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## HAND DRAWING IN THE DEFINITION OF THE FIRST DIGITAL CURVES

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| **Abstract:** | The first digitization systems successfully used in curves with origin in mathematical calculations or experimental tests proved useless when they were the result of heuristic methods developed under aesthetic criteria. Renault engineer Pierre Bézier and Citroën engineer Paul de Casteljau found the solution by focusing on the period of ideation of shapes and not on their subsequent translation and integration into the digital domain, studying the working methods of designers and incorporating in the mathematical definition of their curves the own actions of drawing, understood as basic instrumental support in creative processes. This article traces and analyzes the strong relationship between Bézier's original approach and the procedures driving the manual stroke as the main reason for the consolidation of his graphic proposal as the most effective model for the hand drawing of digital curves and its inclusion since that time and to the present day in the software used by architects. |
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# Hand Drawing in the Definition of the First Digital Curves

## 1 Introduction

The first application of digital methods to the definition of graphic curves was carried out in the military aircraft industry during World War II. Working for the North American Aviation (NAA), Roy Liming in his book of 1944, *Practical analytical geometry with applications to aircraft*, proposed the computational definition of the conic curves used in the fuselages of airplanes (Farin 2002: 2). There was interest in this digital conversion for two reasons: on the one hand, greater accuracy was achieved in the mass production of airplanes; on the other, it made it possible to store the curves in numerical tables, enabling their encrypted transmission and facilitating their protection (Alastair Townsend 2014: 54). The shapes were a direct consequence of the technical needs of and the strict compliance with the conditions set in the programs elaborated by the teams of engineers, very far from design processes focused on obtaining a suggestive perceptive experience. Once the geometry was defined, the function of the digital system was simply translating the curves to the production process.

In the late 1950s the automotive companies incorporated the new computers into their assembly lines, "facilitating the calculation of the own geometric transformations of the materials; behavior under stress, thermodynamics, aerodynamics of parts ..." (Bézier 1971: 207). However, the fairings were still designed by the traditional method, with paper drawings and models[1] because the traditional systems of the aeronautical industry did not prove effective for the translation of these curves and surfaces to digital entities easily mechanized. The problem was of great importance for the research and development teams, since its resolution meant being at the leading edge of the automation and control of tasks frontier within the industry. The technicians of Renault and Citroën, Pierre Étienne Bézier and Paul de Casteljau, sought the answer in the process of ideation of the shapes and not in their subsequent translation and integration, focusing on the development of algorithms capable of defining the geometric elements in a similar way to what the creative teams[2] of their companies had being doing.

The failure to implement the use of computer systems in the manufacture of aircrafts should not be attributed exclusively to the complexity of the curves and surfaces used in the exteriors of airplanes since the aeronautical industry was working with free-form curves of similar complexity since the mid-1950s. Their development in General Motors led to the definition of the B-splines in the late 1960s (Farin 2002: 7). The difficulty was the need to define geometric elements whose shapes were the result of the creative intentions of designers and not of mathematical calculations (Rogers 2001: 17), with a requirement for precision and development of the layout that, at that time, was not easy

[1] The design teams adapted their working methods to the complexity of the curves and surfaces demanded by the new industrial image, extending the method developed by the General Motors engineer Harley J. Earl, based, in addition to the traditional preliminary sketches, on the use of clay models as a suitable means to facilitate the development of the continuous surfaces characteristic of the new image, far from the mechanistic sincerity.

[2] The success of the proposal was based on the novelty of the approach and not on its technical complexity. Proof of this is that the management teams of the two companies questioned its feasibility because of the simplicity of the arguments on which it rested. In the case of Renault, given the simplicity of the method, the management team considered that, if valid, it would have already been developed by other teams. Rogers David quotes the famous phrase they used to address Bézier when dealing with the issue "if your system were that good, the Americans would have invented it first!" (Rogers 2001: 36). In Citroen, as reflected in his autobiography, at the time when de Casteljau presents the first results in parallel to those being developed in Renault, the criticisms from superiors and supervisors focus, among other issues, on considering his approach too simple to be a line of work worthy to follow (de Casteljau 1999).

to translate to the digital space in an automated way.

The mathematician Paul de Casteljau, hired by Citroën in 1959, developed a type of digital curve that could be controlled from external points, allowing its smooth deformation. As a result, he obtained curves similar to those defined by flexible templates, in an attempt to replace a physical tool with its digital equivalent.[3]

In parallel to and independently, the Renault engineer Bézier also proposed to introduce the digital definition in the design process of the curves that defined the shapes of fairings. Possibly as a result of the special sensitivity that his position as head of the company's design team brought to him, his proposal is closer to the working methods of the designers, assuming the analysis of the manual layout as the basic tool in the creative processes beyond the templates used for the geometric adjustment of the curves.

We have numerous studies on the operational capacity and the mathematical properties of the curves proposed by Bézier and their subsequent evolution. In 1970 the French engineer published an initial approximation to his work on the application of numerical analysis in the definition of curves and surfaces (Bézier 1970). But it was in 1971 when the principles governing the UNISURF system, developed by Renault under its direction, were published (Bézier 1971). From this publication, Bézier curves are present in many of the investigations on digital geometry. In addition to the articles by Bézier himself (Bézier 1972; 1974), noteworthy of mention are the works published A. Robin Forrest on the definition of curved surfaces (Forrest 1972a; 1972b) and the article published by William J. Gordon specifically dedicated to Bézier curves applied to free-form curves and surfaces (Gordon and Riesenfeld 1974). The evolution of the digital systems of mathematical determination of the curves moved Bézier's proposals to a marginal situation, but discussion of them remained present during the 1980s and 1990s in numerous studies[4] and even in subsequent investigations[5], although in this latter case from a historical point of view as a result of its pioneering character.

Therefore, from the point of view of analytical geometry, there is little to add to the study of Bezier's curves as regards either their definition or their historical evolution. However, there is no research on the implications of the drawing in their mathematical definition. The contribution of this article is to provide a new analysis of the origin of the mathematical structure of the first digital curves, focusing attention on the inspiration of their creators, and not so much on the application of certain well-known mathematical algorithms.

## 2 Bézier´s innovative approach

Thanks to the article on the UNISURF system (Bézier 1971), in which Bézier defines the general approach of his proposal for drawing digital curves, it is possible to investigate the situation that surrounded his work during the 1960s and the intentions underlying his strategy.

At first, the efforts focused on the attempt to translate the forms contained in full-scale clay models, developed following traditional methods, into digital geometrical entities. It was necessary to acquire the geometric data about these models using different types of peripherals and integrate it into the computers in order to use it in production processes.

[3]As Alastair Townsend (2014: 52) states, there seems to be a relationship with the drawing templates, although we do not have direct information about the author´s intentions.

[4] The compilation work carried out in (Bézier 1986) stands out among the published studies.

[5] (Forrest 1991) (Rogers 2001) (Biswas and Lovell 2008) (Farin 2002) and (Townsend 2014) are examples of the numerous publications that incorporate the analysis of Bézier curves and surfaces.

> The shape of a car body is first of all defined by means of a full-scale clay-model, very carefully hand-built. Drawings, master model and stamping tools must be in perfect accordance with the model; the process is costly and time-consuming. Accuracy and lag-time have been improved with help of numerical control which plays an important part in recording coordinates on models or drawings, marking off points on drawings, lofting, fairing curves and also defining and milling surfaces. ... There are some well-known devices to perform these operations: measuring machines ... curve followers, photogrammetric scanners, or mathematical methods to fair curves and surfaces (Bézier 1971: 207-209).

Nevertheless, this method was not providing the expected results. The model was a communicative artifact that had proved very effective. It was the expression of the designers' thoughts and contained a discourse full of intentions. However, the digital geometry extracted from the model was an automated interpretation of the physical object, therefore partial and limited. The digitally deduced geometric entities did not serve as transmission mechanics for the intentions embodied in the model, which were diluted in the distance that separated the tangible object from its supposedly objective digital expression. The result was a poor translation of the design team projects.

In general, companies focused their research on improving interpolation systems and extracting geometric information from the models, without understanding that the intentions and desires of the designers were hidden in a real object, camouflaged under its complexity, and that for its translation was necessary to be aware of their existence within that reality difficult to embrace. Bézier raised the issue from a more open approach, from a broader framework. He came to the conclusion that the problem of translating the ideas from the physical model to the digital geometry was unsolvable. It was not possible, at least at that time, to extract the intentions of the designers implicit in the sketches and clay models produced in the design departments.

> Most numerical methods tend to use n.c. [numerical control] to translate into figures the shape of a previously hand-made model, or its graphic definition by means of curves lofted with templates or splines. … This translation work always raises heavy difficulties because no infallible algorithm exists that will choose automatically the conditions that must be complied with by the solution looked for.
> An algorithmic method <u>should also be able to account for the smallest detail that expresses an intention or a want, sometimes implicit</u>, yet eliminate the details caused by fate or fault. Such a method would be very expensive and perhaps <u>impossible to create</u> (Bézier 1971: 209; my emphasis).

It was necessary for designers to express their ideas directly in the digital environment, just as they did with their sketches and models. If the whole process was gestated in this way, it would be guaranteed the presence of all the intentions of the design team in the digital expression, which would act directly as a mediating expression.

> On the top of this, Renault now uses it [numerical analysis] to help stylists define, through figures, any shapes they have devised. <u>So, numerical control is used in their conception process instead of taking part into their translation only</u>. …Renault have wanted to do without any translation work and have created a numerical definition method for the use of draughtsmen as well as designers (Bézier 1971: 207-209; my emphasis).

This was a simple approach, but it meant a completely new way of addressing the issue. The focus shifted from the final moment of the process, from the finished design, to the ideation process itself.

It was not the translation of a shape already designed; it was about moving the ideation process to the digital world.

Bézier used the action carried out by the designers during the ideation, which was none other than the act of drawing, as reference process. He knew that it had to be analyzed and simulated, just as other behaviors were simulated digitally. Behind the analytical expressions that describe the development of these digital curves, the concept of 'path' remains present as an intentional movement, an action that leaves the mark of the subject on the support, a link between thought and its expression through the hand.

