ON THE DESCRIPTION OF THE LEIBNIZ ALGEBRAS WITH NILINDEX $n-3$

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Abstract. In this paper we present the classification of a subclass of naturally graded Leibniz algebras. These n-dimensional Leibniz algebras have the characteristic sequence equal to $(n-3, 3)$. For this purpose we use the software *Mathematica*.

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1. Introduction

Leibniz algebras are one of the new algebras introduced by Loday [11], [12] in connection with the study of periodicity phenomena in algebraic K-theory. Leibniz algebras have been introduced as a "non-antisymmetric" analogue of Lie algebras. A Leibniz algebra L is a vector space equipped with a bracket [-,-] satisfying the identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

If the antisymmetric relation is assumed, this identity is equivalent to the Jacobi identity. Hence, a Lie algebra is a Leibniz algebra. It is well known that the natural gradation of nilpotent Lie and Leibniz algebras is very helpful in investigating their structural properties. A remarkable fact of the naturally graded algebras is the relative simplicity of the study of the cohomological properties, (see for example [6]- [10] and [13]).

Recently, some papers are focused to the study of some interesting families of Leibniz algebras, such as p-filiform and quasi-filiform Leibniz algebras. These algebras have their characteristic sequences equal to $(n-p, 1, 1, ..., 1)$ and $(n-2, 2)$ with $dim(L) = n$, [4]–[5].

Naturally graded p-filiform Leibniz algebras are already classified in [2] and [4]. The classification of naturally graded nul-filiform and filiform Leibniz algebras reader can find in [1]. The quasi-filiform n-dimensional Leibniz algebras have characteristic sequence $(n-2, 1, 1)$ (the case of 2-filiform) or $(n-2, 2)$ [3] and [5].

For a given Leibniz algebra L we define the descending central series as follows:

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1.$$

If there exists a natural number s such that $L^s = 0$, then the Leibniz algebra L is said to be nilpotent and minimal such number is called the nilindex of the algebra L.

Bellow we present a gradation closely related to the descending central series.

Let L be a nilpotent Leibniz algebra with nilindex s. We put $L_i = L^i/L^{i+1}$ for $1 \leq i \leq s-1$, and $grL = L_1 \oplus L_2 \oplus \cdots \oplus L_{s-1}$. It is easy to check embedding

$[L_i, L_j] \subseteq L_{i+j}$ and therefore, the algebra grL is graded algebra, which is called the naturally graded Leibniz algebra.

Let x be a nilpotent element of the set $L \setminus L^2$. For the nilpotent operator of right multiplication R_x we define a decreasing sequence $C(x) = (n_1, n_2, \dots, n_k)$, which consists of the dimensions of Jordan blocks of the operator R_x. On the set of such sequences we consider the lexicographic order, that is, $C(x) = (n_1, n_2, \dots, n_k) \leq C(y) = (m_1, m_2, \dots, m_s) \Longleftrightarrow$ there exists $i \in \mathbb{N}$ such that $n_j = m_j$ for any $j < i$ and $n_i < m_i$.

The sequence $C(L) = max\ C(x)_{x \in L \setminus L^2}$ is called characteristic sequence of the algebra L. If $C(L) = (1, 1, \dots, 1)$ then evidently, the algebra L is abelian.

The set $R(L) = \{x \in L \mid [y, x] = 0 \text{ for any } y \in L\}$ is said to be a right annihilator of the algebra L.

In this work we classify a subclass of naturally graded Leibniz algebras with nilindex $n - 3$. In case of Leibniz algebras with nilindex equal to $n - 3$, for the characteristic sequence we have the following tree possibilities:

$$(n-3, 1, 1, 1), \ (n-3, 2, 1) \text{ and } (n-3, 3).$$

The first one is 3-filiform case. We will focus our attention on the study of those with characteristic sequence $(n - 3, 3)$. Throughout all the work, we use the software $Mathematica$. Since in the case of non-Lie Leibniz algebras the skew-symmetric identity is not valid, this classification is very complex and we should overcome the difficulties, which need a lot of computations. Using computer programs is very helpful for computing the Leibniz identity in low dimension and formulate the generalizations of the calculations, which are proved for arbitrary finite dimension. The used program can be find in [5]. Some examples of the programs for various types of Leibniz algebras classes are in the following Web site: http://personal.us.es/jrgomez.

2. Naturally graded Leibniz algebras with characteristic sequence $(n-3, 3)$.

Let L be a naturally graded n-dimensional Leibniz algebra which characteristic sequence equal to $(n - 3, 3)$. From the definition of the characteristic sequence, it follows the existence of a basis $\{e_1, e_2, \dots, e_n\}$ such that element $e_1 \in L \backslash L^2$ and the operator of right multiplication R_{e_1} has one of the following forms:

$$\begin{pmatrix} J_{n-3} & 0 \\ 0 & J_3 \end{pmatrix}, \quad \begin{pmatrix} J_3 & 0 \\ 0 & J_{n-3} \end{pmatrix}$$

Definition 2.1. *A naturally graded Leibniz algebra L which characteristic sequence is equal to $(n-3, 3)$, is called algebra of the second type if there exists a basic element $e_1 \in L \backslash L^2$ such that the operator R_{e_1} has the form:*

$$\begin{pmatrix} J_{n-3} & 0 \\ 0 & J_3 \end{pmatrix};$$

if R_{e_1} has the other form, then it is called algebra of the second type.

Since the classification of Leibniz algebras of the second type is more complicated and it needs to use more original technics, first we present the description of the second type.

Theorem 2.1. *Let L be an n-dimensional naturally graded Leibniz algebra of the second type ($n \geq 9$). Then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

