

ON COMPLEX NILPOTENT LEIBNIZ SUPERALGEBRAS OF NILINDEX $N+M$

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ABSTRACT. We present the description up to isomorphism of Leibniz superalgebras with characteristic sequence $(n|m_1, \dots, m_k)$ and nilindex $n+m$, where $m = m_1 + \dots + m_k$, n and m ($m \neq 0$) are dimensions of even and odd parts, respectively.

Mathematics Subject Classification 2000: 17A32, 17B30.

Key Words and Phrases: Lie superalgebras, Leibniz superalgebras, nilindex, characteristic sequence.

1. INTRODUCTION

Extensive investigations of Lie algebras theory have lead to appearance of more general algebraic objects, such as Mal'cev algebras, Lie superalgebras, Leibniz algebras, and others.

The well-known Lie superalgebras are generalizations of Lie algebras and for many years they attract the attention of both the mathematicians and physicists. The systematical exposition of basic Lie superalgebras theory can be found in the monograph [12] and the papers related with the nilpotent Lie superalgebras are [5], [8], [10], and [11].

Leibniz superalgebras are the generalization of Leibniz algebras and, on the other hand, they naturally generalize Lie superalgebras.

Recall that Leibniz algebras are a "non antisymmetric" generalization of Lie algebras [13]. The study of nilpotent Leibniz algebras [1]–[3] shows that many nilpotent properties of Lie algebras can be extended for nilpotent Leibniz algebras. The results of the nilpotent Leibniz algebras may help us to investigate the nilpotent Leibniz superalgebras.

In the description of Leibniz superalgebras structure the crucial task is to prove the existence of a suitable basis (so-called the adapted basis) in which the multiplication of the superalgebra has the most convenient form.

In contrast to Lie superalgebras for which problem of the description of superalgebras with the maximal nilindex is difficult [10], for nilpotent Leibniz superalgebras it turns out to be comparatively easy and was solved in [1]. The distinctive property of such Leibniz superalgebras is that they are single-generated. The next step - the description of Leibniz superalgebras with the dimensions of even and odd parts, respectively equal to n and m , and with nilindex $n+m$ at this moment seems to be very complicated. Therefore, such Leibniz superalgebras can be studied by applying restrictions on their characteristic sequences [4]–[9]. Following this

Partially supported by the PAICYT, FQM143 of the Junta de Andalucía (Spain). The third author was supported by grant NATO-Reintegration ref. CBP.EAP.RIG.983169.

approach we investigate such Leibniz superalgebras with characteristic sequence $C(L) = (n \mid m_1, \dots, m_k)$, where $m_1 + \dots + m_k = m$.

Taking into account the results of the papers [6] and [9], where some cases of nilpotent Leibniz superalgebras were described, in this work we investigate the rest cases. Thus, we complete the description of Leibniz superalgebras with characteristic sequence $C(L) = (n \mid m_1, \dots, m_k)$ and nilindex $n + m$.

All over the work we consider spaces and algebras over the field of complex numbers. By asterisks (*) we denote the appropriate coefficients at the basic elements of superalgebra.

2. PRELIMINARIES

A \mathbb{Z}_2 -graded vector space $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ is called a *Lie superalgebra* if it is equipped with a product $[-, -]$ which satisfies the following conditions:

1. $[\mathcal{G}_\alpha, \mathcal{G}_\beta] \subseteq \mathcal{G}_{\alpha+\beta \pmod{2}}$,
 2. $[x, y] = -(-1)^{\alpha\beta}[y, x]$,
 3. $(-1)^{\alpha\gamma}[x, [y, z]] + (-1)^{\alpha\beta}[y, [z, x]] + (-1)^{\beta\gamma}[z, [x, y]] = 0$ - *Jacobi superidentity*
- for any $x \in \mathcal{G}_\alpha$, $y \in \mathcal{G}_\beta$, $z \in \mathcal{G}_\gamma$ and $\alpha, \beta, \gamma \in \mathbb{Z}_2$.

A \mathbb{Z}_2 -graded vector space $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ is called a *Leibniz superalgebra* if it is equipped with a product $[-, -]$ which satisfies the following conditions:

1. $[\mathcal{L}_\alpha, \mathcal{L}_\beta] \subseteq \mathcal{L}_{\alpha+\beta \pmod{2}}$,
 2. $[x, [y, z]] = [[x, y], z] - (-1)^{\alpha\beta}[[x, z], y]$ - *Leibniz superidentity*,
- for any $x \in \mathcal{L}$, $y \in \mathcal{L}_\alpha$, $z \in \mathcal{L}_\beta$ and $\alpha, \beta \in \mathbb{Z}_2$.

The vector spaces \mathcal{L}_0 and \mathcal{L}_1 are said to be even and odd parts of the superalgebra \mathcal{L} , respectively.

Note that if in \mathcal{L} the graded identity $[x, y] = -(-1)^{\alpha\beta}[y, x]$ holds, then the Leibniz superidentity and Jacobi superidentity coincide. Thus, Leibniz superalgebras are a generalization of Lie superalgebras.

For examples of Leibniz superalgebras we refer to [1].

Denote by $Leib^{n,m}$ the set of Leibniz superalgebras with dimensions of the even part and the odd part equal to n and m , respectively.

Let $V = V_0 \oplus V_1$, $W = W_0 \oplus W_1$ be two \mathbb{Z}_2 -graded spaces. A linear map $f : V \rightarrow W$ is called of degree α (denoted as $deg(f) = \alpha$), if $f(V_\beta) \subseteq W_{\alpha+\beta}$ for all $\beta \in \mathbb{Z}_2$.