The developed method was not completely applied from the time of its formulation due to the lack of adequate peripherals and, possibly, to the slow response of the computers at that time. For this reason, the team of engineers from Renault went on using the models to obtain the connecting points between the different sections of the compound curves.

> At the moment, coordinates are supplied to the computer through decade switches, a typewriter keyboard, and a tape reader. We will add the possibility of supplying the computer with the coordinates read on the encoders related with the motion of the pencil and optical viewer carrier (Bézier 1971: 211).

On these points, already in the digital field, the designers developed the curves that defined each section.

> Once the coordinates of a few points are picked up on each feature line, the designer transfers them on his drawing machine and chooses the vectors defining each curve segment that will help to calculate the path of the tracer (Bézier 1971: 211).

These shortcomings made it impossible to correctly carry out the first part of the method, but the validity of its approach was such that the type of curves developed by Bézier was used, from then on, as a procedure for drawing digital curves by hand in much of the vector design software. It was included in the PostScript code for the printing of curves, and in programs such as Adobe Illustrator, Corel Draw, Adobe Flash, Photoshop, etc. Even today, similar algorithms can be found in Corel Draw itself, or in recent software such as Rhinoceros, coexisting with the modern NURBS curves (Townsend 2014: 53).

An example of the validity of this type of curves in contemporary graphic design is the publication of the well-known manual dedicated to possible applications of Bezier´s curves, written by one of the most prestigious graphic designers, Von Glitschka, with the title *Vector Basic Training, a systematic creative process for building precision vector artwork* (Glitschka 2011). Among the numerous illustrations, I reproduce here the curious and paradoxical portrait of Bézier, made with his own curves (Fig. 1).

**Fig. 1** Portrait of Pierre Bézier. Image: (Glitschka 2011: 5)

As an example of the application of the Bezier´s curves in architecture, Fig. 2, 3 and 4 reproduces the design of the deck of a footbridge in Seville elaborated in 2016 by the architecture laboratory included in my professional studio. Once located, the flat curves, drawn one by one using Bezier´s curves, define the evolution of the surface

**Fig. 2** Deck of a footbridge in Seville, elaborated in 2016 by the architecture laboratory. Definition of Bezier´s curves

**Fig. 3** Deck of a footbridge in Seville, elaborated in 2016 by the architecture laboratory. Bezier´s curves located in position along the footbridge

**Fig. 4** Deck of a footbridge in Seville, elaborated in 2016 by the architecture laboratory. Resulting surface using Bezier´s curves as supports

## 3 Mathematical Formulation of the Movement of the Hand: Path and Time in Bézier's curves

Bézier himself published the basic mathematical formulation of his curves, first of a summary form (Bézier 1974) and later in a complete way (Bézier 1986). We know the simple mathematical development behind these curves,[6] and can look for the elements that confirm the exposed relationship between stroke and curve digital through movement.

As head of the design team, the engineer was familiar with the work of Renault designers, and it is even possible that he maintained a personal relationship with them during the development of the graphic method. This relationship justifies the knowledge about the curve tracing that emerges from the analysis of his proposal. Bézier understood that the drawing of a curve is an open action, in which, although maybe unconsciously, there are strategies focused on its control and adjustment that facilitate an execution close to the thought that gave rise to the gesture.

We know the author's interest in the aspects that control the plotting because it appears in the original text in which he analytically describes the curves. There are clear references to the importance of continuity between the different sections that make up the entire curve, to the tangency that determines the curve development, and to the possible adjustment of the curvature thanks to its relationship with the graphic method (Bézier 1974). Although in the development of the procedure the engineer reaches out to the definition of warped surfaces, the general approach is developed on flat curves composed of cubic curves, such as those included in his first article (Bézier 1971: 210).

The strategies studied are simple and we can summarize them in the following sequence shown in Fig. 5. The planned plotting is organized by defining the singular points, including the beginning and the end ($p$). We decide the limits that tangentially approach us to the definition of the line between points ($t$), as well as the greater or lesser tension in the curve when approaching those limits, defined geometrically through its curvature ($c$). The stroke is the manual action that runs through this *territory of conditions*; a dynamic action, open and without imposed geometries, translated into variable speeds of the gesture, executed paying attention to the singular areas, and adjusting to a greater or lesser extent to the suggested limits. The proposed digital trace (Fig. 6) assumes its open character in its formulation, tries to minimize the presence of previous geometries, and works with the *topography of conditions* described by this territory. The open character is translated into its construction by sections, the geometric freedom in the flexible mathematical equation of each section and the dynamics of the gesture in the definition of the interpolation support. The singular zones are defined by the successive marked of points that make up the interpolation ($p_1$, $p_2$, $p_3$), while the tangential approach (t) and its influence (c) are adjusted fixing the direction of the tangents and their module, through the definition of the non-interpolated points (1, 2, 2′, 3).

[6] The mathematical development is trivial, not only from the current perspective but also for any mathematician of that time. Keep in mind that the basis used by Bézier and de Casteljau was enunciated by Bernstein in 1912, and that the mathematical definition of spline curves was developed by Isaac Jakob Schoenberg in 1946. Therefore, the mathematical concepts used were chosen from an existing material.

**Fig. 5** Characteristics of the manual stroke

**Fig. 6** Characteristics of the digital trace

### 3.1 Composition by Sections and Flexible Mathematical Equations: Open Action and Geometric Freedom

To simulate the process without determining a closed result, the algorithm should allow the dynamic definition of the curve and in this way it should be generated by space-time sections, without the need for an accurate knowledge of the whole curve. Composite Bézier curves are constructed from sections defined by simple Bézier curves, allowing the modification of the trajectory during their plotting just as it occurs in manual plotting. The definition of the limit points of a certain section and of the tangents in those points determines the section and, only partially, the contiguous sections. The influence of some sections on others is due exclusively to the continuity conditions imposed on the connection between them ($G^1$).

Bézier defines the curves using Bernstein's functions:

$$P_1(u) = \sum_{i=0}^{m} S_i B_i(u)$$

*$B_i$´s being Bernstein's function*

$$B_i = C_i\, u^i (1-u)^{m-i}$$ (Bézier 1974: 139).

He also defines the $G^1$ link conditions:

> …the only requirement consists in having the first leg of a polygon collinear with the first leg of the other. Supposing the two first vectors are respectively $a_1$, $a_2$, $a'_1$ and $a'_2$, the conditions to fulfil to ensure osculation are:
>
> $$a'_1 = g \times a_1 \qquad for\ g < 0$$
>
> $$a'_2 = ha_1 + ka_2 \quad for\ k > 0$$
>
> $$k = \frac{n'}{n} \times \frac{n-1}{n'-1} \times g^2 \quad (*)$$
>
> *(*) n and n´ being the number of the polygon legs* (Bézier 1974: 143).

The geometric condition applied to the connection between the curves partially conditions the second curve with respect to the first. Beyond the coplanar character of the vectors that define the polygon, it is verified that the tangent is common and therefore that the points of the polygon are aligned on both sides of the contact point.

As I mentioned earlier, Bézier designed this procedure to be carried out graphically, using the light pen on the screen. In this way, the flexibility of the method makes sense, since the designer could choose the elements that define each section and build the curve dynamically.

The curves designed and drawn by the designers were not directed by previous mathematical relationships. They were not drawn by means of drawing instruments prepared to force geometric conditions, whether they were rulers or compasses; they were plotted with the movement of the hand, with greater freedom and less accuracy.[7] The traditional curves, mainly the conics, did not adapt well to these new entities whose origin was not a specific geometric place. Hence the need to choose flexible functions, even though they were not very precise:

> To be acceptable, this method must comply with several requirements: … It must allow the use of a large variety of curves. Straight lines, circles, conics or cubics are not enough to solve easily all problems met in our industry (Bézier 1971: 209).

### 3.2 Dynamic Interpolation and Interpolation Support: Approximation to the Imagined Curve and Variations in the Speed of Hand Movement

Since the objective was the translation of data into the digital domain, efforts were focused on finding the best curve to interpolate a series of chosen points on the clay models. It was a classic problem of analytic geometry: an approximation of an existing complex curve through an interpolation process by the use of a simple and manageable function. To apply the conventional systems, the curve had to exist as one of the initial data, in a static and finalist view of the problem, which was well suited to the forms resulting from mathematical calculations (Rogers 2001: 17).

The simulation proposed by the Renault team should also address the problem of obtaining an interpolated curve, but the difference was that, for Bézier, it was about approaching a curve that only existed in the thought of the artist. Interpolation was part of an expressive process rather than a simple translation. The draftsman projected his thought through the movement of the hand onto the digital support, thanks to the light pen. However, instead of leaving a continuous trace as the line of graphite left by a pencil, the action left marked milestones, singular places of transit, which composed the imagined curve step by step.

The interpolation, understood from a dynamic interpretation, required to make a series of decisions in the mathematical development of the interpolation system that distanced it from the usual methods. Conventional interpolation systems choose the position of the fixed points of the interpolated curve over the curve to be simulated. It starts from a set of abscissa s+1 called interpolation support, corresponding to s+1 ordinates. Together they define the endpoints of the segments to be interpolated (nodes). We define a vector space and on it a basis of n+1=s+1 functions in order to have the required number of equations to solve the problem:

$$B = \{B_0(x), B_1(x), \dots, B_n(x)\}.$$

The sought-after curve will be:

$$C(x) = a_0 B_o(x) + a_1 B_1(x) + a_2 B_2(x) + \cdots + a_n B_n(x).$$

Applying the conditions of interpolation, we have a system of $n$+1 equations and $n$+1 unknowns, in matrix form $B\ x\ a = y$.