	λ	μ	$dim(L)$
$L^{0,1}_{(0,0,0,0,0)}$			*odd or even*
$L^{0,2}_{(0,0,0,\lambda,-1)}$	$\lambda \in \{0,1\}$		*odd or even*
$L^{0,3}_{(1,0,0,\lambda,-1)}$	$\lambda \in \mathbb{C}$		*odd or even*
$L^{0,4}_{(1,0,1/4,\lambda,-1)}$	$\lambda \in \mathbb{C}$		*odd or even*
$L^{0,5}_{(0,0,1,\lambda,-1)}$	$\lambda \in \mathbb{C}$		*odd or even*
$L^{0,6}_{(0,1,0,\lambda,-1)}$	$\lambda \in \{0,1\}$		*odd or even*
$L^{0,6}_{(\mu,1,0,\lambda,-1)}$	$\lambda \in \mathbb{C}$	$\mu \in \{1,2\}$	*odd or even*
$L^{0,7}_{(0,1,\mu,\lambda,-1)}$	$\lambda \in \mathbb{C}$	$\mu \in \mathbb{C} \setminus \{0\}$	*odd or even*
$L^{0,8}_{(-2\lambda,1,-\lambda,2,-1)}$	$\lambda \in \{-2,-4/3\}$		*odd or even*
$L^{0,9}_{(2\lambda,1,\lambda,0,-1)}$	$\lambda \in \mathbb{C} \setminus \{0,1\}$		*odd or even*
$L^{0,10}_{(1,1,1/4,1/4,-1)}$			*odd or even*
$L^{0,10}_{(1,1,1/4,1/2,-1)}$			*odd or even*
$L^{0,10}_{(2,1,1,1,-1)}$			*odd or even*
$L^{0,10}_{(2,1,1,0,-1)}$			*odd or even*
$L^{0,11}_{(1,\lambda,1/4,0,-1)}$	$\lambda \in \mathbb{C} \setminus \{0,1/2\}$		*odd or even*
$L^{1,2}_{(0,0,0,\lambda,-1)}$	$\lambda \in \{0,1\}$		*even*
$L^{1,3}_{(1,0,0,\lambda,-1)}$	$\lambda \in \mathbb{C}$		*even*
$L^{1,4}_{(1,0,1/4,\lambda,-1)}$	$\lambda \in \mathbb{C}$		*even*
$L^{1,6}_{(\mu,1,0,\lambda,-1)}$	$\lambda \in \mathbb{C}$	$\mu \in \mathbb{C}$	*even*
$L^{1,7}_{(0,\gamma,\mu,\lambda,-1)}$	$\lambda \in \mathbb{C}$	$\gamma,\ \mu \in \mathbb{C} \setminus \{0\}$	*even*
$L^{1,9}_{(-2\lambda,1,\lambda,\mu,-1)}$	$\lambda \in \mathbb{C} \setminus \{0,1\}$	$\mu \in \mathbb{C}$	*even*
$L^{1,11}_{(\lambda,1,\lambda^2/4,\mu,-1)}$	$\lambda \in \mathbb{C} \setminus \{-2,0\}$	$\mu \in \mathbb{C}$	*even*
$L^{1,12}_{(-1,0,0,\lambda,-1)}$	$\lambda \in \{0,1\}$		*even*
$L^{1,13}_{(-2,0,1,\lambda,-1)}$	$\lambda \in \mathbb{C}$		*even*
$L^{1,14}_{(-4,0,2,\lambda,-1)}$	$\lambda \in \mathbb{C}$		*even*
$L^{1,15}_{(0,0,-1,\lambda,-1)}$	$\lambda \in \mathbb{C}$		*even*
$L^{1,16}_{(-2,0,-1,\lambda,-1)}$	$\lambda \in \mathbb{C}$		*even*
$L^{1,17}_{(0,-1,0,\lambda,-1)}$	$\lambda \in \{0,1\}$		*even*
$L^{1,18}_{(-1,-1,0,\lambda,-1)}$	$\lambda \in \mathbb{C}$		*even*
$L^{1,19}_{(-2,-1,0,1,-1)}$			*even*
$L^{1,20}_{(1,-1,0,\lambda,-1)}$	$\lambda \in \mathbb{C} \setminus \{-1/2\}$		*even*
$L^{1,21}_{(1,1/3,0,\lambda,-1)}$	$\lambda \in \mathbb{C}$		*even*
$L^{1,22}_{(-2,-1,1,\lambda,-1)}$	$\lambda \in \{0,1\}$		*even*
$L^{1,23}_{(1,1/2,1/4,\lambda,-1)}$	$\lambda \in \mathbb{C}$		*even*

	λ	$\gamma,\ \mu$	$dim(L)$
$L^{1,24}_{(-4,-1,2,\lambda,-1)}$	$\lambda\in\mathbb{C}$		*even*
$L^{1,25}_{(-3,-4/3,2,\lambda,-1)}$	$\lambda\in\mathbb{C}$		*even*
$L^{1,26}_{(2/5,2,2/5,\lambda,-1)}$	$\lambda\in\mathbb{C}$		*even*
$L^{1,27}_{(2/\lambda,\lambda,1,\mu,-1)}$	$\lambda\in\mathbb{C}\setminus\{-1,0,1\}$	$\mu\in\mathbb{C}$	*even*
$L^{1,28}_{(8/5,1/2,-4/5,\lambda,-1)}$	$\lambda\in\mathbb{C}$		*even*
$L^{1,29}_{(\lambda,-1,\lambda^2/4,0,-1)}$	$\lambda\in\mathbb{C}\setminus\{-2,0\}$		*even*
$L^{1,30}_{(1,-1,1/4,\lambda,-1)}$	$\lambda\in\{-1/2,1/4\}$		*even*
$L^{1,31}_{(-8,2,16,\lambda,-1)}$	$\lambda\in\mathbb{C}$		*even*
$L^{1,32}_{(-2,\lambda,1,0,-1)}$	$\lambda\in\mathbb{C}\setminus\{-1,0\}$		*even*
$L^{1,33}_{(-2,1,1,\lambda,-1)}$	$\lambda\in\{-1,1\}$		*even*

where the algebra

$$L^{\epsilon,j}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta)}:\quad \epsilon\in\{0,1\},\quad 1\leq j\leq 33,\quad \beta\in\{-1,0\}$$

has the following multiplication:

$$\begin{cases}[e_i,e_1]=e_{i+1},\ 1\leq i\leq n-1,\ i\neq 3\\ [e_1,e_4]=\alpha_1e_2+\beta e_5,\\ [e_2,e_4]=\alpha_2e_3,\\ [e_4,e_4]=\alpha_3e_2,\\ [e_5,e_4]=\alpha_4e_3,\\ [e_1,e_5]=(\alpha_1-\alpha_2)e_3-e_6,\\ [e_4,e_5]=(\alpha_3-\alpha_4)e_3,\\ [e_1,e_i]=\beta e_{i+1},\ 6\leq i\leq n-1,\\ [e_i,e_{n+3-i}]=\epsilon(-1)^ie_n,\ 4\leq i\leq n-1.\end{cases}$$

Proof. From the condition of the theorem we have the following multiplication of the basic element e_1 on the right side:

$$[e_i,e_1]=e_{i+1},\ 1\leq i\leq n-1,\ i\neq 3,\ [e_3,e_1]=[e_n,e_1]=0.$$

From these products we conclude that

$$L_1=<e_1,e_4>,\ L_2=<e_2,e_5>,\ L_3=<e_3,e_6>,\ L_i=<e_{i+3}>,\ 4\leq i\leq n-3$$

and $e_2,e_3\in R(L)$.

Let us introduce denotations

$$\begin{array}{lll}[e_1,e_4]=\alpha_1e_2+\beta_1e_5, & [e_2,e_4]=\alpha_2e_3+\beta_2e_6, & [e_3,e_4]=\beta_3e_7,\\ [e_4,e_4]=\alpha_3e_2+\beta_4e_5, & [e_5,e_4]=\alpha_4e_3+\beta_5e_6, & \\ [e_i,e_4]=\beta_ie_{i+1},\ 6\leq i\leq n-1, & [e_n,e_4]=0. & \end{array}$$

The equalities $[e_i,e_5]=[[e_i,e_4],e_1]-[[e_i,e_1],e_4],\ 1\leq i\leq n$ derive

$$\begin{array}{lll}[e_1,e_5]=(\alpha_1-\alpha_2)e_3+(\beta_1-\beta_2)e_6, & [e_2,e_5]=(\beta_2-\beta_3)e_7, & [e_3,e_5]=\beta_3e_8,\\ [e_4,e_5]=(\alpha_3-\alpha_4)e_3+(\beta_4-\beta_5)e_6, & [e_5,e_5]=(\beta_5-\beta_6)e_7, & \\ [e_i,e_5]=(\beta_i-\beta_{i+1})e_{i+2},\ 6\leq i\leq n-2 & [e_{n-1},e_5]=[e_n,e_5]=0. & \end{array}$$

Using induction on j for any value i it can be proved that

$$[e_i,e_j]=\left(\sum_{k=0}^{j-4}(-1)^k\binom{j-4}{k}\beta_{i+k}\right)e_{i+j-3},\quad 5\leq i\leq n-3,\quad 6\leq j\leq n+3-i.$$

In the case of $e_4\in R(L)$ we obtain the algebra $L^{0,1}_{(0,0,0,0,0)}$.

