} \|\partial_t U\|^s_{(\mathbb{V}^p)^*} dt \le & \int_h^{h+1} \|u\|^p_{W^{1,p}(\Omega)} + 8 \left(C_1^{r_1'} + 1 \right) \int_h^{h+1} \|u\|^{r_1}_{L^{r_1}(\Omega)} dt \\ & + \left(8 C_2^{r_2'} \right) \int_h^{h+1} \|\gamma(u)\|^{r_2}_{L^{r_2}(\Gamma)} dt + 2\kappa \|G\|^2_a + \tilde{K},\end{aligned}$$

where $\tilde{K}$ depends on $|\Omega|, S(\Gamma), C_1^{r_1'}, C_2^{r_2'}, C_{\kappa_1}$ and C_{κ_2}.

Taking appropriate constants, it follows from (4.3) that

$$\int_h^{h+1} \|\partial_t U\|^s_{(\mathbb{V}^p)^*} dt \le \tilde{M} \|U_\tau\|^2_{\mathbb{X}^2} e^{\theta(\tau - h)} + \tilde{M} \|G\|^2_a + \tilde{C}.$$

■

4.2. **Trajectory and Global Attractors for the Problem** (P)**.** Let us look at some aspects of functions f_i and g_i, for $i = 1, 2$, in order to define the appropriate set Σ for Problem (P).

Note that, as $1 \le p', r_1', r_2' < 2$, hypothesis (H4) ensures that

$$\sup_{h\in\mathbb{R}} \left(\int_h^{h+1} \left(\|g_1(l)\|^{p'}_{L^{r_1'}(\Omega)} + \|g_2(l)\|^{p'}_{L^{r_2'}(\Gamma)} \right) dl \right) < \infty; \tag{4.9}$$

From (4.7) and (4.9), we have that

$$\sup_{h\in\mathbb{R}} \left(\int_h^{h+1} \|G\|^s_{(\mathbb{V}^p)^*} dl \right) \le \sup_{h\in\mathbb{R}} \left(2 \left[\int_h^{h+1} \|g_1(l)\|^{p'}_{L^{r_1'}(\Omega)} + \|g_2(l)\|^{p'}_{L^{r_2'}(\Gamma)} dl \right] + 4 \right) < \infty.$$

This ensures that G is a *translation compact* function in $L^{s,w}_{loc}(\mathbb{R}; (\mathbb{V}^p)^*)$ (i.e., $\mathcal{H}(G) = \overline{\{T(h)G;\ h \in \mathbb{R}\}}$ is compact in $L^{s,w}_{loc}(\mathbb{R}; (\mathbb{V}^p)^*)$), see Proposition 4.1 in Chapter V of [11].

Remark 4.6. *The estimates on hypotheses (H2) and (H3) are uniform with respect to translation of functions f_i in the first variable, for $i = 1, 2$, which means,*

(1) $a_0|s|^{r_i} - k_i \le f_i(h + t, s)s = T(h)f_i(t, s)s,\ \forall h \in \mathbb{R}$;
(2) $|T(h)f_i(t, s)| = |f_i(h + t, s)| \le C_i(|s|^{r_i-1} + 1),\ \forall h \in \mathbb{R}$.

We define, for each $(t, s) \in \mathbb{R}^2$, the following function

$$\tilde{\mathcal{F}}(t, s) = \begin{pmatrix} f_1(t, s) \\ f_2(t, s) \end{pmatrix} \in \mathbb{R}^2.$$

Note that the function $\tilde{\mathcal{F}}$ is the nonautonomous part of the function $\mathcal{F}$, and then it is the only part of the function $\mathcal{F}$ which is affected by translations. This allows us to deal with $\sigma_0 = (\tilde{\mathcal{F}}, G)$ instead of $(\mathcal{F}, G)$ when it is convenient. Besides, from the continuity of the operators involved, for each $t \in \mathbb{R}$ we have $\tilde{\mathcal{F}}(t, \cdot) \in C_{loc}(\mathbb{R}; \mathbb{R}^2)$, and then $\tilde{\mathcal{F}} \in C_{loc}(\mathbb{R}; C_{loc}(\mathbb{R}; \mathbb{R}^2))$.