Let \mathcal{L} and \mathcal{L}' be Leibniz superalgebras. A linear map $f : \mathcal{L} \rightarrow \mathcal{L}'$ is called a *homomorphism* of Leibniz superalgebras if

1. $f(\mathcal{L}_0) \subseteq \mathcal{L}'_0$ and $f(\mathcal{L}_1) \subseteq \mathcal{L}'_1$, i.e. $deg(f) = 0$;
2. $f([x, y]) = [f(x), f(y)]$ for all $x, y \in \mathcal{L}$.

Moreover, if f is bijection then it is called an *isomorphism* of Leibniz superalgebras \mathcal{L} and \mathcal{L}' .

For a given Leibniz superalgebra \mathcal{L} we define a descending central sequence in the following way:

$$\mathcal{L}^1 = \mathcal{L}, \quad \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}^1], \quad k \geq 1.$$

Definition 2.1. A Leibniz superalgebra \mathcal{L} is called *nilpotent*, if there exists $s \in \mathbb{N}$ such that $\mathcal{L}^s = 0$. The minimal number s with this property is called *nilindex* of the superalgebra \mathcal{L} .

The sets

$$\mathcal{R}(\mathcal{L}) = \{z \in \mathcal{L} \mid [\mathcal{L}, z] = 0\} \text{ and } \mathcal{Z}(\mathcal{L}) = \{z \in \mathcal{L} \mid [\mathcal{L}, z] = [z, \mathcal{L}] = 0\}$$

are called *the right annihilator* and *the center of a superalgebra* \mathcal{L} , respectively.

Using the Leibniz superidentity it is not difficult to see that $\mathcal{R}(\mathcal{L})$ is an ideal of the superalgebra \mathcal{L} . Moreover, the elements of the form $[a, b] + (-1)^{\alpha\beta}[b, a]$, ($a \in \mathcal{L}_\alpha$, $b \in \mathcal{L}_\beta$) belong to $\mathcal{R}(\mathcal{L})$.

The following theorem describes the nilpotent Leibniz superalgebras with maximal nilindex.

Theorem 2.2. [1] *Let \mathcal{L} be a Leibniz superalgebra of the variety $Leib^{n,m}$ with nilindex equal to $n + m + 1$. Then \mathcal{L} is isomorphic to one of the following non-isomorphic superalgebras:*

$$[e_i, e_1] = e_{i+1}, \quad 1 \leq i \leq n - 1; \quad \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n + m - 1, \\ [e_i, e_2] = 2e_{i+2}, & 1 \leq i \leq n + m - 2, \end{cases}$$

(omitted products are equal to zero).

It should be noted that for the second superalgebra we have $m = n$ when $n + m$ is even and $m = n + 1$ if $n + m$ is odd. Moreover, it is clear that the Leibniz superalgebra has the maximal nilindex if and only if it is single-generated.

Let $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ be a nilpotent Leibniz superalgebra. For an arbitrary element $x \in L_0$, the operator of right multiplication R_x is a nilpotent endomorphism of the space \mathcal{L}_i , where $i \in \{0, 1\}$. Denote by $C_i(x)$ ($i \in \{0, 1\}$) the descending sequence of the dimensions of Jordan blocks of the operator R_x . Consider the lexicographical order on the set $C_i(\mathcal{L}_0)$.

Definition 2.3. *A sequence*

$$C(\mathcal{L}) = \left(\max_{x \in L_0 \setminus \{\mathcal{L}_0, \mathcal{L}_0\}} C_0(x) \mid \max_{\tilde{x} \in \mathcal{L}_0 \setminus \{\mathcal{L}_0, \mathcal{L}_0\}} C_1(\tilde{x}) \right)$$

is said to be the characteristic sequence of the Leibniz superalgebra \mathcal{L} .

Similar as in case of Lie superalgebras [7] (Corollary 3.0.1) it can be proved that the characteristic sequence is an invariant under isomorphisms.

For Leibniz superalgebras we introduce the analogue of the zero-filiform Leibniz algebras.

Definition 2.4. *A Leibniz superalgebra $\mathcal{L} \in Leib^{n,m}$ is called zero-filiform if $C(\mathcal{L}) = (n|m)$*

Denote by $\mathcal{ZF}^{n,m}$ the set of all zero-filiform Leibniz superalgebras from $Leib^{n,m}$.

From [2] it can be concluded that the even part of a zero-filiform Leibniz superalgebra is a zero-filiform Leibniz algebra, therefore a zero-filiform superalgebra is not a Lie superalgebra.

Further, we need the result on existence of an adapted basis for zero-filiform Leibniz superalgebras.

Theorem 2.5. [9] *In an arbitrary superalgebra from $\mathcal{ZF}^{n,m}$ there exists a basis $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m\}$ which satisfies the following conditions:*

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n - 1, \\ [x_n, x_1] = 0, \\ [x_i, x_k] = 0, & 1 \leq i \leq n, \quad 2 \leq k \leq n, \\ [y_j, x_1] = y_{j+1}, & 1 \leq j \leq m - 1, \\ [y_m, x_1] = 0, \\ [y_j, x_k] = 0, & 1 \leq j \leq m, \quad 2 \leq k \leq n. \end{cases}$$

3. ZERO-FILIFORM LEIBNIZ SUPERALGEBRA WITH NILINDEX EQUAL TO $n+m$

This section is devoted to the description of zero-filiform Leibniz superalgebras with nilindex equal to $n+m$.

Let $\mathcal{L} \in \mathcal{ZF}^{n,m}$ with nilindex equal to $n+m$. Evidently, \mathcal{L} has two generators. Moreover, from Theorem 2.5 it follows that one generator lies in \mathcal{L}_0 and the second generator lies in \mathcal{L}_1 . Without loss of generality it can be assumed that in an adapted basis the generators are x_1 and y_1 .

In the adapted basis of \mathcal{L} we introduce the notations.

$$[x_i, y_1] = \sum_{j=2}^m \alpha_{i,j} y_j, \quad 1 \leq i \leq n, \quad [y_i, y_1] = \sum_{j=2}^n \beta_{i,j} x_j, \quad 1 \leq i \leq m.$$

In the above notation the following lemma holds.