Let now $e_4 \notin R(L)$. Then we consider the following cases:

$\boxed{e_5 \in R(L)}$

Then $e_i \in R(L)$ for $2 \leq i \leq n,\ i \neq 4$.

From the equalities $[[e_i, e_1], e_4] = [[e_i, e_4], e_1],\ 1 \leq i \leq n$, we have

$$\alpha_2 = \alpha_1,\ \alpha_4 = \alpha_3,\ \beta_3 = \beta_2 = \beta_1,\ \beta_i = \beta_4,\ 5 \leq i \leq n-1.$$

For $n \geq 8$ we have also $\beta_1 = 0$.

The change of basis taken as

$$e'_i = e_i,\ 1 \leq i \leq n,\ i \neq 4,5,6, \quad e'_j = e_j - \beta_4 e_{j-3},\ 4 \leq j \leq 6$$

deduces $\beta_4 = 0$.

If we take the change of basis in the following way:

$$e'_1 = Ae_1 + Be_4,\ e'_{n-2} = e_1,\ e'_j = [e'_{j-1}, e'_1],\ 2 \leq j \leq n,\ j \neq n-2$$

with condition $AB(A + \alpha_1 B) \neq 0$, then we obtain the algebra of the first type. Therefore, this case is impossible for the algebra of the second type.

$\boxed{e_5 \notin R(L)}$

The embedding $[e_4, e_4] \in R(L)$ implies $\beta_4 = 0$ and from $[e_i, [e_4, e_1]] = -[e_i, [e_1, e_4]]$, with $1 \leq i \leq n$ we obtain$\beta_1 = -1$.

If $e_6 \in R(L)$, then for $n \geq 9$ it follows $\beta_1 = 0$, which is a contradiction with the condition $\beta_1 = -1$. Therefore, $e_6 \notin R(L)$.

It is easy to check that $[e_i, e_j] + [e_j, e_i] \in R(L)$ for any values of i, j. Applying this for $i = 1$ and $j = 5$ we obtain $\beta_2 = 0$.

The following equalities:

$$[e_1, e_i] = -e_{i+1}, \quad [e_2, e_i] = [e_3, e_i] = 0,\ 6 \leq i \leq n-1$$

are proved by induction on i.

From $[e_1, [e_4, e_{2j+1}]] = -[e_5, e_{2j+1}] + [e_{2j+2}, e_4],\ j \geq 2$, we have that

$$2\beta_{2j+2} = \beta_5 + \beta_{2j+1} + \sum_{k=1}^{2j-4} (-1)^k \begin{pmatrix} 2j-3 \\ k \end{pmatrix} (\beta_{5+k} - \beta_{4+k}),\ j \geq 2.$$

Similar as in [5] we derive

$$\begin{cases} \beta_j = \beta_5, & 6 \leq j \leq n-1, \text{ for n odd,} \\ \beta_j = \beta_5, & 6 \leq j \leq n-2, \text{ for n even} \end{cases}$$

and

$$\begin{array}{l} [e_4, e_{n-1}] = -\beta_5 e_n \text{ for } n \text{ odd,} \\ [e_4, e_{n-1}] = (\beta_{n-1} - 2\beta_5)e_n \text{ for } n \text{ even,} \\ [e_i, e_{n+3-i}] = (-1)^i(\beta_{n-1} - \beta_5)e_n,\ 5 \leq i \leq n-2, \text{ for } n \text{ even.} \end{array}$$

If $\beta_{n-1} = \beta_5$, then by the change of basis defined as $e'_i = e_i,\ 1 \leq i \leq n, i \neq 4,5,6$, and $e'_i = e_i - \beta_5 e_{i-3}$, $4 \leq i \leq 6$ we can assume $\beta_5 = 0$.

If $\beta_{n-1} \neq \beta_5$ (the case of n even), then by using the change of basis:

$$\begin{cases} e'_i = (\beta_{n-1} - \beta_5)^i e_i, & 1 \leq i \leq 3, \\ e'_4 = e_4 - \beta_5 e_1, & \\ e'_5 = (\beta_{n-1} - \beta_5)(e_5 - \beta_5 e_2), & \\ e'_6 = (\beta_{n-1} - \beta_5)^2 (e_6 - \beta_5 e_3), & \\ e'_i = (\beta_{n-1} - \beta_5)^{i-4} e_i, & 7 \leq i \leq n \end{cases}$$

we obtain $[e_i, e_{n+3-i}] = (-1)^i e_n$ for $4 \le i \le n-1$. Thus, multiplication in L is as follows:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-1,\ i \ne 3, \\ [e_1, e_4] = \alpha_1 e_2 - e_5, \\ [e_2, e_4] = \alpha_2 e_3, \\ [e_4, e_4] = \alpha_3 e_2, \\ [e_5, e_4] = \alpha_4 e_3, \\ [e_1, e_5] = (\alpha_1 - \alpha_2) e_3 - e_6, \\ [e_4, e_5] = (\alpha_3 - \alpha_4) e_3, \\ [e_1, e_i] = -e_{i+1}, & 6 \le i \le n-1, \\ [e_i, e_{n+3-i}] = \epsilon(-1)^i e_n, & 4 \le i \le n-1,\ \epsilon \in \{0,1\}. \end{cases}$$

Case 1. $\epsilon = 0$ (n odd or even)

Applying the general change of generators of the basis:

$$e_1' = \sum_{i=1}^{n} A_i e_i, \qquad e_{n-2}' = \sum_{i=1}^{n} B_i e_i,$$

we determine the other elements of the new basis and the products in this basis. Then the new parameters are the following:

$$\alpha_1' = \frac{(\alpha_1 A_1 + 2\alpha_3 A_4) B_4}{A_1^2 + \alpha_1 A_1 A_4 + \alpha_3 A_4^2}, \qquad \alpha_2' = \frac{\alpha_2 B_4}{A_1 + \alpha_2 A_4},$$

$$\alpha_3' = \frac{\alpha_3 B_4^2}{A_1^2 + \alpha_1 A_1 A_4 + \alpha_3 A_4^2}, \quad \alpha_4' = \frac{(\alpha_4 A_1 + \alpha_2 \alpha_3 A_4) B_4^2}{(A_1 + \alpha_2 A_4)(A_1^2 + \alpha_1 A_1 A_4 + \alpha_3 A_4^2)},$$

satisfying the restriction $A_1 (A_1 + \alpha_2 A_4)(A_1^2 + \alpha_1 A_1 A_4 + \alpha_3 A_4^2) B_4 \ne 0$.