For node interpolation support $N_i(x_i,y_i)$, $i$=0,1,…,$s$

[7] Tools such as flexible material strips were used for the forced plotting of curves not analytically defined. Their use allowed a clean stroke on curves already defined by the freehand stroke. They were instruments designed to force previous geometric relationships, implicit in the previous manual stroke, although they were not analytical relationships

$$a_0B_o(x_0) + a_1B_1(x_0) + a_2B_2(x_0) + \cdots + a_nB_n(x_0) = y_0.$$
…….
$$a_0B_o(x_s) + a_1B_1(x_s) + a_2B_2(x_s) + \cdots + a_nB_n(x_s) = y_s.$$

$$\begin{bmatrix} B_0(x_0) & \dots & B_n(x_0) \\ \dots & \dots & \dots \\ B_0(x_s) & \dots & B_n(x_s) \end{bmatrix} \begin{bmatrix} a_0 \\ \dots \\ a_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \dots \\ y_n \end{bmatrix}.$$

The selection of the interpolation support is free and prior to the solution of the problem and it influences the interpolation error. Depending on the values of the support, the solution of the system of equations changes, resulting in different sets of coefficients and consequently, different curves of interpolation $C(x)$, among which the optimal is selected.

In Fig. 7 we have chosen, on the curve that we want to simulate, an interpolation support formed by three nodes [$N_0$ (0,0), $N_1$ (1,1) and $N_2$ (1.5,0.5)], and a base of polynomials on which we define the interpolation curve $C(x) = a_0 + a_1x + a_2x^2$, with a number of coefficients that matches the number of nodes and allows solving the system of equations, whose result is the curve $C(x) = \frac{1}{0.75}(1.75x - x^2)$.

**Fig 7** Curve drawing by conventional interpolation

This method allows us to compute an optimized polynomial interpolation curve, but it continues responding to a global view of the problem. I start from a curve to simulate and then define the optimal support to solve the system and calculate the polynomial curve that fits best.

In the case of the composite Bézier curves, the parameter evolves taking concrete values for the points of connection between the different segments, points already fixed when drawing. The function that defines this support determines the value of the parameter at the points, not the points, and with it the relation between the times of plotting the curve for each segment. It is interesting to note that this is a process that occurs as the drawing advances. The first segment defines the reference value from which the rest of values are calculated. Each segment to be interpolated is a different curve, connected at the fixed points under restrictive conditions that define the continuity of the complete plotting

We start with the parametric formulation of the curve:[8]

For a segment $C(t) = \sum_{k=0}^{n} P_kB_k(t) \qquad t \in [0,1] \quad where\ \ P_k\ is\ a\ point\ (U_k, V_k)$

For several segments the formulation is the same, but it must be defined for an interval t∈ [$u_0$, $u_s$], each segment being ($u_0$, $u_1$), ($u_1$, $u_2$), ..., ($u_{s-1}$, $u_s$):

$$C_1(t) = \sum_{k=0}^{n} P_{1k}B_k\left(\frac{u-u_0}{u_1-u_0}\right) \qquad t \in [u_0, u_1].$$

$$C_2(t) = \sum_{k=0}^{n} P_{2k}B_k\left(\frac{u-u_1}{u_2-u_1}\right) \qquad t \in [u_1, u_2].$$

…..

$$C_s(t) = \sum_{k=0}^{n} P_{sk}B_k\left(\frac{u-u_{s-1}}{u_s-u_{s-1}}\right) \qquad t \in [u_{s-1}, u_s].$$

[8] Bézier curves can be developed in any of the usual systems of equations, in their implicit, explicit or parametric form, but they are designed to work in the latter. Bézier makes direct reference to this formulation in his texts as a feature that makes the curve independent of spatial coordinates, facilitating its reading as a geometric object (Bézier 1971:211; Bézier 1974: 130).

Generally:

$$C_i(t) = \sum_{k=0}^{n} P_{ik} B_k \left(\frac{u-u_{i-1}}{u_i-u_{i-1}}\right) \quad t \in [u_{i-1}, u_i]$$ where $i$ is the segment and $P_k$ is a point $(U_k,V_k)$.

Deriving:

$$C'_i(t) = \frac{1}{u-u_{i-1}} \sum_{k=0}^{n} P_{ik} B'_k \left(\frac{u-u_{i-1}}{u_i-u_{i-1}}\right) \ t \in [u_{i-1}, u_i].$$

We deduce the conditions of continuity $C^1$ by matching the derivatives corresponding to the end of one of the segments and at the beginning of the following:

$$C'_i(u_i) = \frac{1}{u-u_{i-1}} \sum_{k=0}^{n} P_{ik} B'_k(u_i) = \frac{1}{u_i-u_{i-1}} n(P_{in} - P_{i(n-1)}) \quad t \in [u_{i-1}, u_i].$$

$$C'_{i+1}(u_i) = \frac{1}{u-u_i} \sum_{k=0}^{n} P_{ik} B'_k(u_i) = \frac{1}{u_{i+1}-u_i} n(P_{(i+1)1} - P_{(i+1)0}) \quad t \in [u_i, u_{i+1}].$$

Matching:

$$\frac{1}{u_i-u_{i-1}} n\left(P_{in} - P_{i(n-1)}\right) = \frac{1}{u_{i+1}-u_i} n(P_{(i+1)1} - P_{(i+1)0}) \quad t \in [u_{i-1}, u_i].$$

The continuity conditions at the joints between segments are:

$\left(P_{in} - P_{i(n-1)}\right)\mu = \left(P_{(i+1)1} - P_{(i+1)0}\right)$, which is the common tangent at the link point.

$\frac{u_{i+1}-u_i}{u_i-u_{i-1}} = \frac{\|P_{(i+1)1}-P_{(i+1)0}\|}{\|P_{in}-P_{i(n-1)}\|} = U_i,$ ratio between values or $t$ in adjacent segments.

As in the previous case, the choice of the support is free, but we choose $s$+1 nodes in the variable $t$, defined by the ratio $U_i$. We may choose different ratios of $t$ without affecting the points that define the segments, so that the choice of the support does not change their positions. It only changes the transit speed of the parameter by those points and with it the curvature in the joining zones.

The choice of the support is based on the geometric relationship between the positions of the points. A uniform support can be used, so that the change in the parameter is the same for each section of the support. However, this election assumes that for sections with close extreme points, the curve must be opened a lot to compensate the space traveled with the elapsed time. To correct these problems and simulate better the manual stroke, either we work with step values for the parameter proportional to the distances between the positions of the points (Chord length), or we make the distances proportional to the square of the step value (centripetal support).[9] Its use is due to the fact

[9] Functions of parameter $t$:

1. Uniform support: equal time periods between segments:
   $U_i$=1.
2. Chord length support: different times in the plotting of each segment of the curve depending on the position of the end points, achieving a movement adapted to the dimensions of each segment, approximating spaces and times:
   $U_i = \frac{\|P_{(i+1)n}-P_{(i+1)0}\|}{\|P_{in}-P_{i0}\|}$.
3. Centripetal support: it captures the "centripetal" force exerted by the artist to compensate for the centrifugal force caused by the drawings of the curve in the short segments, so that the trajectory opens somewhat less than in the case of a continuous movement based on a uniform support, but something more than the rigorous and excessively homogeneous trajectory generated by the chord length support. Although seldom used due to its high computational cost, this support function goes a step further in the interpretation of the digital curve as a drawing, simulating the control that the draftsman exerts on his hand through the rectification of the speed applied to the stroke and with it on its curvature:

that, at present, the curves are plotted exclusively from the points that define the interpolation segments, leaving the rest of the curve in the hands of the equation of the support. However, in the case of Bézier, the behavior of the curve between the connecting points was controlled through the points of the polygon outside the curve. Therefore, the support was not relevant, and the choice was the simple uniform support of low computational cost, as implied by author's texts (Bézier 1974: 143).

In Fig. 8, (Phase 1), I have drawn a first segment between the first two points of the previous interpolation support ($N_0 \equiv P_{10}$ and $N_1 \equiv P_{12}$) by adding a control point to shape the curve ($P_{11}$).

The formulation corresponds to a Bézier curve of grade 2 with control points $P_{10}(0,0)$, $P_{11}(0.5,1)$, $P_{12}(1,1)$:

In Fig. 9, (Phase 2), I have drawn the second segment directly, marking the end, point $P_{22}$, since the interpolation is directed by the variation of the support, in this case uniform.

To maintain $C^1$ continuity between the two segments, the tangent must be the same in $P_{12} \equiv P_{20}$ and the following relationship must be fulfilled:

$$U_1 = \frac{u_2 - u_1}{u_1 - u_0} = \frac{\|P_{21} - P_{20}\|}{\|P_{12} - P_{11}\|} = \frac{\|P_{21} - P_{20}\|}{0.5}$$ where $u_i$ are step values of the parameter $t$.

For uniform support, $U_1 = 1 = (u_2 - u_1)/(u_1 - u_0)$, therefore $\|P_{21} - P_{20}\| = 0{,}5$. The choice of the control point $P_{21}$ is implicit in the choice of the uniform support. The points $P_{11}$, $P_{12} \equiv P_{20}$ and $P_{21}$ must be aligned and the distance between $P_{20}$ and $P_{21}$ must be 0.5, which defines the point $P_{21}(1.5,1)$.

It is only necessary to define the curve of the second segment, according to the control points $P_{20}(1,1)$, $P_{21}(1.5,1)$, $P_{22}(1.5,0.5)$.

In Fig. 10, (Phase 3), the second segment is drawn again, but for different support functions:
In the case of support based on the chord length, the same reasoning would apply, but with another support relationship between segments. The distance between points $P_{10}$ and $P_{12}$, would be $\|P_{12} - P_{10}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$. When marking the end point of the second section $P_{22}$, we know the distance between points $P_{20}$ and $P_{22}$, $\|P_{22} - \mathrm{P}_{20}\| = \sqrt{0.5^2 + 0.5^2} = \sqrt{0.5}$.