Lemma 3.1.

$$[y_i, y_j] = \sum_{s=0}^{\min\{i+j-1, m\}-i} (-1)^s C_{j-1}^s \sum_{t=2}^{n-j+s+1} \beta_{i+s,t} x_{t+j-s-1}, \quad (1)$$

where $1 \leq i, j \leq m$.

Proof. The proof is deduced by induction on j at any value of i . \square

Since in the work [9] the set $\mathcal{ZF}^{n,2}$ was already described, we consider the set $\mathcal{ZF}^{n,m}$ ($m \geq 3$).

Case $\mathcal{ZF}^{2,m}$ ($m \geq 3$).

Theorem 3.2. *Let \mathcal{L} be a Leibniz superalgebra with the nilindex $m+2$ from $\mathcal{ZF}^{2,m}$ ($m \geq 3$). Then m is odd and \mathcal{L} is isomorphic to the following superalgebra:*

$$\begin{aligned} [x_1, x_1] &= x_2, \\ [y_i, x_1] &= y_{i+1}, & 1 \leq i \leq m-1, \\ [x_1, y_i] &= -y_{i+1}, & 1 \leq i \leq m-1, \\ [y_i, y_{m+1-i}] &= (-1)^{j+1} x_2, & 1 \leq i \leq m-1. \end{aligned}$$

Proof. From (1) we easily obtain

$$\begin{aligned} [y_i, y_j] &= (-1)^{j-1} \beta_{i+j-1,2} x_2, & 2 \leq i+j \leq m+1, \\ [y_i, y_j] &= 0, & m+2 \leq i+j \leq 2m. \end{aligned} \quad (2)$$

It should be noted that $\beta_{m,2} \neq 0$. Indeed, if $\beta_{m,2} = 0$, then $\mathcal{L}^{m-1} = \{x_2, y_{m-1}, y_m\}$, $\mathcal{L}^m = \{x_2, y_m\}$ and $\mathcal{L}^{m+1} = \{y_m\}$ which imply that $[y_{m-1}, y_1] = ax_2$ and $[x_2, y_1] = by_m$, where $ab \neq 0$.

The chain of the equalities

$$aby_m = [ax_2, y_1] = [[y_{m-1}, y_1], y_1] = \frac{1}{2} [y_{m-1}, [y_1, y_1]] = 0$$

implies a contradiction to the property $ab \neq 0$. Therefore, $\beta_{m,2} \neq 0$.

The simple analysis of the products leads to $x_2 \in \mathcal{Z}(\mathcal{L})$ (since $x_2 \in \mathcal{L}^{m+1} \subseteq \mathcal{Z}(\mathcal{L})$).

Using the Leibniz superidentity we have

$$[x_1, y_i] = \alpha_{1,2} y_{i+1} + \cdots + \alpha_{1,m-i+1} y_m, \quad 1 \leq i \leq m-1.$$

The expression $[y_1, x_1] + [x_1, y_1]$ lies in $\mathcal{R}(\mathcal{L})$. Hence

$$(1 + \alpha_{1,2})y_2 + \alpha_{1,3}y_3 + \cdots + \alpha_{1,m}y_m \quad (3)$$

belongs to $\mathcal{R}(\mathcal{L})$, as well.

If either $\alpha_{1,2} \neq -1$ or there exists i ($3 \leq i \leq m$) such that $\alpha_{1,i} \neq 0$, then multiplying the linear combination (3) from the right side required times to x_1 we deduce $y_m \in \mathcal{R}(\mathcal{L})$. However, by (2) we have $[y_1, y_m] = (-1)^{m-1}\beta_{m,2}x_2$ which implies that $\beta_{m,2} = 0$ and we get a contradiction with condition $\beta_{m,2} \neq 0$.

If $\alpha_{1,2} = -1$ and $\alpha_{1,i} = 0$ ($3 \leq i \leq m$), then by applying the Leibniz superidentity for the basic elements $\{x_1, y_i, y_i\}$ we obtain $\beta_{2i,2} = 0$ for $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$.

Note that in case m is even we obtain $\beta_{m,2} = 0$ which is a contradiction. Therefore, m is odd.

Let us introduce new notations

$$\gamma_s = \beta_{2s-1,2}, \quad 1 \leq s \leq \frac{m+1}{2}.$$

Then we obtain the family $L(\gamma_1, \gamma_2, \dots, \gamma_{\frac{m+1}{2}})$:

$$\begin{cases} [x_1, x_1] = x_2, \\ [y_i, x_1] = y_{i+1}, & 1 \leq i \leq m-1, \\ [x_1, y_i] = -y_{i+1}, & 1 \leq i \leq m-1, \\ [y_i, y_j] = (-1)^{j-1} \gamma_{\frac{i+j}{2}, 2} x_2, & i+j \text{ is even, } 2 \leq i+j \leq m+1, \text{ } m \text{ is odd.} \end{cases}$$

Make the following general transformation of the generator basic elements:

$$x'_1 = b_1 x_1, \quad y'_1 = \sum_{s=1}^{\frac{m+1}{2}} a_{2s-1} y_{2s-1}.$$

Then $x'_2 = b_1^2 x_2$ and

$$y'_{2i-1} = b_1^{2(i-1)} \sum_{s=1}^{\frac{m-2(i-1)+1}{2}} a_{2s-1} y_{2s+2i-3}, \quad 1 \leq i \leq \frac{m+1}{2},$$

$$y'_{2i} = b_1^{2i-1} \sum_{s=1}^{\frac{m-2(i-1)-1}{2}} a_{2s-1} y_{2s+2i-2}, \quad 1 \leq i \leq \frac{m-1}{2}.$$