Note that for new parameters we have

$$\alpha_1'^2 - 4\alpha_3' = \frac{(\alpha_1^2 - 4\alpha_3) A_1^2 B_4^2}{(A_1^2 + \alpha_1 A_1 A_4 + \alpha_3 A_4^2)^2},$$

$$\alpha_1' \alpha_2' - 2\alpha_3' = \frac{(\alpha_1 \alpha_2 - 2\alpha_3) A_1 B_4^2}{(A_1 + \alpha_2 A_4)(A_1^2 + \alpha_1 A_1 A_4 + \alpha_3 A_4^2)},$$

$$\alpha_1' \alpha_2' - 2\alpha_4' = \frac{(\alpha_1 \alpha_2 - 2\alpha_4) A_1 B_4^2}{(A_1 + \alpha_2 A_4)(A_1^2 + \alpha_1 A_1 A_4 + \alpha_3 A_4^2)}.$$

Consequently, the nullity of $\alpha_1^2 - 4\alpha_3$ is invariant in the following sense:

if $\alpha_1^2 - 4\alpha_3 = 0$, then ${\alpha'}_1^2 - 4{\alpha'}_3 = 0$ and if $\alpha_1^2 - 4\alpha_3 \ne 0$, then ${\alpha'}_1^2 - 4{\alpha'}_3 \ne 0$. Analogously, the expressions $\alpha_1\alpha_2 - 2\alpha_3$ and $\alpha_1\alpha_2 - 2\alpha_4$ are nullity invariants.

Consider the following subcases:

$\alpha_2 = 0,\ \alpha_3 = 0$

Then, $\alpha_1' = \dfrac{\alpha_1 B_4}{A_1 + \alpha_1 A_4}$, $\alpha_2' = 0$, $\alpha_3' = 0$ and $\alpha_4' = \dfrac{\alpha_4 B_4^2}{A_1(A_1 + \alpha_1 A_4)}$.

- $\alpha_1 = 0$.
  If $\alpha_4 = 0$, then the algebra $L^{0,2}_{(0,0,0,\lambda,-1)}$ with $\lambda = 0$ is obtained.
  If $\alpha_4 \ne 0$, then we obtain the algebra $L^{0,2}_{(0,0,0,\lambda,-1)}$ with $\lambda = 1$.
- $\alpha_1 \ne 0$.
  If $\alpha_4 = 0$, then we easily obtain $\alpha_1' = 1$. Thus, we have the algebra $L^{0,3}_{(1,0,0,\lambda,-1)}$ with $\lambda = 0$.

If $\alpha_4 \neq 0$, then choosing appropriate values of A_4 and B_4 we derive $\alpha_1' = \alpha_4' = 1$. Hence, the algebra $L^{0,3}_{(1,0,0,\lambda,-1)}$ with $\lambda = 1$ is obtained.

$\boxed{\alpha_2 = 0,\ \alpha_3 \neq 0}$

Then, $$\alpha_1' = \frac{(\alpha_1 A_1 + 2\alpha_3 A_4)B_4}{A_1^2 + \alpha_1 A_1 A_4 + \alpha_3 A_4^2}, \quad \alpha_2' = 0,$$
$$\alpha_3' = \frac{\alpha_3 B_4^2}{A_1^2 + \alpha_1 A_1 A_4 + \alpha_3 A_4^2}, \quad \alpha_4' = \frac{\alpha_4 B_4^2}{A_1^2 + \alpha_1 A_1 A_4 + \alpha_3 A_4^2}.$$

- If $\alpha_1^2 - 4\alpha_3 = 0$, then taking adequate value of B_4 we obtain $\alpha_1' = 1$, $\alpha_3' = 1/4$ and $\alpha_4' = \frac{\alpha_4}{\alpha_1^2} = \lambda$. So, we obtain the family of algebras $L^{0,4}_{(1,0,1/4,\lambda,-1)}$ with $\lambda \in \mathbb{C}$.
- If $\alpha_1^2 - 4\alpha_3 \neq 0$, then taking suitable values of A_4 and B_4 we deduce $\alpha_1' = 0$, $\alpha_3' = 1$ and $\alpha_4' = \frac{\alpha_4}{\alpha_3} = \lambda$. The family $L^{0,5}_{(0,0,1,\lambda,-1)}$, $\lambda \in \mathbb{C}$ is obtained.

$\boxed{\alpha_2 \neq 0,\ \alpha_3 = 0}$

Then,

$$\alpha_1' = \frac{\alpha_1 B_4}{A_1 + \alpha_1 A_4}, \quad \alpha_2' = \frac{\alpha_2 B_4}{A_1 + \alpha_2 A_4}, \ \alpha_3' = 0, \ \alpha_4' = \frac{\alpha_4 B_4^2}{(A_1 + \alpha_1 A_4)(A_1 + \alpha_2 A_4)}.$$

- $\alpha_1 = 0$.
  If $\alpha_4 = 0$, then the choosing appropriate B_4 leads $\alpha_2' = 1$. Thus, we obtain $L^{0,6}_{(0,1,0,\lambda,-1)}$, $\lambda = 0$.
  If $\alpha_4 \neq 0$, then taking adequate A_4 and B_4 we derive $\alpha_2' = \alpha_4' = 1$. The algebra $L^{0,6}_{(0,1,0,\lambda,-1)}$, $\lambda = 1$ is obtained.
- $\alpha_1 \neq 0$.
  - ✓ $\alpha_4 = 0$.
    If $\alpha_1 - \alpha_2 = 0$, then for suitable B_4 we have $\alpha_1' = \alpha_2' = 1$, i.e. we obtain the algebra $L^{0,6}_{(\mu,1,0,\lambda,-1)}$ with $\mu = 1,\ \lambda = 0$.
    If $\alpha_1 - \alpha_2 \neq 0$, then for adequate A_4 and B_4 it follows that $\alpha_1' = 2$, $\alpha_2' = 1$. The algebra $L^{0,6}_{(\mu,1,0,\lambda,-1)}$, with $\mu = 2, \lambda = 0$ is obtained.
  - ✓ $\alpha_4 \neq 0$.
    If $\alpha_1 - \alpha_2 = 0$, then for appropriate value of B_4 we have $\alpha_1' = \alpha_2' = 1$ and $\alpha_4' = \frac{\alpha_4}{\alpha_1^2} = \lambda$. Therefore, we obtain the family of algebras $L^{0,6}_{(\mu,1,0,\lambda,-1)}$, where $\mu = 1,\ \lambda \in \mathbb{C} \setminus \{0\}$.
    If $\alpha_1 - \alpha_2 \neq 0$, then taking suitable values of A_4 and B_4 we obtain $\alpha_1' = 2$, $\alpha_2' = 1$, $\alpha_4' = \frac{2\alpha_4}{\alpha_1\alpha_2} = \lambda$, i.e., the family $L^{0,6}_{(\mu,1,0,\lambda,-1)}$, $\mu = 2, \lambda \in \mathbb{C} \setminus \{0\}$ is obtained.