If we take support values proportional to the chords:

$$\frac{u_2 - u_1}{u_1 - u_0} = \sqrt{\frac{0.5}{2}} = \sqrt{0.25} = 0.5, \quad \|P_{21} - P_{20}\| = 0.25, \ and\ \ P_{21}^{B}(1.25,1).$$

In the case of centripetal support, the squared supports are proportional to the length of the chords:

$$\frac{(u_2 - u_1)^2}{(u_1 - u_0)^2} = 0.5, \qquad \frac{u_2 - u_1}{u_1 - u_0} = \sqrt{0.5} = 0.707 \quad \|P_{21} - P_{20}\| = 0.3535\,, \ \ P_{21}^{C}(1.3535,1).$$

**Fig. 8** Curve drawing by interpolation proposed by Bezier. Phase 1. First segment

**Fig. 9** Curve drawing by interpolation proposed by Bezier. Phase 2. Second segment with uniform support

$$U_i = \frac{\sqrt{\|P_{(i+1)n} - P_{(i+1)0}\|}}{\sqrt{\|P_{in} - P_{i0}\|}}.$$

**Fig. 10** Curve drawing by interpolation proposed by Bezier. Phase 3. Second segment with others supports

### 3.3 Tangent Direction: Control of the Feasible Range of Hand Movements

While conventional interpolation closed the question, although obviously wrongly, the *dynamic* interpolation raised by Bézier made it possible to intentionally define the imaginary join points and the speeds of the gesture reproduced in the support function, but left open the stroke adjustment at the beginning, the end and the passing through singular points.

As in the case of drawing the curve on a conventional physical support, the control of the drawing of the digital curve in the critical points was carried out thanks to the plotting – at least imagined – of the tangents in those points. As is well known, the control of the curved paths by using polygons tangent to them is a common operation in drawing, especially if you look for a certain precision in the freehand paths. Bézier directly simulates this form of control and incorporates it into the logic of its digital curves as one of its basic elements, using the endpoints of the tangents to the curve at their ends ($P_1$ y $P_{n-1}$), as multiplying coefficients of the Bernstein functions ($B_k$).

The coefficients of the linear combination that define simple Bezier curves form a polygon that marks the tangents at their extreme points (Bézier 1971: 210). When combining these simple curves into composite Bézier curves, the last and first segments of the different polygons should belong to the same line, so that the direction of the tangent at the point of passage is equal in the two adjacent sections (Bézier 1986: 41). This provides smoothness and geometric continuity ($G^1$) to the composite curve (see the mathematical formulation described in Section 2). "There is no difficulty in blending two curves because the only requirement consists in having the first leg of a polygon collinear with the first leg of the other" (Bézier 1974: 143). In this way the curve is traced following the imaginary join points, at the speed defined by the support and adjusting to those tangents that control the delicate areas of the gesture: at the beginning, the transit and the end.

### 3.4 Module of the tangent - control of the softness in the movement of the hand

When we perform the adjustment in the tracings on conventional supports through tangents, we follow our mental image of the curve to decide the level of influence that the tangent exerts on it. We choose the distance to the control point; we begin to adjust the curve to, giving it greater smoothness or tension, all in an approximate way and without any calculation. In the case of Bézier curves, the influence of the tangent is reflected in the value of its module in the proximities of the control point, acting as a controller of the change of curvature at that point. The relationship between the influence of the tangent and the curvature is simple. The greater the influence, the closer to the tangent the curve is so that its curvature reduces, approaching the zero-limit value corresponding to the curvature of the tangent line.

The choice of the direction and module of the tangent is made during the drawing of the curve, with the draughtsman defining the position of the two external control points of the cubic curve. Given the continuity properties of composite Bézier curves, the positions of these points contiguous to the connecting point between segments directly determine the direction of the tangent and its module: "Functions $f_{i,m}$ are such that, on the initial point ($u$=0), the curve is tangent to $a_1$, and its curvature is only related with $a_1$ and $a_2$" ($a_i$ are points of the non-interpolated polygon) (Bézier 1974: 132).[10]

[10] There is a different treatment of this issue in the 1974 and 1986 documents. In (Bézier 1974) he proposed the control of the curvature through the non-interpolated points, while in (Bézier 1986: 41) he seems to assume the modifications

The curvature at any point can be expressed as:

$$C_r = \frac{\|C'(x)\ X\ C''(x)\|}{\|C'(x)\|^3} = \frac{1}{\|C'(x)\|^2}\ \|C''(x)\| \sin w.$$

As we saw in section 3.2, starting from the formulation of the Bézier curve and taking into account the change of variable adapting it to the composite curve, we can compute the derivatives at the connecting point $u=u_1$.

The first derivative at the extreme point:

$$C'_i(u_i) = \frac{1}{u-u_{i-1}} \sum_{k=0}^{n} P_{ik} B'_k(u_i) = \frac{1}{u_i-u_{i-1}} n(P_{in} - P_{i(n-1)}) \quad t \in [u_{i-1}, u_i].$$

For the concrete curve $C_1$:

$$C'_1(u_1) = \frac{\mathrm{n}}{u_1-u_0}(P_\mathrm{n} - P_{\mathrm{n}-1}) = \frac{\mathrm{n}}{u_1-u_0} B.$$

The second derivative, deriving the first $C'_1$:

$$C''_1(u_1) = \frac{n(n-1)}{(u_1-u_0)^2}\ (P_n - 2P_{n-1} + P_{n-2}) = \frac{n(n-1)}{(u_1-u_0)^2}\ [(P_n - P_{n-1}) + (P_{n-2} - P_{n-1})] = \frac{n(n-1)}{(u_1-u_0)^2}\ \mathrm{A}.$$

Substituting:

$$C_r = \frac{1}{n^2\|B\|^2 \big/ (u_1-u_0)^2}\ \frac{n(n-1)}{(u_1-u_0)^2} \|A\|\ \sin w = \frac{(u_1-u_0)^2}{n^2\|B\|^2}\ \frac{n(n-1)}{(u_1-u_0)^2} \|A\|\ \sin w = \frac{(n-1)}{n}\ \frac{1}{\|B\|^2} \|A\|\ \sin w = \boldsymbol{\frac{(n-1)}{n}\ \frac{1}{\|B\|^2} A_p}\ .$$

Where:

$$A = (P_n - P_{n-1}) + (P_{n-2} - P_{n-1}), \qquad B = (P_n - P_{n-1}), \qquad A_p = \|A\| \sin w.$$

In Fig. 11, we show vectors $A$ and $B$. The distance from point $P_{n-1}$ to the position $P'_{n-1}$, decreases the curvature at point $P_n$ since it increases the module of the vector $B$ ($P_n$-$P_{n-1}$), $\|B\|$, and, although vector A is transformed into A`, its projection over the normal line, represented by $A_p = \|A\| \sin w$, does not change. Therefore, when moving the non-interpolated control point ($P_{n-1}$), the second curve, $C_2$, presents less curvature at $P_n$ than the first, $C_1$, increasing the influence of the tangent on the path.

**Fig. 11** Relationship between the curvature and the position of the control points

## 4 The Commitment to Flexibility: Splines Curves

Bézier curves were surpassed by the spline curves, and with that development, the relationship with manual plotting vanished. The structure of the curve in segments, the common tangents and the value of its module, which had been basic elements in the simulation, lose their meaning in the new curves, so that the control points are freed from the points that define the different segments (their alignment is not necessary), and they compose a polygon attached to the curve that allows its local modification by simply moving its vertices. There are no segments connected by continuity conditions but intervals of influence of the transformative actions of the draughtsman. The geometry of the curve is governed as if an existing flexible element were deformed. Tangents are only present at the ends of the curves, not inside them, and as long as they are open curves.

proposed by the new conditions of the spline curves, considering the equality of the curvatures in the links between curves.

The moment of the layout is of no interest in spline curves. What is interesting is the suitability of the geometric entity to later transformation. The breakpoints, which substitute the connecting points of the sections, only define intervals of influence of the control points, delimiting the extension of the curve deformation caused by the action of the draftsman. The geometry of each section is conditioned by the number of B-spline basis functions active in the interval, and, therefore, by the coefficients of the linear combination that multiply these functions, or what is the same, by the corresponding control points (Townsend 2014: 54).

However, we continue to count on Bézier curves in cases where we need to manually draw digital curves. This is why we still find them in current design programs. We may find them under the name of curve handles, Bézier tools or directly as Bézier curves, but in any case we face a mathematical formulation with almost forty years of history. This impressive record is a consequence of the success in emulating the ancient and imperishable act of drawing.

**Acknowledgments**

All figures are by the author unless specifically noted.

## References

Bézier, Pierre Étienne. 1970. Use of numerical control in the car industry for body drawing and stamping tool milling. In: *Advances in machine tool design and research 1969.* Proceedings of the 10th International M.T.D.R. Conference, University of Manchester Institute of Science and Technology, September 1969, eds. S. A. Tobias and F. Koenigsberger, 677-685. Pergamon Press.

Bézier, Pierre Étienne. 1971. Example of an Existing System in the Motor Industry: The UNISURF System. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 321(1545): 207-218.

Bézier, Pierre Étienne. 1972. Numerical control in automobile design and manufacture of curved surfaces. In: *Curved Surfaces in Engineering*, ed. I. J. Brown. London: IPC Science and Technology Press.

Bézier, Pierre Étienne. 1974. Mathematical and practical possibilities of UNISURF. In: *Computer Aided Geometric Design*, eds. Robert E. Barnhill and richard F. Riesenfeld, 127-152. London: Academic Press.

Bézier, Pierre Étienne. 1986. *The mathematical basis of the UNISURF CAD system*. London: Butterworths.

Biswas, Sambhunath and Brian Lovell. 2008. *Bézier and splines in image processing and machine vision*. London: Springer-Verlag London Limited.

De Casteljau, Paul de Faget. 1999. *De Casteljau's autobiography: My time at Citroën*. Computer Aided Geometric Design 16: 583-586.

Farin, Gerald E. 2002. A History of Curves and Surfaces in CAGD. In: *Handbook of Computer Aided Geometric Design*, eds. M.-S. Kim, J. Hoschek, and G. Farin, 1-22. Elsevier.

Forrest, A. R. 1972a. *Interactive Interpolation and Approximation by Bezier Polynomials*. The Computer Journal. Oxford University Press, vol. 15, nº.1: 71-79.

Forrest, A. R. 1972b. *On coons and other methods for the representation of curved surfaces*. Computer Graphics and Image Processing 1(4): 341-359.