Choosing the parameters a_i as follows

$$a_1 = \sqrt{\frac{1}{b_1^{m-3} \gamma_{\frac{m+1}{2}}}}, \quad a_3 = -\frac{a_1 \gamma_{\frac{m-1}{2}}}{2 \gamma_{\frac{m+1}{2}}},$$

$$a_i = -\frac{a_1^2 \gamma_{\frac{m-i+2}{2}} + 2a_1 a_3 \gamma_{\frac{m-i+4}{2}} + \cdots + (2a_1 a_{i-2} + \cdots + 2a_{\frac{i-3}{2}} a_{\frac{i+1}{2}}) \gamma_{\frac{m-1}{2}}}{2a_1 \gamma_{\frac{m+1}{2}}}$$

$$-\frac{(2a_3 a_{i-2} + \cdots + 2a_{\frac{i-3}{2}} a_{\frac{i+5}{2}} + a_{\frac{i+1}{2}}^2) \gamma_{\frac{m+1}{2}}}{2a_1 \gamma_{\frac{m+1}{2}}}, \quad \text{for } \frac{i+1}{2} \text{ odd.}$$

$$a_i = -\frac{a_1^2 \gamma_{\frac{m-i+2}{2}} + 2a_1 a_3 \gamma_{\frac{m-i+4}{2}} + \cdots + (2a_1 a_{i-2} + \cdots + 2a_{\frac{i-5}{2}} a_{\frac{i+3}{2}} + a_{\frac{i-1}{2}}^2) \gamma_{\frac{m-1}{2}}}{2a_1 \gamma_{\frac{m+1}{2}}}$$

$$-\frac{(2a_3 a_{i-2} + \cdots + 2a_{\frac{i-1}{2}} a_{\frac{i+3}{2}}) \gamma_{\frac{m+1}{2}}}{2a_1 \gamma_{\frac{m+1}{2}}}, \quad \text{for } \frac{i+1}{2} \text{ even.}$$

when $4 \leq i \leq m$, we obtain $[y'_m, y'_1] = x'_2$, $[y'_i, y'_1] = 0$ for $1 \leq i \leq m-1$.

Then applying Leibniz superidentity get the rest brackets

$$\begin{aligned} [y'_i, y'_j] &= 0, & 1 \leq i, j \leq m, \quad i+j \neq m+1, \\ [y'_i, y'_j] &= (-1)^{j-1} x'_2, & 1 \leq i, j \leq m, \quad i+j = m+1. \end{aligned}$$

Thus, we obtain the superalgebra of the theorem. \square

Case $\mathcal{ZF}^{n,m}$ ($n \geq 3, m \geq 3$).

Lemma 3.3. *Any Leibniz superalgebra from $\mathcal{ZF}^{n,m}$ ($n \geq 3, m \geq 3$) has nilindex less than $n+m$.*

Proof. Let us assume the contrary, i.e. \mathcal{L} is a Leibniz superalgebra from $\mathcal{ZF}^{n,m}$ ($n \geq 3, m \geq 3$) and \mathcal{L} has the nilindex equal to $n+m$. Then in the adapted basis we have

$$\begin{aligned} \mathcal{L} &= \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m\}, \\ \mathcal{L}^2 &= \{x_2, \dots, x_n, y_2, \dots, y_m\}, \\ \mathcal{L}^3 &\supset \{x_3, \dots, x_n, y_3, \dots, y_m\}. \end{aligned}$$

Let us suppose that $\mathcal{L}^3 = \{x_3, \dots, x_n, y_2, \dots, y_m\}$, i.e. $x_2 \notin \mathcal{L}^3$ and $y_2 \in \mathcal{L}^3$. Then there exists i_0 ($2 \leq i_0 \leq n$) such that $[x_{i_0}, y_1] = \alpha_{i_0,2}y_2 + \dots + \alpha_{i_0,m}y_m$ with $\alpha_{i_0,2} \neq 0$. Since $x_i \in \mathcal{R}(\mathcal{L})$ for $2 \leq i \leq n$ and $\mathcal{R}(\mathcal{L})$ is an ideal, then $\alpha_{i_0,2}y_2 + \dots + \alpha_{i_0,m}y_m \in \mathcal{R}(\mathcal{L})$.

Multiplying the product $[x_{i_0}, y_1]$ on the right side consequently to the basic element x_1 ($m-1$)-times we easily obtain that $y_2, y_3, \dots, y_m \in \mathcal{R}(\mathcal{L})$, that is $\mathcal{L}^2 = \mathcal{R}(\mathcal{L})$.

By induction one can prove the following

$$[x_i, y_1] = \begin{cases} \sum_{j=2}^{m+1-i} \alpha_{1,j} y_{j+i-1}, & \text{if } i+1 \leq m, \\ 0, & \text{if } i+1 > m. \end{cases} \quad (4)$$

Since $\mathcal{L}^2 = \mathcal{R}(\mathcal{L})$ then y_2 can appear only in the products $[x_i, y_1]$ for $2 \leq i \leq n$) or $[y_j, x_1]$ for $2 \leq j \leq m-1$). However, in the first case from (4) we conclude that y_2 does not lie in \mathcal{L}^3 and in the second case the element y_2 can not be obtained, i.e. in both cases we have a contradiction with the assumption $\mathcal{L}^3 = \{x_3, \dots, x_n, y_2, \dots, y_m\}$.

Thus, $\mathcal{L}^3 = \{x_2, \dots, x_n, y_3, \dots, y_m\}$. Let s be a natural number such that $x_2 \in \mathcal{L}^s \setminus \mathcal{L}^{s+1}$.

Suppose $s \leq m$. Then we have

$$\begin{aligned} \mathcal{L}^i &= \{x_2, \dots, x_n, y_i, \dots, y_m\}, \quad 2 \leq i \leq s, \\ \mathcal{L}^{s+1} &= \{x_3, \dots, x_n, y_s, \dots, y_m\} \end{aligned}$$

and in the equality $[y_{s-1}, y_1] = \sum_{j=2}^n \beta_{s-1,j} x_j$ the coefficient $\beta_{s-1,2}$ is not zero.