$\boxed{\alpha_2 \neq 0,\ \alpha_3 \neq 0}$

- $\alpha_1^2 - 4\alpha_3 \neq 0$, $\alpha_1\alpha_2 - 2\alpha_3 \neq 0$. Taking appropriate A_4 and B_4 we derive
$$\alpha_1' = 0, \quad \alpha_2' = 1, \quad \alpha_3' = -\frac{(\alpha_1\alpha_2 - 2\alpha_3)^2}{\alpha_2^2(\alpha_1^2 - 4\alpha_3)} = \mu,$$

$$\alpha_4' = -\frac{(\alpha_1\alpha_2 - 2\alpha_3)(\alpha_1\alpha_2 - 2\alpha_4)}{\alpha_2^2(\alpha_1^2 - 4\alpha_3)} = \lambda.$$

Hence, we obtain the family of algebras $L^{0,7}_{(0,1,\mu,\lambda,-1)}$, where $\mu \in \mathbb{C}\setminus\{0\},\ \lambda \in \mathbb{C}$.

- $\alpha_1^2 - 4\alpha_3 \neq 0,\ \alpha_1\alpha_2 - 2\alpha_3 = 0.$
  It yields

$$\alpha_3' - \alpha_4' = \frac{(\alpha_3 - \alpha_4)\alpha_2 A_1 B_4^2}{(A_1 + \alpha_2 A_4)(\alpha_2 A_1^2 + 2\alpha_3 A_1 A_4 + \alpha_2\alpha_3 A_4^2)},$$

$$2\alpha_3'\alpha_4' - \alpha_2'^2\alpha_3' - \alpha_4'^2 = \frac{(2\alpha_3\alpha_4 - \alpha_2^2\alpha_3 - \alpha_4^2)\alpha_2^2 A_1^2 B_4^4}{(A_1 + \alpha_2 A_4)^2(\alpha_2 A_1^2 + 2\alpha_3 A_1 A_4 + \alpha_2\alpha_3 A_4^2)^2}.$$

  ✓ $\alpha_3 - \alpha_4 = 0.$

  Therefore, $2\alpha_3\alpha_4 - \alpha_2^2\alpha_3 - \alpha_4^2 \neq 0$ and taking the suitable values of A_4 and B_4 we obtain $\alpha_1' = 4,\ \alpha_2' = 1,\ \alpha_3' = 2,\ \alpha_4' = 2$. Thus, the algebra $L^{0,8}_{(-2\lambda,1,-\lambda,2,-1)}$ with $\lambda = -2$ is obtained.

  ✓ $\alpha_3 - \alpha_4 \neq 0.$

  If $2\alpha_3\alpha_4 - \alpha_2^2\alpha_3 - \alpha_4^2 = 0$, then $\alpha_4 \neq 0,\ \alpha_3 = \dfrac{\alpha_4^2}{2\alpha_4 - \alpha_2^2},\ \alpha_4 \neq \dfrac{\alpha_2^2}{2}$. Choosing adequate values of A_4 and B_4 we obtain $\alpha_1' = 8/3,\ \alpha_2' = 1,\ \alpha_3' = 4/3,\ \alpha_4' = 2$, i.e., we derive the algebra $L^{0,8}_{(-2\lambda,1,-\lambda,2,-1)}$ with $\lambda = -4/3$.

  If $2\alpha_3\alpha_4 - \alpha_2^2\alpha_3 - \alpha_4^2 \neq 0$, then as before we deduce $\alpha_1' = 2\alpha_3',\ \alpha_2' = 1,\ \alpha_3' = -\dfrac{(\alpha_3 - \alpha_4)^2}{2\alpha_3\alpha_4 - \alpha_2^2\alpha_3 - \alpha_4^2} = \lambda,\ \alpha_4' = 0$ and the family $L^{0,9}_{(2\lambda,1,\lambda,0,-1)}$ with $\lambda \in \mathbb{C}\setminus\{0,1\}$ is obtained.

- $\alpha_1^2 - 4\alpha_3 = 0,\ \alpha_1\alpha_2 - 2\alpha_3 \neq 0.$
  Then, $\alpha_1 \neq 2\alpha_2,\quad \alpha_1'^2 - 4\alpha_4' = \dfrac{(\alpha_1^2 - 4\alpha_4)4A_1B_4^2}{(2A_1 + \alpha_1 A_4)^2(A_1 + \alpha_2 A_4)}.$

  ✓ $\alpha_1^2 - 4\alpha_4 = 0.$
  Then, $\alpha_1\alpha_2 - 2\alpha_4 \neq 0$ and from the above we deduce $\alpha_1' = 1, \alpha_2' = 1, \alpha_3' = 1/4, \alpha_4' = 1/4$. So, we obtain the algebra $L^{0,10}_{(\lambda,1,\lambda^2/4,\mu,-1)}$ with $\lambda = 1, \mu = 1/4$.

  ✓ $\alpha_1^2 - 4\alpha_4 \neq 0,\ \alpha_1\alpha_2 - 2\alpha_4 = 0 \Rightarrow\ \alpha_1' = 1,\ \alpha_2' = 1,\ \alpha_3' = 1/4,\ \alpha_4' = 1/2$, i.e., we obtain $L^{0,10}_{(\lambda,1,\lambda^2/4,\mu,-1)}$ with $\lambda = 1, \mu = 1/2$.

  ✓ $\alpha_1^2 - 4\alpha_4 \neq 0,\ \alpha_1\alpha_2 - 2\alpha_4 \neq 0 \Rightarrow\ \alpha_1' = 1,\ \alpha_2' = \dfrac{\alpha_1\alpha_2 - 2\alpha_4}{\alpha_1^2 - 4\alpha_4},\ \alpha_3' = 1/4,\ \alpha_4' = 0$. The family $L^{0,11}_{(1,\lambda,1/4,0,-1)}$, where $\lambda \in \mathbb{C}\setminus\{0,1/2\}$ is obtained.

- $\alpha_1^2 - 4\alpha_3 = 0,\ \alpha_1\alpha_2 - 2\alpha_3 = 0.$

Then, $\alpha_1 = 2\alpha_2, \quad \alpha_3 = \alpha_2^2, \quad \alpha_2'^2 - \alpha_4' = \dfrac{(\alpha_2^2-\alpha_4)A_1B_4^2}{(A_1+\alpha_2A_4)^3}.$

✓ $\alpha_2^2 - \alpha_4 = 0.$

Taking an appropriate value of B_4 it follows that $\alpha_1' = 2$, $\alpha_2' = 1$, $\alpha_3' = 1$, $\alpha_4' = 1$. Hence, we obtain $L^{0,10}_{(\lambda,1,\lambda^2/4,\mu,-1)}$ with $\lambda = 2, \mu = 1$.

✓ $\alpha_2^2 - \alpha_4 \neq 0.$

Choosing adequate A_4 and $B_4 = \dfrac{(\alpha_2^2-\alpha_4)A_1}{\alpha_2^3}$ yields $\alpha_1' = 2$, $\alpha_2' = 1$, $\alpha_3' = 1$ and $\alpha_4' = 0$. Thus, the algebra $L^{0,10}_{(\lambda,1,\lambda^2/4,\mu,-1)}$ with $\lambda = 2, \mu = 0$ is obtained.