With the hypothesis (H5), for all $R > 0$ and $(t_1, s_1), (t_2, s_2) \in Q(R)$, we deduce

$$\|\tilde{\mathcal{F}}(t_1, s_1) - \tilde{\mathcal{F}}(t_2, s_2)\|^2_{\mathbb{R}^2} \le \alpha_1(|t_1 - t_2| + |s_1 - s_2|, R)^2 + \alpha_2(|t_1 - t_2| + |s_1 - s_2|, R)^2.$$

Taking $\alpha_3(l, R) := \sqrt{\alpha_1(l, R)^2 + \alpha_2(l, R)^2}$, we have

$$\|\tilde{\mathcal{F}}(t_1, s_1) - \tilde{\mathcal{F}}(t_2, s_2)\|_{\mathbb{R}^2} \le \alpha_3(|t_1 - t_2| + |s_1 - s_2|, R), \tag{4.10}$$

with $\alpha_3(l, R) \to 0$, when $l \to 0^+$, and $\tilde{\mathcal{F}}$ is bounded in $Q(R)$.

Therefore, $\tilde{\mathcal{F}}$ is a translation compact function in $C_{loc}(\mathbb{R}; C_{loc}(\mathbb{R}; \mathbb{R}^2))$ (i.e., $\mathcal{H}(\tilde{\mathcal{F}}) = \overline{\{T(h)\tilde{\mathcal{F}};\ h \in \mathbb{R}\}}$ is compact in $C_{loc}(\mathbb{R}; C_{loc}(\mathbb{R}; \mathbb{R}^2))$), see Proposition 2.5 in Chapter V of [11].

Take $\Xi = C_{loc}(\mathbb{R}; C_{loc}(\mathbb{R}; \mathbb{R}^2)) \times L^{s,w}_{loc}(\mathbb{R}; (\mathbb{V}^p)^*)$. Thus, from Section V.5 of [11], the function $\sigma_0 = (\tilde{\mathcal{F}}, G)$ is a translation compact function in Ξ. This ensures that the set $\Sigma := \mathcal{H}(\sigma_0)$ is a compact metric subspace of Ξ with $T(t)$ continuous in Σ and $T(t)\Sigma = \Sigma$, $\forall\ t \in \mathbb{R}$.

Proposition 4.7. *For all $\sigma_1 = (\tilde{\mathcal{F}}^{(1)}, G^{(1)}) \in \Sigma$, we have that*

(i) $\|G^{(1)}\|^s_a = \sup_{h\in\mathbb{R}} \int_h^{h+1} \|G^{(1)}\|^s_{(\mathbb{V}^p)^*} dl \le \|G\|^s_a$;

(ii) *the function* $\tilde{\mathcal{F}}^{(1)}$ *satisfy* (4.10) *with the same function* α_3.

Proof: It follows directly from Propositions 4.2 and 2.3 of Chapter V in [11], respectively. ■

Note that for each $\sigma \in \Sigma$ there is $h \in \mathbb{R}$ such that $T(h)\sigma_0 = \sigma$. Let U be a weak solution to Problem (P) with functions f_i and g_i, $i = 1, 2$, taken at the beginning, we say that U is a solution associated with functions $\tilde{\mathcal{F}}$ and G. Note that, $T(h)U$ is a solution to Problem (P) associated with functions $T(h)\tilde{\mathcal{F}}$ and $T(h)G$. We will denote this problem associated with functions $T(h)\tilde{\mathcal{F}}$ and $T(h)G$ by Problem (P_σ).

Proposition 4.8. *The family* $\mathscr{G} = \{\mathscr{G}_\sigma(\tau)\}_{\tau\in\mathbb{R},\sigma\in\Sigma}$*, where*

$$\mathscr{G}_\sigma(\tau) := \left\{ \begin{array}{l} U : [\tau, +\infty) \to \mathbb{X}^2;\ U \text{ is weak solution of} \\ (P_\sigma) \text{ with initial condition in } \tau \end{array} \right\},$$

is an exact generalized process in $\mathbb{X}^2$.

The proof of this proposition is analogous to the proof of Proposition 4.12 in [32].

Proposition 4.9. *The generalized process* $\mathscr{G}$ *is LUUS.*

Proof: Let $\{U_n\}_{n\in\mathbb{N}} \subset \mathscr{G}_\Sigma(\tau)$ and $\{\sigma_n\}_{n\in\mathbb{N}} \subset \Sigma$ sequences such that $U_n \in \mathscr{G}_{\sigma_n}(\tau)$, for each $n \in \mathbb{N}$. Suppose that $U_n(\tau) \to z \in \mathbb{X}^2$.