From Lemma 3.1 we have

$$[y_1, y_s] = \sum_{i=0}^{s-1} (-1)^i C_{s-1}^i \sum_{t=2}^{n-s+i+1} \beta_{1+i,t} x_{t+s-i-1},$$

in which the coefficient $\beta_{s-1,2}$ occurs. Taking into account the equality $[y_s, y_1] = \sum_{j=2}^n \beta_{s,j} x_j$ we conclude that $x_3 \in \text{lin} \langle [y_1, y_s], [y_s, y_1], x_4, \dots, x_n \rangle$. Therefore $\mathcal{L}^{s-2} = \langle x_3, \dots, x_n, y_{s+1}, \dots, y_m \rangle$, i.e. $y_s \in \mathcal{L}^{s+1} \setminus \mathcal{L}^{s+2}$ and $\alpha_{2,s} \neq 0$.

Consider the equalities

$$[y_{s-1}, [y_1, y_1]] = 2[[y_{s-1}, y_1], y_1] = 2 \left[\sum_{t=2}^n \beta_{s-1,t} [x_t, y_1] \right] = 2\beta_{s-1,2} [x_2, y_1] + \sum_{i \geq s+1} (*) y_i.$$

On the other hand $[y_{s-1}, [y_1, y_1]] = 0$, because $[y_1, y_1] \in \mathcal{R}(\mathcal{L})$.

The basic element y_s appears only in the product $[x_2, y_1]$. Hence we have that $\beta_{s-1,2} \alpha_{2,s} y_s + \sum_{i \geq s+1} (*) y_i = 0$ which implies $\beta_{s-1,2} \alpha_{2,s} = 0$. This contradicts to the assumption $s \leq m$.

Let us consider now the case $s = m + 1$. Then we have

$$\begin{aligned} \mathcal{L} &= \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m\}, \\ \mathcal{L}^i &= \{x_2, x_3, \dots, x_n, y_i, y_{i+1}, \dots, y_m\}, \quad 2 \leq i \leq m, \\ \mathcal{L}^{m+i-1} &= \{x_i, x_{i+1}, \dots, x_n\}, \quad 2 \leq i \leq n. \end{aligned}$$

Since $x_2 \in \mathcal{L}^{m+1}$ we have $[y_m, y_1] = \sum_{i=2}^n \beta_{m,i} x_i$ with $\beta_{m,2} \neq 0$.

The sum $[y_1, x_1] + [x_1, y_1]$ lies in $\mathcal{R}(\mathcal{L})$ since $[y_1, x_1] + [x_1, y_1] = (1 + \alpha_{1,2})y_2 + \alpha_{1,3}y_3 + \dots + \alpha_{1,m}y_m \in \mathcal{R}(\mathcal{L})$.

If $[y_1, x_1] + [x_1, y_1] = 0$, then using the Leibniz superidentity we have

$$[x_1, [y_m, y_1]] = [[x_1, y_m], y_1] + [[x_1, y_1], y_m] = -[y_2, y_m] = \beta_{m,2} x_3 + \sum_{i \geq 4} (*) x_i.$$

On the other hand

$$[x_1, [y_m, y_1]] = \sum_{i=2}^n \beta_{m,i} [x_1, x_i] = 0.$$

Hence, $\beta_{m,2} = 0$ which is a contradiction.

Thus, $[y_1, x_1] + [x_1, y_1] \neq 0$. Continuing the same argumentation as in the proof of Theorem 3.2 we obtain $y_m \in \mathcal{R}(\mathcal{L})$. Therefore

$$[y_1, y_m] = \sum_{i=0}^{m-1} (-1)^i C_{m-1}^i \sum_{t=2}^{n-m+i+1} \beta_{1+i,t} x_{t+m-i-1} = 0.$$

The minimal value of the expression $t + m - i - 1$ is reached when $i = m - 1$ and $t = 2$. Thus, $[y_1, y_m] = (-1)^{m-1} C_{m-1}^{m-1} \beta_{m,2} x_2 + \sum_{i \geq 4} (*) x_i$ which implies that $\beta_{m,2} = 0$. That is a contradiction with the assumption that nilindex of \mathcal{L} is equal to $n + m$. \square

4. LEIBNIZ SUPERALGEBRAS WITH THE CHARACTERISTIC SEQUENCE

$(n|m_1, m_2, \dots, m_k)$ AND NILINDEX $n + m$

Leibniz superalgebras with the characteristic sequence equal to $(n|m-1, 1)$ and with the nilindex $n + m$ were examined in [6]. Therefore, in this section we shall consider the Leibniz superalgebras \mathcal{L} of nilindex $n + m$ with the characteristic sequence equal to $(n|m_1, m_2, \dots, m_k)$ with conditions $m_1 \leq m - 2$.

From the definition of characteristic sequence there exists a basis $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m\}$ in which the operator $R_{x_1|_{\mathcal{L}_1}}$ has the following form:

$$R_{x_1|_{\mathcal{L}_1}} = \begin{pmatrix} J_{m_{j_1}} & 0 & \cdots & 0 \\ 0 & J_{m_{j_2}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & J_{m_{j_k}} \end{pmatrix},$$

where $(m_{j_1}, m_{j_2}, \dots, m_{j_k})$ is a permutation of (m_1, m_2, \dots, m_k) . Without loss of generality, by a shifting of the basic elements we can assume that operator $R_{x_1|_{\mathcal{L}_1}}$ has the following form

$$R_{x_1|_{\mathcal{L}_1}} = \begin{pmatrix} J_{m_1} & 0 & \cdots & 0 \\ 0 & J_{m_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & J_{m_k} \end{pmatrix}.$$

It means that the basis $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m\}$ satisfies the following conditions:

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n-1, \\ [y_j, x_1] = y_{j+1}, & \text{for } j \notin \{m_1, m_1 + m_2, \dots, m_1 + m_2 + \dots + m_k\}, \\ [y_j, x_1] = 0, & \text{for } j \in \{m_1, m_1 + m_2, \dots, m_1 + m_2 + \dots + m_k\}. \end{cases} \quad (5)$$

It is clear that two generators can not lie in \mathcal{L}_0 . In fact, in [2] the table of multiplication of the Leibniz algebra \mathcal{L}_0 is presented and it has only one generator.