Now, we consider the other case.

Case 2. $\epsilon = 1$ (n even)

Similar to the case 1, we apply the general change of generators of basis. Then, we obtain all products and the following expressions for $\alpha_i', 1 \le i \le 4$:

$$\alpha_1' = \frac{(A_1-A_4)(\alpha_1A_1+2\alpha_3A_4)}{A_1^2+\alpha_1A_1A_4+\alpha_3A_4^2}, \qquad \alpha_2' = \frac{\alpha_2(A_1-A_4)}{A_1+\alpha_2A_4},$$

$$\alpha_3' = \frac{\alpha_3(A_1-A_4)^2}{A_1^2+\alpha_1A_1A_4+\alpha_3A_4^2}, \qquad \alpha_4' = \frac{(A_1-A_4)^2(\alpha_4A_1+\alpha_2\alpha_3A_4)}{(A_1+\alpha_2A_4)(A_1^2+\alpha_1A_1A_4+\alpha_3A_4^2)},$$

verifying the restriction $A_1(A_1-A_4)(A_1+\alpha_2A_4)(A_1^2+\alpha_1A_1A_4+\alpha_3A_4^2) \neq 0$.

Note that for these parameters we have

$$\begin{aligned}
\alpha_1'^2 - 4\alpha_3' &= \frac{(\alpha_1^2-4\alpha_3)A_1^2(A_1-A_4)^2}{(A_1^2+\alpha_1A_1A_4+\alpha_3A_4^2)^2},\\
\alpha_1'\alpha_2' - 2\alpha_3' &= -\frac{(\alpha_1\alpha_2-2\alpha_3)A_1(A_1-A_4)^2}{(A_1+\alpha_2A_4)(A_1^2+\alpha_1A_1A_4+\alpha_3A_4^2)},\\
\alpha_1'\alpha_2' - 2\alpha_4' &= \frac{(\alpha_1\alpha_2-2\alpha_4)A_1(A_1-A_4)^2}{(A_1+\alpha_2A_4)(A_1^2+\alpha_1A_1A_4+\alpha_3A_4^2)},\\
\alpha_1' + 2\alpha_3' &= \frac{(\alpha_1+2\alpha_3)(A_1-A_4)A_1}{A_1^2+\alpha_1A_1A_4+\alpha_3A_4^2}.
\end{aligned}$$

Consequently, the nullity of the expressions $\alpha_1^2-4\alpha_3, \alpha_1\alpha_2-2\alpha_3, \alpha_1\alpha_2-2\alpha_4, \ \alpha_1+2\alpha_3$ are invariants.

Applying arguments as in the case 1 for the following subcases:

$\alpha_2 = 0 \ \alpha_3 = 0$, $\alpha_2 = 0, \ \alpha_3 \neq 0$, $\alpha_2 \neq 0 \ \alpha_3 = 0$, $\alpha_2 \neq 0, \ \alpha_3 \neq 0$

we obtain the rest algebras and families of the theorem. □

The next theorem completes the classification of naturally graded Leibniz algebras with characteristic sequence $(n-3,3)$.

Theorem 2.2. *Let L be an n-dimensional naturally graded Leibniz algebra of the first type ($n \geq 9$). Then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$L^{34}_{(0,\lambda,0)}$:

$$\begin{cases} [e_i,e_1]=e_{i+1}, \ 1\leq i\leq n-1, \ i\neq n-3, \\ [e_1,e_{n-2}]=\lambda e_{n-1}, \\ [e_2,e_{n-2}]=\lambda e_n, \ \lambda\in\mathbb{C}. \end{cases}$$

$L^{35}_{(\mu,\lambda,1)}$:

$$\begin{cases} [e_i,e_1]=e_{i+1}, \ 1\leq i\leq n-1, \ i\neq n-3, \\ [e_1,e_{n-2}]=\mu e_2+\lambda e_{n-1}, \\ [e_2,e_{n-2}]=\mu e_3+\lambda e_n, \ \lambda,\mu\in\{0,1\} \\ [e_i,e_{n-2}]=\mu e_{i+1}, \ 3\leq i\leq n-4, \\ [e_i,e_{n-2}]=e_{i+1}, \ n-2\leq i\leq n-1, \ . \end{cases}$$

$L^{36}_{(1,\lambda,0)}$:

$$\begin{cases} [e_i,e_1]=e_{i+1}, \ 1\leq i\leq n-1, \ i\neq n-3, \\ [e_1,e_{n-2}]=e_2+\lambda e_{n-1}, \\ [e_2,e_{n-2}]=e_3+\lambda e_n, \ \lambda\in\{-1,0\} \\ [e_i,e_{n-2}]=e_{i+1}, \ 3\leq i\leq n-4. \end{cases}$$

$L^{37}_{(1,\lambda,2)}$:

$$\begin{cases} [e_i,e_1]=e_{i+1}, \ 1\leq i\leq n-1, \ i\neq n-3, \\ [e_1,e_{n-2}]=e_2+\lambda e_{n-1}, \\ [e_2,e_{n-2}]=e_3+\lambda e_n, \ \lambda\in\mathbb{C} \\ [e_i,e_{n-2}]=e_{i+1}, \ 3\leq i\leq n-4, \\ [e_i,e_{n-2}]=2e_{i+1}, \ n-2\leq i\leq n-1. \end{cases}$$

$L^{38}_{(0,0,\lambda)}$:

$$\begin{cases} [e_i,e_1]=e_{i+1}, \ 1\leq i\leq n-1, \ i\neq n-3, \\ [e_1,e_{n-2}]=-e_{n-1}, \\ [e_2,e_{n-2}]=-(1+\lambda)e_n, \\ [e_1,e_{n-1}]=\lambda e_n, \ \lambda\in\mathbb{C}. \end{cases}$$

$L^{39}_{(0,1,\lambda)}$:

$$\begin{cases} [e_i,e_1]=e_{i+1}, \ 1\leq i\leq n-1, \ i\neq n-3, \\ [e_1,e_{n-2}]=-e_{n-1}, \\ [e_2,e_{n-2}]=-(1+\lambda)e_n, \\ [e_{n-1},e_{n-2}]=-e_n, \\ [e_1,e_{n-1}]=\lambda e_n, \\ [e_{n-2},e_{n-1}]=e_n, \ \lambda\in\{-1,0\}. \end{cases}$$

$L^{40}_{(1,\lambda,\mu)}$:

$$\begin{cases} [e_i,e_1]=e_{i+1}, \ 1\leq i\leq n-1, \ i\neq n-3, \\ [e_1,e_{n-2}]=e_2-e_{n-1}, \\ [e_2,e_{n-2}]=e_3-(1+\mu)e_n, \\ [e_i,e_{n-2}]=e_{i+1}, \ 3\leq i\leq n-4, \\ [e_{n-1},e_{n-2}]=-\lambda e_n, \\ [e_1,e_{n-1}]=\mu e_n, \\ [e_{n-2},e_{n-1}]=\lambda e_n, \ \lambda\in\{0,1\}, \ \mu\in\mathbb{C}. \end{cases}$$