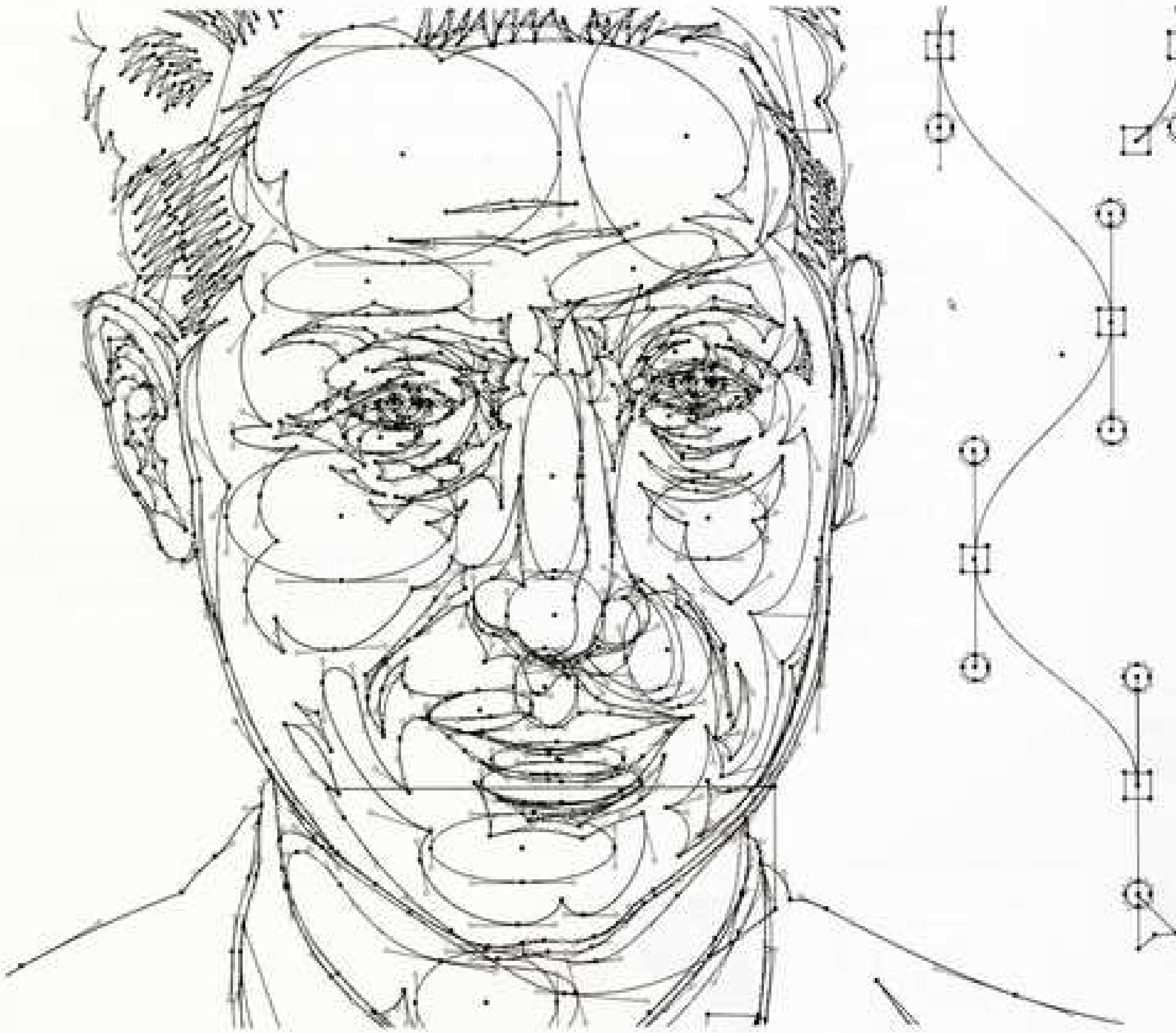
Forrest, A. R. 1991. Computational Geometry in Practice. In: *Fundamental Algorithms for Computer Graphics*, ed. Rae Earnshaw, 707-724. Berlin/Heidelberg: Springer.

Glitschka, Von. 2011. *Vector Basic Training, a systematic creative process for building precision vector artwork*. Berkeley: New Riders.

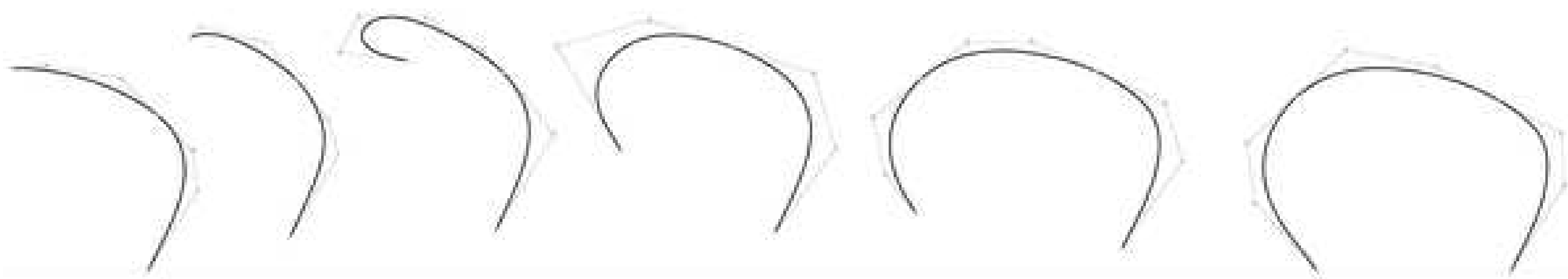
Gordon, William J. and Richard F. Riesenfeld. 1974. Bernstein-Bézier Methods for the Computer-Aided Design of Free-Form Curves and Surfaces. *Journal of the ACM* 21(2): 293-310.

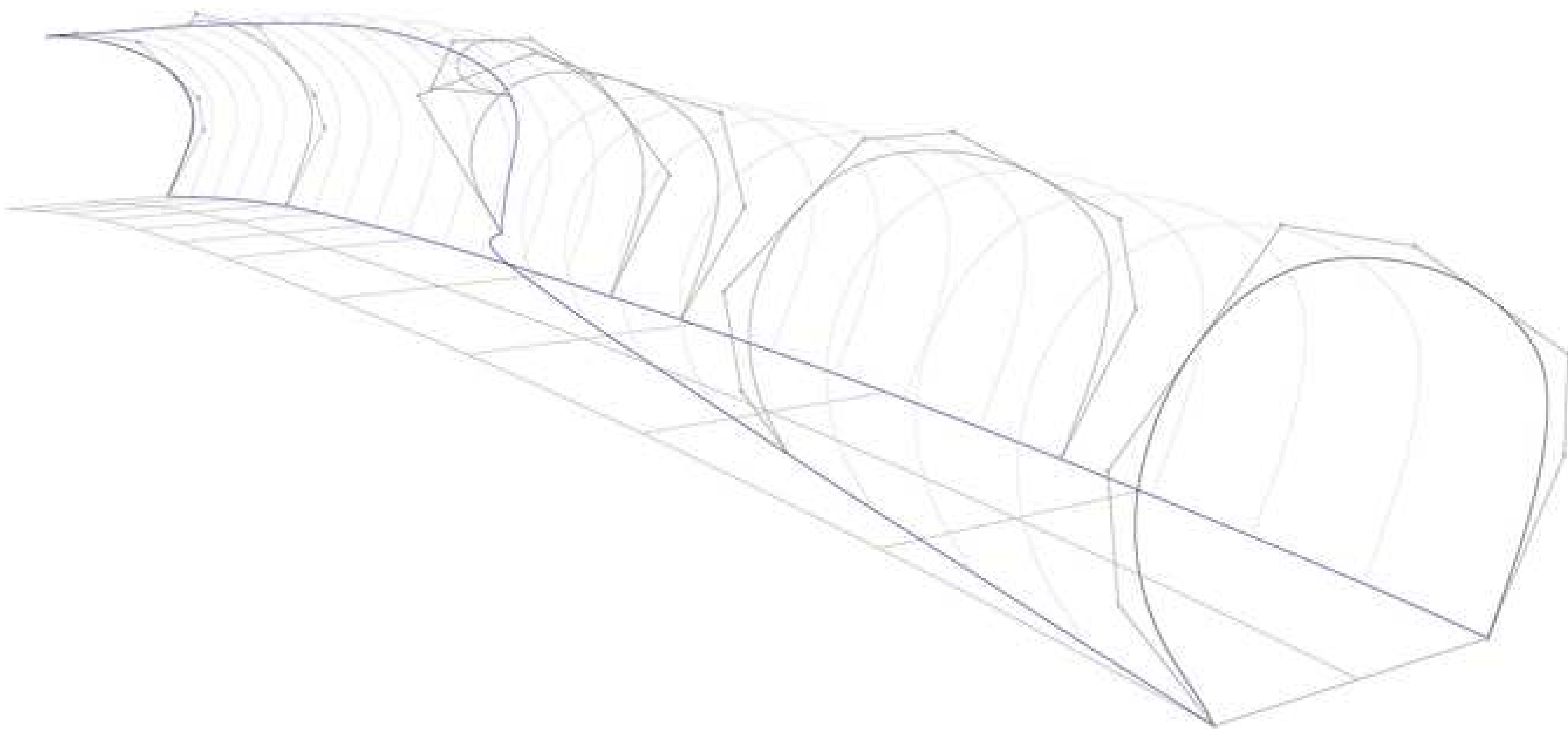
Rogers, David F. 2001. *An Introduction to NURBS With Historical Perspective*. Morgan Kaufmann Publishers.

Townsend, Alastair. 2014. On the spline: A Brief History of the Computational Curve. *International journal of interior architecture + spatial design* 3: 49-59.



Thousands of ironic Bézier curves make up my Pierre Bézier illustration.