Theorem 4.1. *Let \mathcal{L} be a Leibniz superalgebra of nilindex $n+m$ with characteristic sequence $(n|m_1, m_2, \dots, m_k)$, where $m_1 \leq m-2$. Then both generators can not belong to \mathcal{L}_1 at the same time.*

Proof. Let Leibniz superalgebra $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ has nilindex $n+m$ and let $\{x_1, x_2, \dots, x_n\}$ be a basis of \mathcal{L}_0 and $\{y_1, y_2, \dots, y_m\}$ a basis of \mathcal{L}_1 . Suppose that two generators lie in \mathcal{L}_1 . Then they should be from the set

$$\{y_1, y_{m_1+1}, y_{m_1+m_2+1}, \dots, y_{m_1+m_2+\dots+m_{k-2}+1}, y_{m_1+m_2+\dots+m_{k-1}+1}\}$$

Without loss of generality, the generators can be chosen as $\{y_1, y_{m_1+1}\}$.

Consider the following cases:

Case 1. Let $[y_1, y_1] \in \mathcal{L}_0 \setminus \mathcal{L}_0^2$. Then consider the Leibniz superalgebra generated by the element $\langle y_1 \rangle$. Since $[y_1, y_1] \in \mathcal{L}_0 \setminus \mathcal{L}_0^2$ we can assume $[y_1, y_1] = x_1$. Then from the products in (5) we deduce $\{x_1, x_2, \dots, x_n, y_2, y_3, \dots, y_{m_1}\} \subseteq \langle y_1 \rangle$. It is easy to see that $y_{m_1+1} \notin \langle y_1 \rangle$. Indeed, if $y_{m_1+1} \in \langle y_1 \rangle$, then $\{y_{m_1+2}, \dots, y_{m_1+m_2}\} \subseteq \langle y_1 \rangle$ which implies $C(\mathcal{L}) \geq (n | m_1 + m_2, m_3, \dots, m_k)$. That is a contradiction to the condition of characteristic sequence of \mathcal{L} , because $C(\mathcal{L}) = (n|m_1, m_2, \dots, m_k)$.

Thus, the Leibniz superalgebra generated by the basic element y_1 consist of

$$\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_{m_1}\}.$$

Since the superalgebra $\langle y_1 \rangle$ is single-generated then from Theorem 2.2 we have that either $m_1 = n$ or $m_1 = n+1$ and the multiplication in $\langle y_1 \rangle$ has the following form:

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n-1, \\ [y_j, x_1] = y_{j+1}, & 1 \leq j \leq m_1-1, \\ [x_i, y_1] = \frac{1}{2}y_{i+1}, & 1 \leq i \leq m_1-1, \\ [y_j, y_1] = x_j, & 1 \leq j \leq n. \end{cases}$$

Case $m_1 = n$. Since y_1 and y_{n+1} are generators we have

$$\begin{aligned}\mathcal{L} &= \{x_1, x_2, \dots, x_n, y_1, \dots, y_n, y_{n+1}, \dots, y_m\}, \\ \mathcal{L}^2 &= \{x_1, x_2, \dots, x_n, y_2, \dots, y_n, y_{n+2}, \dots, y_m\}.\end{aligned}$$

Besides, $x_1 \notin \mathcal{L}^3$. Otherwise, if $x_1 \in \mathcal{L}^3$, then there exists $z \in \mathcal{L}_1$ such that $z \in \mathcal{L}^2/\mathcal{L}^3$. Thereby $z \in \text{lin} \langle [y_1, y_1], [y_1, y_{n+1}], [y_{n+1}, y_1], [y_{n+1}, y_{n+1}] \rangle$ and taking into account that $[y_i, y_j] \in \mathcal{L}_0$ we obtain $z \in \mathcal{L}_0$ which is a contradiction. Thus,

$$\mathcal{L}^3 = \{x_2, \dots, x_n, y_2, \dots, y_n, y_{n+2}, \dots, y_m\}.$$

If $\mathcal{L}^{2k} = \{x_i, x_{i+1}, \dots, x_n, y_j, \dots, y_n, y_{n+2}, \dots, y_m\}$, then by a similar way one can prove that $\mathcal{L}^{2k+1} = \{x_{i+1}, \dots, x_n, y_j, \dots, y_n, y_{n+2}, \dots, y_m\}$. In fact, if $z \in \mathcal{L}^{2k}/\mathcal{L}^{2k+1}$, then z has to be generated by $2k$ products of the generators (but they are from \mathcal{L}_1). Hence this products belong to \mathcal{L}_0 , and we have $z \in \mathcal{L}_0$.