$L^{41}_{(1,-1,\lambda)}$:

$$\begin{cases} [e_i,e_1]=e_{i+1}, \ 1\leq i\leq n-1, \ i\neq n-3, \\ [e_1,e_{n-2}]=e_2-e_{n-1}, \\ [e_2,e_{n-2}]=e_3-(1+\lambda)e_n, \\ [e_i,e_{n-2}]=e_{i+1}, \ 3\leq i\leq n-4, \\ [e_{n-1},e_{n-2}]=e_n, \\ [e_1,e_{n-1}]=\lambda e_n, \\ [e_{n-2},e_{n-1}]=-e_n, \ \lambda\in\{-1,0\}. \end{cases}$$

Proof. Let L be a Leibniz algebra of the first type. Then we have the following multiplication:

$$\begin{array}{l} [e_i,e_1]=e_{i+1}, \ 1\leq i\leq n-4, \quad [e_{n-3},e_1]=0, \\ [e_{n-2},e_1]=e_{n-1}, \ [e_{n-1},e_1]=e_n, \ [e_n,e_1]=0. \end{array}$$

It is not difficult to verify that

$L_1=<e_1,e_{n-2}>, \ L_2=<e_2,e_{n-1}>, \ L_3=<e_3,e_n>, \ L_i=<e_i>, \ 4\leq i\leq n-3$

and $\{e_2,e_3,\ldots,e_{n-3}\}\subseteq R(L)$. Therefore, to define the multiplication in L it suffice to study the multiplication of the element e_{n-2} from the right side.

Introduce denotations $[e_1,e_{n-2}]=\alpha_1e_2+\alpha_2e_{n-1}$, $[e_{n-2},e_{n-2}]=\beta_1e_2+\beta_2e_{n-1}$, $[e_2,e_{n-2}]=\gamma_1e_3+\gamma_2e_n$, $[e_{n-1},e_{n-2}]=\delta_1e_3+\delta_2e_n$.

Then to verify the Leibniz identity $[e_i,[e_j,e_k]]=[[e_i,e_j],e_k]-[[e_i,e_k],e_j]$ it suffice to consider

$$\begin{array}{ll} j=1 \text{ and } k=n-2,n-1,n; & j=n-2 \text{ and } k=1,n-1,n; \\ j=n-1 \text{ and } k=1,n-2,n; & j=n \text{ and } k=1,n-2,n-1. \end{array}$$

We consider several cases.

$\boxed{e_{n-2} \in R(L).}$

Then $\{e_2, e_3, \dots, e_n\} \subseteq R(L)$ and, consequently, we have $\alpha_k = \beta_k = \gamma_k = \delta_k = 0,\ 1 \le k \le 2$. Thus, we obtain the algebra $L^{34}_{(0,0,0)}$.

$\boxed{e_{n-2} \notin R(L),\ e_{n-1} \in R(L).}$

Then, $e_n \in R(L)$ and $\gamma_k = \alpha_k,\ \delta_k = \beta_k,\ 1 \le k \le 2,\ \beta_1 = 0$. Thus, the multiplication table of L can be expressed in the form:

$$\begin{array}{lll} [e_i, e_1] & = & e_{i+1}, \qquad\qquad 1 \le i \le n-1,\ i \ne n-3, \\ [e_1, e_{n-2}] & = & \alpha_1 e_2 + \alpha_2 e_{n-1}, \\ [e_2, e_{n-2}] & = & \alpha_1 e_3 + \alpha_2 e_n, \\ [e_i, e_{n-2}] & = & \alpha_1 e_{i+1}, \qquad\qquad 3 \le i \le n-4, \\ [e_i, e_{n-2}] & = & \beta_2 e_{i+1}, \qquad\qquad n-2 \le i \le n-1. \end{array}$$

Taking the general change of generators of basis:

$$e'_1 = \sum_{i=1}^{n} A_i e_i, \quad e'_{n-2} = \sum_{i=1}^{n} B_i e_i$$

we obtain the new basis $\{e'_1, e'_2, \dots, e'_{n-1}, e'_n\}$.

We compute all products and the new parameters are the following:

$$\alpha'_1 = \frac{\alpha_1 B_{n-2}}{A_1 + \alpha_1 A_{n-2}}, \quad \alpha'_2 = \frac{A_1(\alpha_2 A_1 + \beta_2 A_{n-2} - \alpha_1 A_{n-2})}{(A_1 + \alpha_1 A_{n-2})(A_1 + \beta_2 A_{n-2})}, \quad \beta'_2 = \frac{\beta_2 B_{n-2}}{A_1 + \beta_2 A_{n-2}},$$

satisfying the restrictions

$$\begin{array}{ll} A_1 B_{n-2}(A_1 + \alpha_1 A_{n-2})(A_1 + \beta_2 A_{n-2}) \ne 0, & \\ B_i = 0, & 1 \le i \le n-6, \\ (\alpha_1 - \beta_2) B_i = 0, & n-5 \le i \le n-4, \\ \alpha_2 B_i = 0, & n-5 \le i \le n-4, \end{array}$$

and $B_{n-2}(\beta_2 A_{n-1} + \alpha_2 A_2 - \alpha_1 A_{n-1}) = B_{n-1}(\beta_2 A_{n-2} + \alpha_2 A_1 - \alpha_1 A_{n-2})$.

Note that only coefficients A_1, A_{n-2}, B_{n-2} participate in the expressions for the parameters $\alpha'_1,\ \alpha'_2,\ \beta'_2$. Hence, we can suppose that $A_i = 0,\ i \ne \{1, n-2\}$ and $B_j = 0$, with $j \ne n-2$.

It can be proved that the nullity of $\alpha_1 - \beta_2$ is invariant.

If $\alpha_1 = 0$, then the nullity of $1 - \alpha_2$ is invariant and if $\beta_2 = 0$, then the nullity of $1 + \alpha_2$ is invariant, as well.

Similar as in the proof of Theorem 2.1 we consider the possible cases and in each of them we have the following pairwise non-isomorphic algebras of the theorem:

$L^{34}_{(0,\lambda,0)},\ \lambda \in \mathbb{C} - \{0\};\ L^{35}_{(\mu,\lambda,1)},\ \lambda, \mu \in \{0,1\};\ L^{36}_{(1,\lambda,0)},\ \lambda \in \{-1,0\};\ L^{37}_{(1,\lambda,2)},\ \lambda \in \mathbb{C}.$

$\boxed{e_{n-1} \notin R(L),\ e_n \in R(L).}$

Then, $e_{n-2} \notin R(L)$. Therefore, for defining the multiplication of L_1 and L_2 it is enough to study the multiplication of e_{n-2} and e_{n-1} on the right side.