Figure

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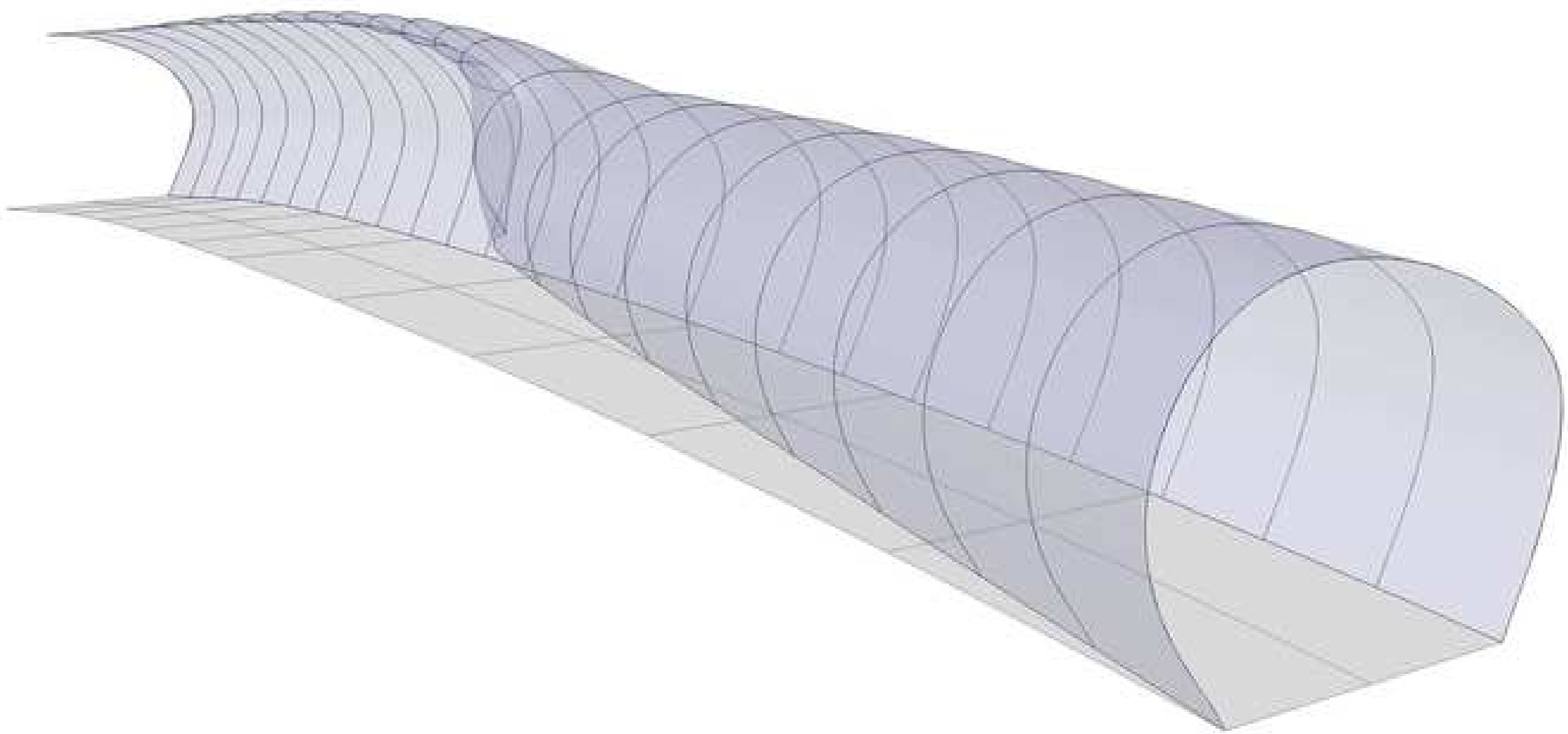
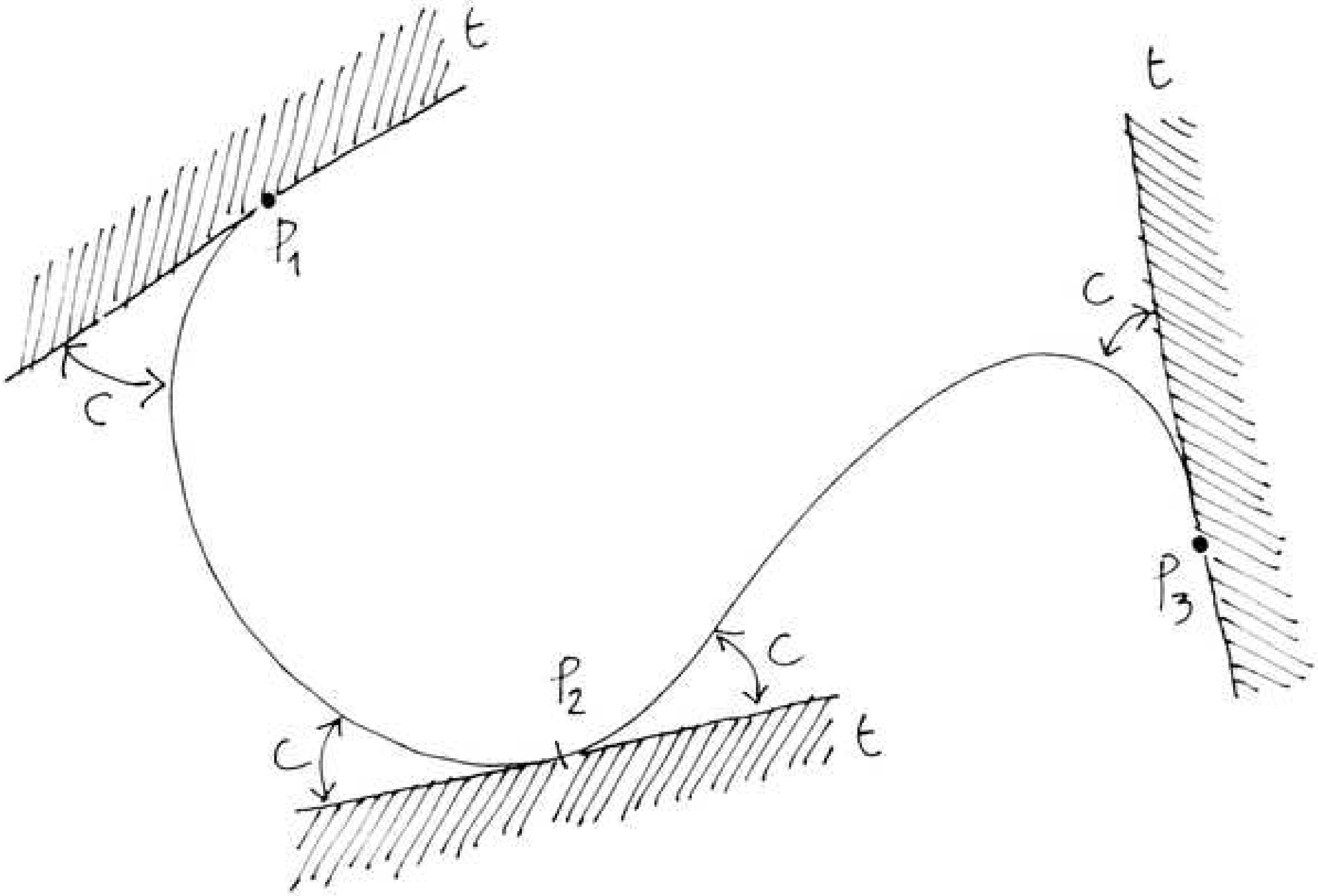
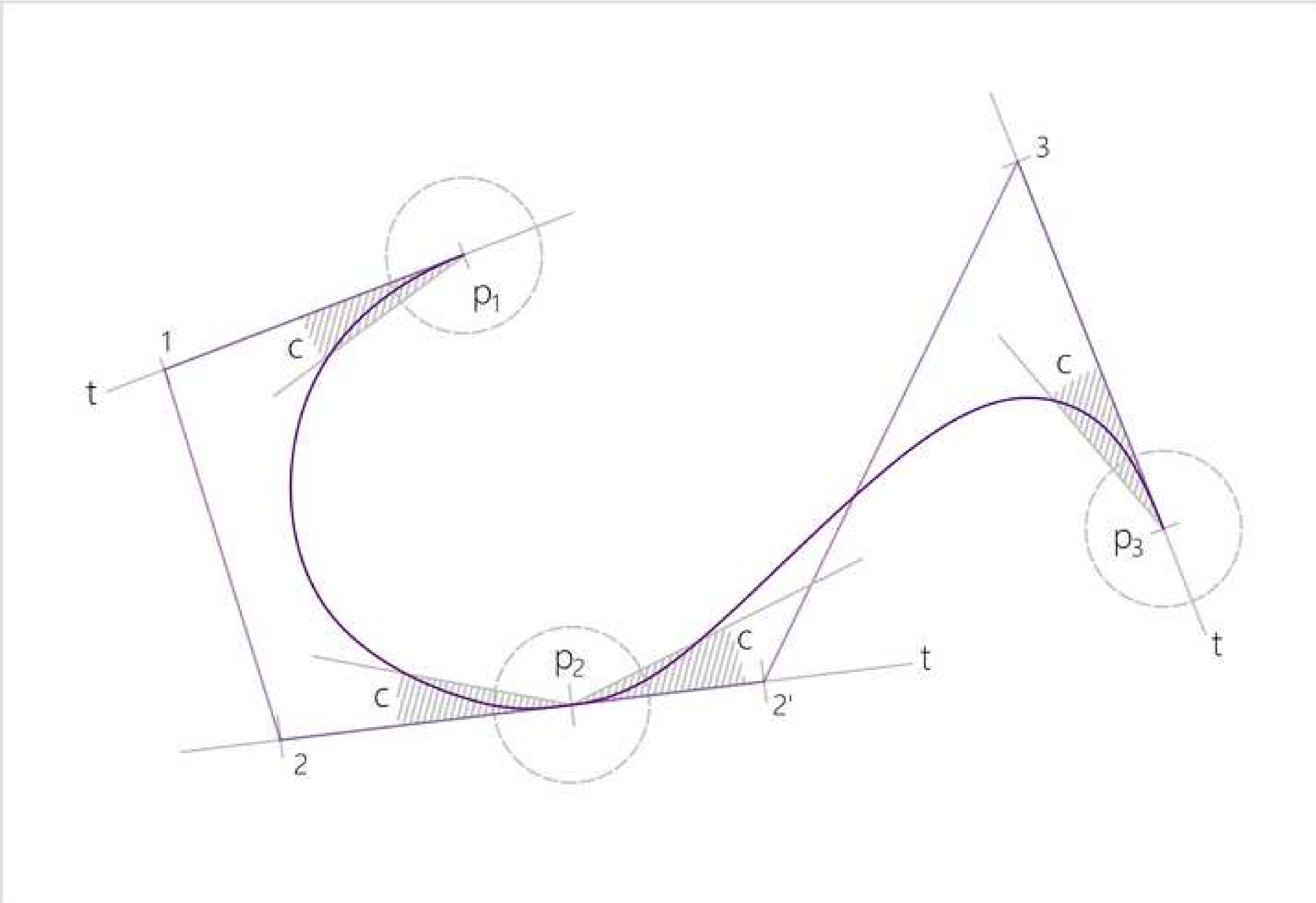
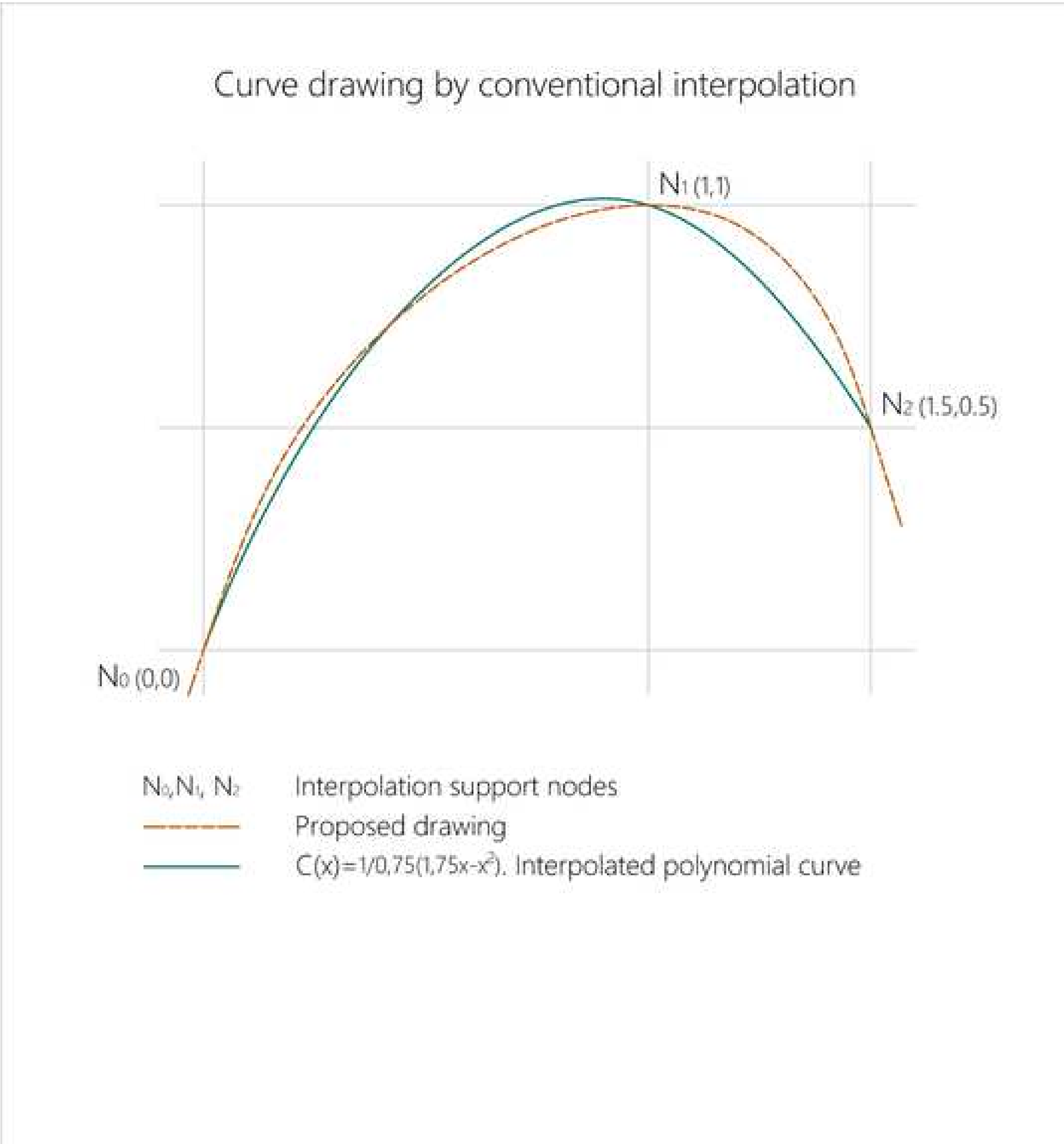