Applying the similar argumentation we get

$$\mathcal{L}^{2k+2} = \{x_{i+1}, \dots, x_n, y_{j+1}, \dots, y_n, y_{n+2}, \dots, y_m\}.$$

Continuing with the process, we obtain that $\mathcal{L}^{2n+1} = \{y_{i_1}, y_{i_2}, \dots, y_{i_k}\}$ and $\mathcal{L}^{2n+2} = 0$. Since $\dim(\mathcal{L}^{2n+1}/\mathcal{L}^{2n+2}) = 1$ then $\mathcal{L}^{2n+1} = \{y_{n+2}\}$ and nilindex should be equal to $2n + 2$. Thus, $m = n + 2$ and we have

$$\begin{aligned}\mathcal{L} &= \{x_1, x_2, \dots, x_n, y_1, \dots, y_n, y_{n+1}, y_{n+2}\}, \\ \mathcal{L}^{2k} &= \{x_k, \dots, x_n, y_{k+1}, \dots, y_n, y_{n+2}\}, \quad 1 \leq k \leq n-1, \\ \mathcal{L}^{2k+1} &= \{x_{k+1}, \dots, x_n, y_{k+1}, \dots, y_n, y_{n+2}\}, \quad 1 \leq k \leq n-1, \\ \mathcal{L}^{2n} &= \{x_n, y_{n+2}\}, \quad \mathcal{L}^{2n+1} = \{y_{n+2}\}, \quad \mathcal{L}^{2n+2} = \{0\}.\end{aligned}$$

Furthermore, $\mathcal{L}^{2n} = [\mathcal{L}^{2n-1}, \mathcal{L}] = \langle [x_n, y_1], [x_n, y_{n+1}], [y_n, y_1], [y_n, y_{n+1}], [y_{n+2}, y_1], [y_{n+2}, y_{n+1}] \rangle$. Note that the element y_{n+2} can be obtained only from product $[x_n, y_{n+1}]$ (because $[x_n, y_1] = 0$, otherwise we get a contradiction with the property of characteristic sequence). However,

$$[x_n, y_{n+1}] = [[x_{n-1}, x_1], y_{n+1}] = [x_{n-1}, [x_1, y_{n+1}]] + [[x_{n-1}, y_{n+1}], x_1] = 0,$$

which deduce $\mathcal{L}^{2n} = \{x_n\}$, that is a contradiction with the condition of nilindex.

Case $m_1 = n + 1$. In this case, similar to the previous case, we get a contradiction.

Case 2. Let $[y_1, y_1] \notin \mathcal{L}_0 \setminus \mathcal{L}_0^2$ and $[y_{m_1+1}, y_{m_1+1}] \in \mathcal{L}_0 \setminus \mathcal{L}_0^2$. Then applying the same arguments for y_{m_1+1} as for y_1 in Case 1, we obtain a contradiction with the fact that both generators lie in \mathcal{L}_1 , as well.

Case 3. Let $[y_1, y_1] \notin \mathcal{L}_0 \setminus \mathcal{L}_0^2$ and $[y_{m_1+1}, y_{m_1+1}] \notin \mathcal{L}_0 \setminus \mathcal{L}_0^2$. Then, without loss of generality, we can assume that

$$\begin{aligned}[y_1, y_{m_1+1}] &= x_1, \\ [y_{m_1+1}, y_1] &= \sum_{i=1}^n b_i x_i.\end{aligned}$$

If $b_1 = 1$, then making the change of basis $y'_1 = y_1 + y_{m_1+1}$ we obtain $[y'_1, y'_1] \in \mathcal{L}_0 \setminus \mathcal{L}_0^2$. Therefore this case can be reduced to Case 1.

If $b_1 \neq 1$, then $[y_1, y_{m_1+1}] - [y_{m_1+1}, y_1] = (1 - b_1)x_1 + b_2x_2 + \dots + b_nx_n \in \mathcal{R}(\mathcal{L})$ and since $x_i \in \mathcal{R}(\mathcal{L})$ ($2 \leq i \leq n$) we get $x_1 \in \mathcal{R}(\mathcal{L})$. From the Leibniz superidentity

we obtain $\mathcal{L}^3 = \{0\}$ and therefore $n = 1$, $m = 2$. Obviously, we get a Leibniz algebra, i.e. the Leibniz superalgebra with condition $m = 0$. \square

In other words, Theorem 4.1 claims that one generator lies in \mathcal{L}_0 and another one belongs to \mathcal{L}_1 . Evidently, x_1 is a generator and as a generator in \mathcal{L}_1 we can chose y_1 .

Put

$$\begin{aligned} [x_i, y_1] &= \sum_{t=2}^m \alpha_{i,t} y_t, \quad 1 \leq i \leq n, \\ [y_j, y_1] &= \sum_{s=2}^n \beta_{j,s} x_s, \quad 1 \leq j \leq m. \end{aligned}$$

The following equality can be proved by induction

$$[y_i, y_j] = \sum_{s=0}^{\min\{i+j-1, m_1\}-i} (-1)^s C_{j-1}^s \sum_{t=2}^{n-j+s+1} \beta_{i+s,t} x_{t+j-s-1} \quad (6)$$

where $1 \leq i, j \leq m_1$.

Theorem 4.2. *Let \mathcal{L} be a Leibniz superalgebra with characteristic sequence equal to $(n|m_1, m_2, \dots, m_k)$, where $m_1 \leq m - 2$. Then nilindex of \mathcal{L} is less than $n + m$.*

Proof. Let us suppose the contrary, i.e. the nilindex of \mathcal{L} is equal to $n + m$. Then $\dim \mathcal{L}^k = n + m - k$, $2 \leq k \leq n + m$. In the adapted basis $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m\}$ we have the products

$$\begin{aligned} [x_i, x_1] &= x_{i+1}, & 1 \leq i \leq n-1, \\ [y_j, x_1] &= y_{j+1}, & j \notin \{m_1, m_1 + m_2, \dots, m_1 + m_2 + \dots + m_k\}, \\ [y_j, y_1] &= \beta_{j,2} x_2 + \dots + \beta_{j,n} x_n, & 1 \leq j \leq m, \\ [x_i, y_1] &= \alpha_{i,2} y_2 + \dots + \alpha_{i,m} y_m, & 1 \leq i \leq n. \end{aligned}$$

Suppose that $x_2 \in \mathcal{L}^3$, i.e. $\mathcal{L}^3 = \{x_2, \dots, x_n, y_3, \dots, y_m\}$. Then the element x_2 is generated from the products $[y_j, y_1]$, i.e. there exists j_0 ($2 \leq j_0 \leq m$) such that in $[y_{j_0}, y_1] = \sum_{i=2}^n \beta_{j_0,i} x_i$ the parameter $\beta_{j_0,2} \neq 0$.