Note that, except for a finite number of indices n, there exists a bounded set $B_0 \subset \mathbb{X}^2$, such that $\{U_n(\tau)\}_{n\in\mathbb{N}} \subset B_0$. Then, from Lemma 4.3, given $T > \tau$, we have

$$\|U_n\|_{L^\infty(\tau,T;\mathbb{X}^2)} \leq C \text{ and } \|U_n\|_{L^p(\tau,T;\mathbb{V}^p)} \leq C.$$

As it was done with the sequence generated by Faedo-Galerkin method in Section 4 of [32], we then have that there exists W such that

$$U_n \overset{*}{\rightharpoonup} W \text{ in } L^\infty(\tau, T; \mathbb{X}^2) \text{ and } U_n \rightharpoonup W \text{ in } L^p(\tau, T; \mathbb{V}^p).$$

Since $U_n \in \mathscr{G}_{\sigma_n}(\tau)$ we have

$$\partial_t U_n + \beta_p U_n + \mathcal{F}_n(t, U_n) = G_n(t),$$

in $L^s(\tau, T; (\mathbb{V}^p)^*)$. The fact that $s \leq 2$ ensures that $G_n \overset{*}{\rightharpoonup} G$ in $L^s(\tau, T; (\mathbb{V}^p)^*)$. We also know that $U_n \to W$ a.e. in $\overline{\Omega} \times [\tau, T]$, thus $\mathcal{F}_n(t, U_n) \to \mathcal{F}(t, W)$ a.e. in $\overline{\Omega} \times [\tau, T]$. Note that, each f_{in} satisfies item 2 of Remark 4.6 uniformly. Then, from (4.8) and (4.3) we have that $\mathcal{F}_n$ is uniformly bounded in $L^s(\tau, T; (\mathbb{V}^p)^*)$. From Lemma 8.3 in [31]

$$\mathcal{F}_n(t, U_n) \overset{*}{\rightharpoonup} \mathcal{F}(t, W) \text{ in } L^s(\tau, T; (\mathbb{V}^p)^*).$$

This implies that $\beta_p U_n \overset{*}{\rightharpoonup} \beta_p W$ in $L^s(\tau, T; (\mathbb{V}^p)^*)$, see [32] for details.

Then we can ensure that $W \in \mathscr{G}_\sigma(\tau)$ and $U_n \to W$ in $C(\tau, T; \mathbb{X}^2)$. Therefore, as $T > \tau$ is arbitrary, we have $W \in \mathscr{G}_\sigma(\tau)$, and $\mathscr{G}$ is an LUUS generalized process. ■

Corollary 4.10. *The trajectory space* $\mathscr{G}_\Sigma(\tau)$ *is* $(C_{loc}([\mathbb{R}_\tau, +\infty); \mathbb{X}^2), \Sigma)$*-closed.*

Proof: From the previous proposition, the generalized process $\mathscr{G}$ is LUUS. Then the result follows from Proposition 3.15. ■

Theorem 4.11. *Suppose that hypotheses (H1)-(H5) hold. Then the semigroup* $\{H(t)\}_{t\geq 0}$ *acting in* $\mathscr{G}_\Sigma(\tau)$ *possesses a uniform trajectory attractor, that attracts bounded (in* $L^\infty(\tau,+\infty;\mathbb{X}^2)$ *) subsets of* $\mathscr{G}_\Sigma(\tau)$ *in the topology of* $C_{loc}([\tau,+\infty);\mathbb{X}^2)$.

Proof: In order to apply Theorem 3.14 we need to find an attractor $P\subset\mathscr{G}_\Sigma(\tau)$ bounded in $L^\infty(\tau,+\infty;\mathbb{X}^2)$ and compact in $C_{loc}([\tau,+\infty);\mathbb{X}^2)$. Take

$$P=\left\{U\in\mathscr{G}_\Sigma(\tau);\operatorname*{ess\,sup}_{h\geq\tau}\left(\|U\|_{L^\infty(h,h+1;\mathbb{X}^2)}+\|\partial_t U\|_{L^s(h,h+1;(\mathbb{V}^p)^*)}\right)\leq 2R\right\}, \tag{4.11}$$

where $R:=1+\tilde{M}+2M\|G\|_a^2+\frac{2C}{\theta}$.

We will show that P is an absorbing set, $\mathbb{B}\subset\mathscr{G}_\Sigma(\tau)$ bounded in $L^\infty(\tau,+\infty;\mathbb{X}^2)$, which means that there is $R_0>0$ such that $\|U\|_{L^\infty(\tau,+\infty;\mathbb{X}^2)}<R_0$, for each trajectory $U\in\mathbb{B}$. Choosing $t_0\geq\tau$ such that $R_0e^{\theta(\tau-t_0)}+R\leq 2R$, it follows from estimates (4.2) and (4.5), for all $t>t_0$, that

$$\|U\|_{L^\infty(t,t+1;\mathbb{X}^2)}+\|\partial_t U\|_{L^s(t,t+1;(\mathbb{V}^p)^*)}\leq 2R.$$

Then, $H(t)U\in P$, i.e., $H(t)\mathbb{B}\subset P$ for all $t>t_0$. It is evident that the set P is bounded in $L^\infty(\tau,+\infty;\mathbb{X}^2)$.