Introduce the notations

$$\begin{array}{ll} [e_1, e_{n-2}] = \alpha_1 e_2 + \alpha_2 e_{n-1}, & [e_{n-2}, e_{n-2}] = \beta_1 e_2 + \beta_2 e_{n-1}, \\ [e_2, e_{n-2}] = \gamma_1 e_3 + \gamma_2 e_n, & [e_{n-1}, e_{n-2}] = \delta_1 e_3 + \delta_2 e_n, \\ [e_1, e_{n-1}] = a_1 e_3 + a_2 e_n, & [e_{n-2}, e_{n-1}] = b_1 e_3 + b_2 e_n \end{array}$$

From equality $[e_{i+1}, e_{n-1}] = [[e_i, e_{n-1}], e_1],\ 1 \le i \le n-1$ we obtain

$$\begin{array}{ll} [e_i, e_{n-1}] = a_1 e_{i+2}, & 2 \le i \le n-5, \\ [e_i, e_{n-1}] = 0, & n-4 \le i \le n-3, \\ [e_{n-1}, e_{n-1}] = b_1 e_4, & \\ [e_n, e_{n-1}] = b_1 e_5. & \end{array}$$

The equality $[e_{i+1}, e_{n-2}] = [[e_i, e_{n-2}], e_1] - [e_i, e_{n-1}],\ 1 \le i \le n-1$ yields

$$\alpha_2 = -1,\ \gamma_1 = \alpha_1 - a_1,\ \gamma_2 = -(1+a_2),\ \delta_1 = \beta_1 - b_1,\ \delta_2 = \beta_2 - b_2,$$

$$[e_i, e_{n-2}] = (\alpha_1 - (i-1)a_1)e_{i+1}, \quad 3 \le i \le n-4,$$

$$[e_{n-3}, e_{n-2}] = 0,\ [e_n, e_{n-2}] = (\beta_1 - 2b_1)e_4.$$

Since $[e_1, e_{n-1}] \in R(L)$, then $[[e_i, e_{n-1}], e_1] = [e_{i+1}, e_{n-1}]$, which implies $b_1 = 0$ for $n \ge 9$. From $[e_n, [e_1, e_{n-2}]] = -[[e_n, e_{n-2}], e_1]$, we obtain $\beta_1 = 0$. Consequently, $[e_{n-2}, e_{n-2}] \in R(L)$ and $e_{n-1} \notin R(L)$ we have $\beta_2 = 0$.

Moreover, $[e_{n-2}, e_{n-1}] \in R(L)$. Then $[[e_i, e_{n-2}], e_{n-1}] = [[e_i, e_{n-1}], e_{n-2}]$, with $1 \le i \le n$ and hence, $a_1 = 0$ for $n \ge 9$.

Thus, the multiplication in L is as follows

$$\begin{array}{lcll} [e_i, e_1] & = & e_{i+1}, & 1 \le i \le n-1,\ i \ne n-3, \\ [e_1, e_{n-2}] & = & \alpha_1 e_2 - e_{n-1}, & \\ [e_2, e_{n-2}] & = & \alpha_1 e_3 - (1+a_2)e_n, & \\ [e_i, e_{n-2}] & = & \alpha_1 e_{i+1}, & 3 \le i \le n-4, \\ [e_{n-1}, e_{n-2}] & = & -b_2 e_n, & \\ [e_1, e_{n-1}] & = & a_2 e_n, & \\ [e_{n-2}, e_{n-1}] & = & b_2 e_n. & \end{array}$$

Similar as above we take the general change of generators of basis and then we generate the new basis. After that we determine all products and the new parameters are the following:

$$\alpha_1' = \frac{\alpha_1 B_{n-2}}{A_1 + \alpha_1 A_{n-2}}, \quad b_2' = \frac{b_2 B_{n-2}}{A_1 - b_2 A_{n-2}}, \quad a_2' = \frac{(a_2 A_1 + b_2 A_{n-2})}{A_1 - b_2 A_{n-2}},$$

with the restrictions

$$\begin{array}{l} A_1 B_{n-2}(A_1 + \alpha_1 A_{n-2})(A_1 - b_2 A_{n-2}) \ne 0, \\ B_i = 0, \qquad 1 \le i \le n-3, \\ (1+a_2)(-(A_2 + \alpha_1 A_{n-1})B_{n-2} + (A_1 + \alpha_1 A_{n-2})B_{n-1}) = 0. \end{array}$$

Note that only coefficients A_1, A_{n-2}, B_{n-2} participate in the expressions for the parameters $\alpha_1',\ b_2',\ a_2'$. Therefore, we can assume that $A_i = 0,\ i \ne 1, n-2$ and $B_j = 0,\ j \ne n-2$.

It is proved that the nullity of $1 + a_2,\ \alpha_1 + b_2$ are invariants. Similarly as in the proof of Theorem 2.1 we consider the possible cases and for each of them we have the following pairwise non-isomorphic Leibniz algebras:

$$L^{38}_{(0,0,\lambda)},\ \lambda \in \mathbb{C}; \quad L^{39}_{(0,1,\lambda)},\ \lambda \in \{-1, 0\};$$

$$L^{40}_{(1,\lambda,\mu)},\ \lambda \in \{0,1\},\ \mu \in \mathbb{C}; \quad L^{41}_{(1,-1,\lambda)},\ \lambda \in \{-1, 0\}.$$

$\boxed{e_n \notin R(L).}$

Then, $e_{n-2},\ e_{n-1} \notin R(L)$, as well. We set $[e_1, e_n] = \lambda_1 e_4,\ [e_{n-2}, e_n] = \lambda_2 e_4$.

From $[[e_i,e_1],e_n]=[[e_i,e_n],e_1],\ 1\leq i\leq n$ it follows that

$$\begin{array}{lcll} [e_i,e_n] & = & \lambda_1 e_{i+3}, & 2\leq i\leq n-6,\\ [e_i,e_n] & = & 0, & n-5\leq i\leq n-3,\\ [e_{n-1},e_n] & = & \lambda_2 e_5, & \\ [e_n,e_n] & = & \lambda_2 e_6. & \end{array}$$

However, $[[e_n,e_n],e_1]=0$ implies $\lambda_2=0$ for $n\geq 9$ and hence, $\lambda_1\neq 0$.

If we denote $[e_1,e_{n-1}]=a_1e_3+a_2e_n$ and $[e_{n-2},e_{n-1}]=b_1e_3+b_2e_n$. Then due to $[e_1,e_{n-1}]+[e_{n-1},e_1]\in R(L)$ we have $a_2=-1$.

From $[[e_i,e_1],e_{n-1}]=[[e_i,e_{n-1}],e_1]-[e_i,e_n],\ 1\leq i\leq n$ it follows that $b_1=0$ for $n\geq 9$ and

$$\begin{array}{lcll} [e_i,e_{n-1}] & = & (a_1-(i-1)\lambda_1)e_{i+2}, & 2\leq i\leq n-5,\\ [e_i,e_{n-1}] & = & 0, & n-4\leq i\leq n-3,\\ [e_{n-2},e_{n-1}] & = & b_2e_n, & \\ [e_{n-1},e_{n-1}] & = & 0, & \\ [e_n,e_{n-1}] & = & 0. & \end{array}$$

The equality $[[e_i,e_n],e_{n-1}=[[e_i,e_{n-1}],e_n],\ 1\leq i\leq n$ implies $\lambda_1=0$, which is a contradiction with condition $\lambda_1\neq 0$. Consequently, in this case there does not appear any naturally graded Leibniz algebra. □

Summarizing the results of the Theorems 2.1. and 2.2 we complete the classification of naturally graded Leibniz algebras with the characteristic sequence $(n-3,3)$.

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