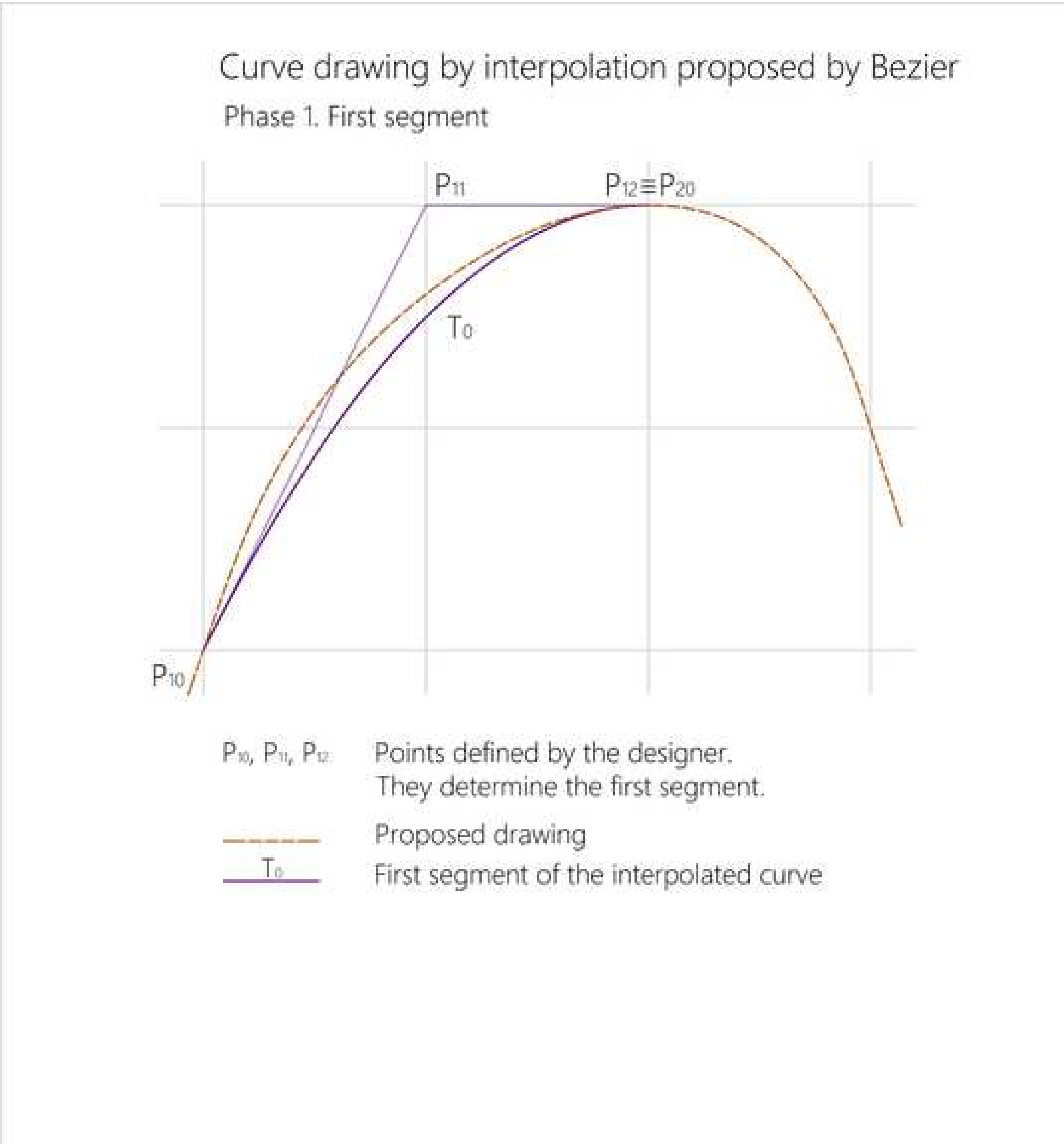

t
P1
c
t
c
P3
c
P2
c
t

Figure



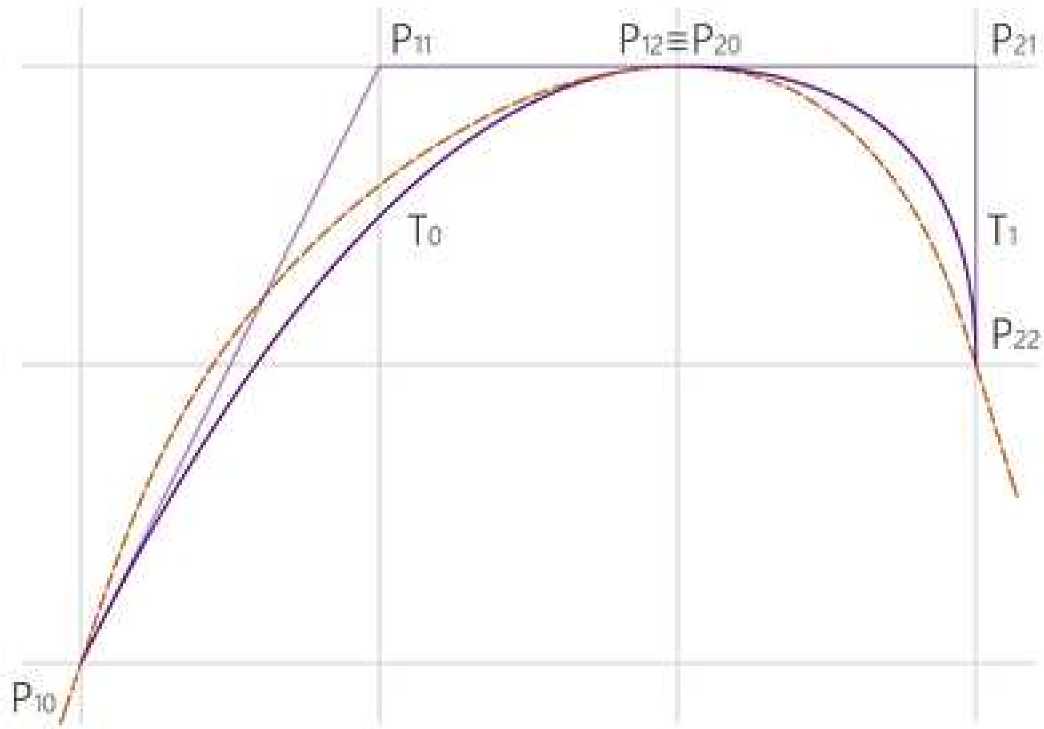



Curve drawing by conventional interpolation
N1 (1,1)
N2 (1.5,0.5)
N0 (0,0)
N0,N1, N2 Interpolation support nodes
Proposed drawing
C(x)=1/0,75(1,75x-x2). Interpolated polynomial curve


Curve drawing by interpolation proposed by Bezier
Phase 1. First segment
P11
P12≡P20
T0
P10
P10, P11, P12
Points defined by the designer.
They determine the first segment.
Proposed drawing
T0
First segment of the interpolated curve

Curve drawing by interpolation proposed by Bezier

Phase 2. Second segment with uniform support




$P_{10}$, $P_{11}$, $P_{12}$ — Points defined by the designer. They determine the first segment.