Finally, we need to show that the set P is compact in $C_{loc}([\tau,+\infty);\mathbb{X}^2)$, for this it is enough to show that $\Pi_{[\tau,T]}P$ is compact in $C([\tau,T];\mathbb{X}^2)$ for all $T\geq\tau$ (see Proposition 2.1 of Chapter V in [11]).

We first show that $\Pi_{[\tau,T]}P$ is pre-compact in $C([\tau,T];\mathbb{X}^2)$ and next we prove that it is also closed. Let $\{U_n\}_{n\in\mathbb{N}}\subset\Pi_{[\tau,T]}P$ be a sequence of solutions, i.e, for each $n\in\mathbb{N}$, U_n is a solution of Problem (P_{σ_n}) associated with a $\sigma_n=(\tilde{\mathcal{F}}_n,G_n)$. From Proposition 4.2 in [32], we have

$$\begin{aligned}\|U_n(t)\|_{\mathbb{X}^2}^2+&K\int_s^t\|u_n(\ell)\|_{W^{1,p}(\Omega)}^p d\ell+\frac{a_0}{2}\int_s^t\|u_n(\ell)\|_{L^{r_1}(\Omega)}^{r_1}d\ell+a_0\int_s^t\|\gamma(u_n)(\ell)\|_{L^{r_2}(\Gamma)}^{r_2}d\ell\\ &\leq\|U_n(s)\|_{\mathbb{X}^2}^2+C\int_s^t\left(\|g_{1n}(t)\|_{L^{r_1'}(\Omega)}^{p'}+\|g_{2n}(t)\|_{L^{r_2'}(\Gamma)}^{p'}\right)dt+M(t-s),\end{aligned}$$

for $t,s\in[\tau,T]$, $t\geq s$, , where K,a_0,C and M are constants independent of n.

Thus, from the uniform boundedness of σ_n, as in proof of Proposition 4.9, and as in the result of existence of solution in [32], we can ensure that (up to a subsequence) U_n converges to a weak solution $U\in\mathscr{G}_\sigma(\tau)$ in the following way:

$$\begin{aligned}&U_n\to U\text{ weak star in }L^\infty(\tau,T;\mathbb{X}^2),\\&U_n\to U\text{ weak in }L^p(\tau,T;\mathbb{V}^p),\\&\partial_t U_n\to\partial_t U\text{ weak star in }L^s(\tau,T;(\mathbb{V}^p)^*),\end{aligned}\tag{4.12}$$

from where we have the strong convergence in $L^p(\tau,T;\mathbb{X}^2)$. Proceeding exactly as in the proof of Proposition 2.5 in [17] (but considering $\mathcal{F}$ and G depending on time) we can conclude that $U\in C([\tau,T];\mathbb{X}^2)$.

We now consider $\{t_n\}_{n\in\mathbb{N}}\subset[\tau,T]$ and $t_0\in[\tau,T]$ such that $t_n\to t_0$. From (4.12) we have $U_n(t_n)\to U(t_0)$ weakly in $\mathbb{X}^2$, and now we can show that this sequence strongly converges in $\mathbb{X}^2$.

From the weak convergence we obtain $\liminf \|U_n(t_n)\|_{\mathbb{X}^2} \geq \|U(t_0)\|_{\mathbb{X}^2}$. It is easy to see that U_n and U satisfy the following inequalities:

$$\|U_n(t)\|_{\mathbb{X}^2} \leq \|U_n(s)\|_{\mathbb{X}^2} + \tilde{K}(t-s) + 2\left(\int_s^t \langle g_{1n}(\ell), u_n(\ell)\rangle_2 + \langle g_{2n}(\ell), \gamma(u_n(\ell))\rangle_{2,\Gamma}\, d\ell\right),$$

$$\|U(t)\|_{\mathbb{X}^2} \leq \|U(s)\|_{\mathbb{X}^2} + \tilde{K}(t-s) + 2\left(\int_s^t \langle g_1(\ell), u(\ell)\rangle_2 + \langle g_2(\ell), \gamma(u(\ell))\rangle_{2,\Gamma}\, d\ell\right)$$

for $t, s \in [\tau, T]$, $t \geq s$, where the constant $\tilde{K}$ does not depend on n. Therefore, the functions

$$J_n(t) = \|U_n(t)\|_{\mathbb{X}^2} - \tilde{K}t - 2\left(\int_s^t \langle g_{1n}(\ell), u_n(\ell)\rangle_2 + \langle g_{2n}(\ell), \gamma(u_n(\ell))\rangle_{2,\Gamma}\, d\ell\right),$$

$$J(t) = \|U(t)\|_{\mathbb{X}^2} - \tilde{K}t - 2\left(\int_s^t \langle g_1(\ell), u(\ell)\rangle_2 + \langle g_2(\ell), \gamma(u(\ell))\rangle_{2,\Gamma}\, d\ell\right)$$

are continuous and nonincreasing on $[\tau, T]$. From (H4) we have taht $(g_{1n}, g_{2n}) \to (g_1, g_2)$ weakly in $L^2(\tau, T; \mathbb{X}^2)$ and, from now on, as it is done in the proof of Lemma 15 of [36], we can guarantee that $\limsup \|U_n(t_n)\|_{\mathbb{X}^2} \leq \|U(t_0)\|_{\mathbb{X}^2}$.

Then, $U_n(t_n) \to U(t_0)$ in $\mathbb{X}^2$, it ensures that $\Pi_{[\tau,T]}P$ is pre-compact in $C(\tau, T; \mathbb{X}^2)$.

Therefore, it remains to ensure that $\Pi_{[\tau,T]}P$ is closed in $C([\tau, T]; \mathbb{X}^2)$. Let $\{U_n\}_{n\in\mathbb{N}} \subset P$ be a sequence such that $U_n \to U$ in $C_{loc}([\tau, +\infty); \mathbb{X}^2)$, and then $\Pi_{[\tau,T]}U_n \to \Pi_{[\tau,T]}U$ in $C([\tau, T]; \mathbb{X}^2)$. The sequence $\{U_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\tau, +\infty; \mathbb{X}^2)$, from Corollary 4.10 we have that $U \in \mathscr{G}_\Sigma(\tau)$, and proceeding in the same way as in the proof of Proposition 4.9, we obtain

$$U_n \overset{*}{\rightharpoonup} U \text{ in } L^\infty(\tau, T; \mathbb{X}^2) \text{ and } \partial_t U_n \overset{*}{\rightharpoonup} \partial_t U \text{ in } L^s(\tau, T; (\mathbb{V}^p)^*).$$

for all $T > \tau$, and thus

$$\|U\|_{L^\infty(h,h+1;\mathbb{X}^2)} + \|\partial_t U\|_{L^s(h,h+1;(\mathbb{V}^p)^*)} \leq \liminf \left(\|U_n\|_{L^\infty(h,h+1;\mathbb{X}^2)} + \|\partial_t U_n\|_{L^s(h,h+1;(\mathbb{V}^p)^*)}\right) \leq 2R.$$

Therefore, $\Pi_{[\tau,T]}U \in \Pi_{[\tau,T]}P$, or in other words, $\Pi_{[\tau,T]}P$ is closed in $C([\tau, T]; \mathbb{X}^2)$ ensuring that P is compact in $C_{loc}([\tau, +\infty); \mathbb{X}^2)$.

■

Theorem 4.12. *Assume hypotheses of Theorem 4.11 hold. Then the family of multivalued processes $\{U_\sigma\}_{\sigma\in\Sigma}$ associated with a generalized process $\mathscr{G}$ possesses a compact invariant uniform global attractor in $\mathbb{X}^2$.*

Proof: Note that the space $\mathbb{X}^2$ and the set Σ satisfy the initial hypotheses of Theorem 3.31, and in the proof of the previous theorem the set P defined in (4.11) is a uniform trajectory attractor.

Then, in order to ensure the existence of invariant compact uniform global attractor for the family $\{U_\sigma\}_{\sigma\in\Sigma}$ associated with $\mathscr{G}$, we need to show that, for all bounded $B \subset \mathbb{X}^2$, the set

$$\mathbb{B} = \{U; U \in \mathscr{G}_\Sigma(\tau) \text{ and } U(\tau) \in B\}$$

is bounded in $L^\infty(\mathbb{R}_\tau; \mathbb{X}^2)$. But this follows directly from estimate (4.2).

From Proposition 4.9 we have that the generalized process $\mathscr{G}$ is LUUS.

Therefore, thanks to Theorem 3.31, there exists the invariant compact uniform global attractor in $\mathbb{X}^2$ for the family of multivalued processes $\{U_\sigma\}_{\sigma\in\Sigma}$ associated with the generalized process $\mathscr{G}$.

■

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