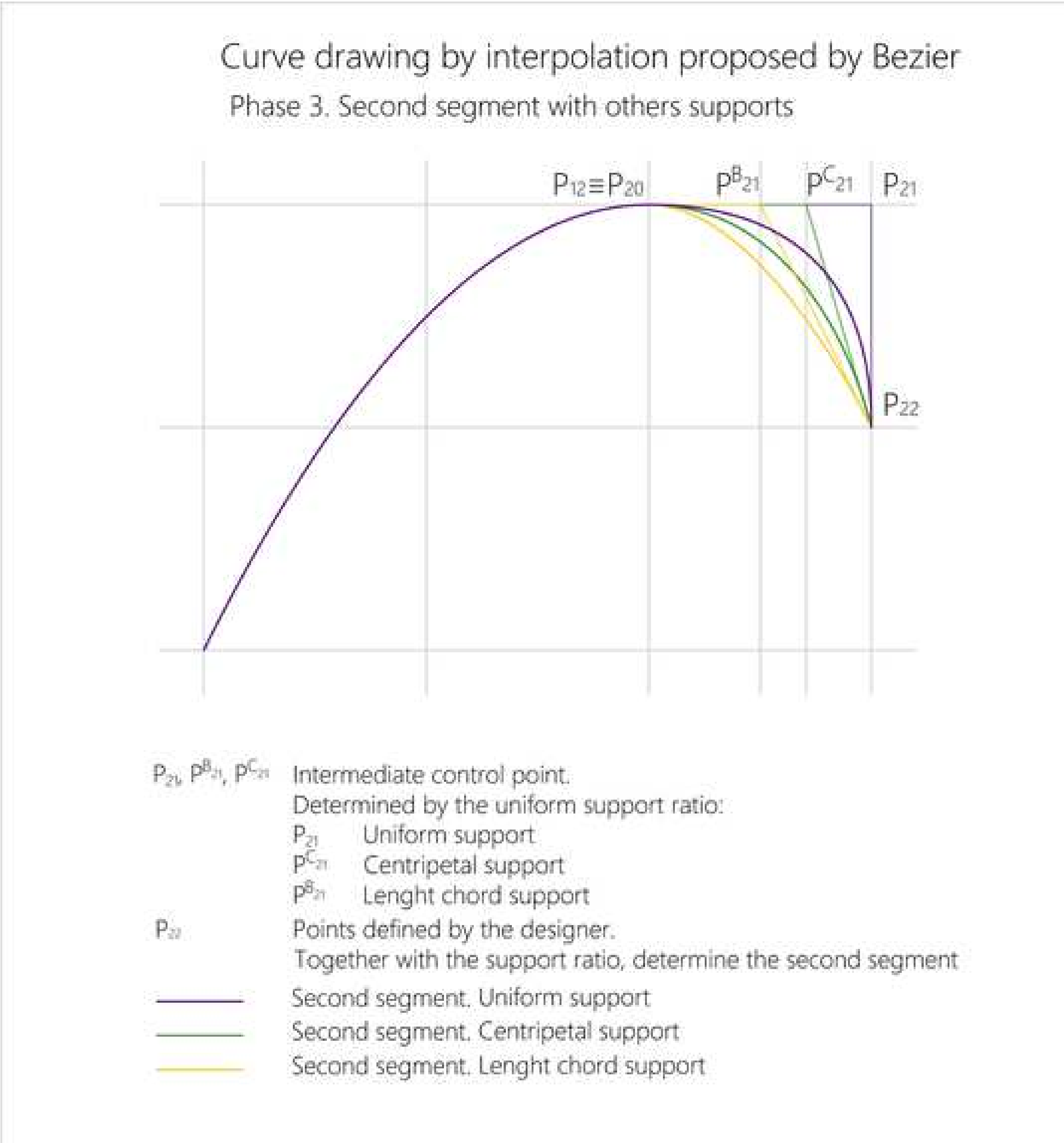
$P_{21}$ — Intermediate control point. Determined by the uniform support ratio

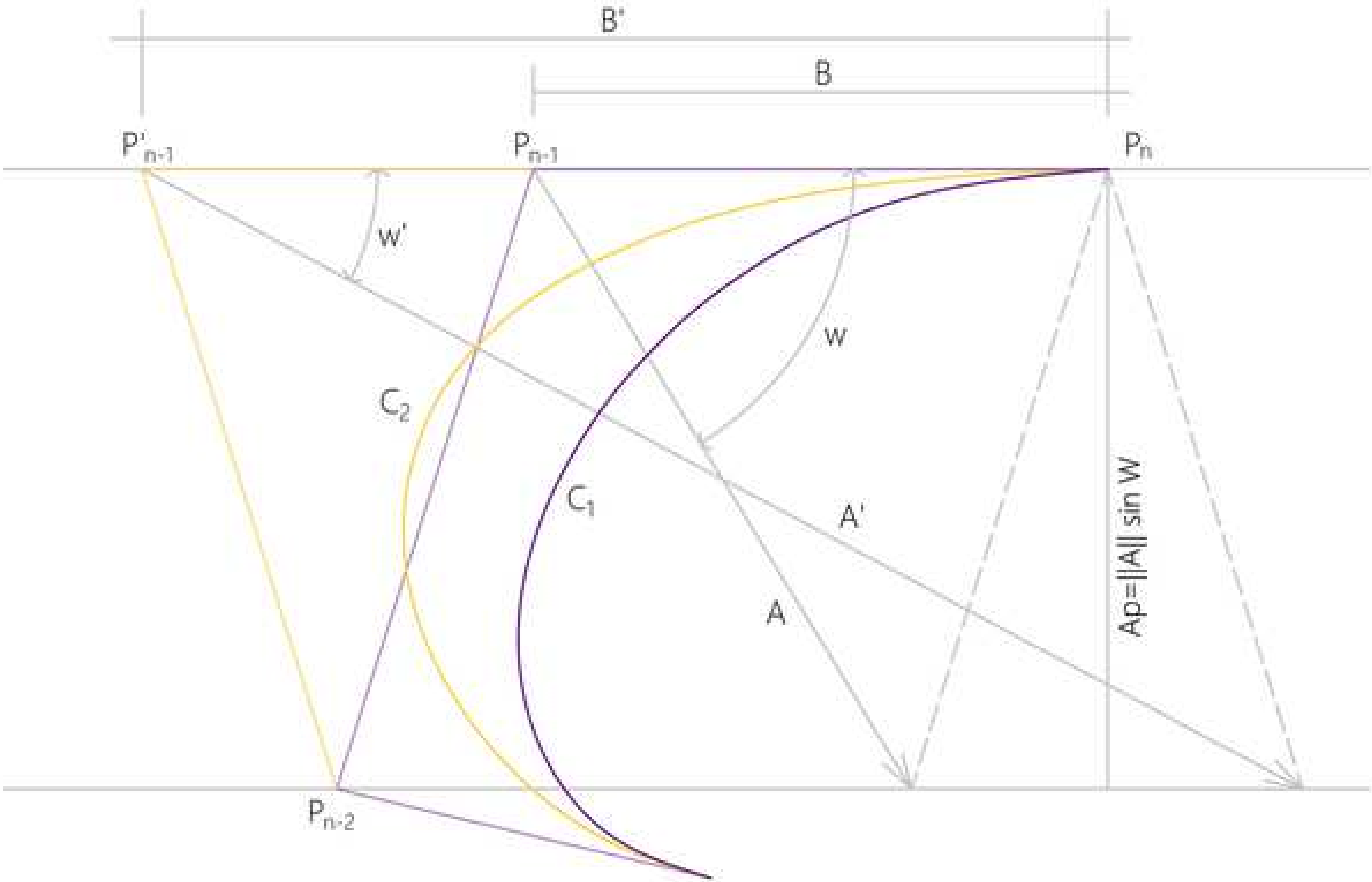
$P_{22}$ — Points defined by the designer. Together with the support ratio, determine the second segment

(dashed orange line) Proposed drawing

$T_0$ — First segment of the interpolated curve

$T_1$ — Second segment of the interpolated curve


Curve drawing by interpolation proposed by Bezier
Phase 3. Second segment with others supports
$P_{12} \equiv P_{20}$
$P^{B}_{21}$
$P^{C}_{21}$
$P_{21}$
$P_{22}$
$P_{21}$, $P^{B}_{21}$, $P^{C}_{21}$ Intermediate control point.
Determined by the uniform support ratio:
$P_{21}$ Uniform support
$P^{C}_{21}$ Centripetal support
$P^{B}_{21}$ Lenght chord support
$P_{22}$ Points defined by the designer.
Together with the support ratio, determine the second segment
Second segment. Uniform support
Second segment. Centripetal support
Second segment. Lenght chord support




Relationship between the curvature and the position of the control points