Taking the change of basis $x'_1 = \frac{1}{\beta_{j_0,2}} \left(\sum_{s=2}^n \beta_{j_0,s} x_{s-1} \right)$ we can assume that $[y_{j_0}, y_1] = x_2$.

The equalities

$$[y_{j_0}, [y_1, y_1]] = 2[[y_{j_0}, y_1], y_1] = 2[x_2, y_1].$$

and

$$[y_{j_0}, [y_1, y_1]] = [y_{j_0}, \sum_{i=2}^n \beta_{1,i} x_i] = 0$$

imply $[x_2, y_1] = 0$.

Since $\mathcal{L}^3 = n + m - 3$ we have $\mathcal{L}^3 = \{x_2, x_3, \dots, x_n, y_3, \dots, y_{m_1}, A_{1,1} y_2 + A_{1,2} y_{m_1+1} + \dots + A_{1,k} y_{m_1+m_2+\dots+m_{k-1}+1}, y_{m_1+2}, \dots, y_{m_1+m_2}, A_{2,1} y_2 + A_{2,2} y_{m_1+1} + \dots + A_{2,k} y_{m_1+m_2+\dots+m_{k-1}+1}, \dots, y_{m_1+m_2+\dots+m_{k-1}}, \dots, A_{k-1,1} y_2 + A_{k-1,2} y_3 + \dots + A_{k-1,k} y_{m_1+m_2+\dots+m_{k-1}+1}, \dots, y_m\}$

If there exists i such that $[x_i, y_1] = c_2 (A_{i,1} y_2 + A_{i,2} y_{m_1+1} + \dots + A_{i,k} y_{m_1+\dots+m_{k-1}+1}) + \sum_{s \geq 3} (*) y_s$ with $c_2 \neq 0$, then applying the Leibniz superidentity for the elements $\{x_i, x_1, y_1\}$ inductively, we obtain $A_{i,1} = 0$ for all i ($1 \leq i \leq k-1$).

Thus,

$$\mathcal{L}^3 = \{x_2, x_3, \dots, x_n, y_3, \dots, y_{m-1}, y_m\}.$$

Consider the product $[x_1, y_1] = \sum_{i=2}^n \alpha_{1,i} y_i$.

Case 1. Let $\alpha_{1,2} \neq 0$. Let us suppose that there exists s from $3 \leq s \leq m_1$ such that

$$\begin{aligned} \mathcal{L}^s &= \{x_2, \dots, x_n, y_s, \dots, y_m\}, \\ \mathcal{L}^{s+1} &= \{x_3, \dots, x_n, y_s, \dots, y_m\}. \end{aligned}$$

Note that $\beta_{s-1,2} \neq 0$, because $x_2 \in \mathcal{L}^s \setminus \mathcal{L}^{s+1}$. The equality $[x_2, y_1] = 0$ implies that the element y_s must be generated by the products $[x_i, y_1]$ for $3 \leq i \leq n$. Therefore,

$$\mathcal{L}^{s+2} = \{x_4, \dots, x_n, y_s, \dots, y_{m-1}, y_m\}.$$

The parameters $\beta_{s,2}, \beta_{s,3} = 0$ are equal to zero because of $y_s \in \mathcal{L}^{s+2}$. Thus, we have

$$[y_{s-1}, y_1] = \sum_{i=2}^n \beta_{s-1,i} x_i \quad \text{and} \quad [y_s, y_1] = \sum_{i=4}^n \beta_{s,i} x_i.$$

From the equality (6) we get

$$[y_2, y_{s-1}] = (-1)^{s-1} \beta_{s-1,2} x_3 + \sum_{t \geq 4} (*) x_t.$$

The following chain of the equalities

$$\begin{aligned} 0 &= [x_1, [y_1, y_{s-1}]] = [[x_1, y_1], y_{s-1}] + [[x_1, y_{s-1}], y_1] = \\ &= \left[\sum_{i=2}^n \alpha_{1,i} y_i, y_{s-1} \right] + \left[\sum_{i=s}^n \gamma_{1,i} y_i, y_1 \right] = \alpha_{1,2} \beta_{s-1,2} x_3 + \sum_{t \geq 4} (*) x_t \end{aligned}$$

implies $\alpha_{1,2} \beta_{s-1,2} = 0$, that is a contradiction with the assumption $s \leq m_1$.

Thus, we have $s > m_1$. Then

$$\mathcal{L}^{m_1} = \{x_2, \dots, x_n, y_{m_1}, \dots, y_m\},$$

$$\mathcal{L}^{m_1+1} = \{x_2, \dots, x_n, y_{m_1+1}, \dots, y_m\}.$$

Since $\mathcal{L}^{m_1+2} = [\mathcal{L}^{m_1+1}, \mathcal{L}]$ then using the equality (6) we conclude that

$$\mathcal{L}^{m_1+2} = \{x_3, \dots, x_n, y_{m_1+1}, \dots, y_m\}.$$

Therefore, $s = m_1 + 1$ and $[y_{m_1}, y_1] = \sum_{i=2}^n \beta_{m_1,2} x_2$ with $\beta_{m_1,2} \neq 0$. From (6) we get $[y_2, y_{m_1}] = \beta_{m_1,2} x_3 + \sum_{i \geq 4} x_i$ and using the Leibniz superidentity we obtain $\beta_{m_1,2} = 0$, but it is a contradiction. Hence this case is not possible.

Case 2. Let $\alpha_{1,2} = 0$. Then we have that

$$\begin{aligned} [x_1, y_1] &= \alpha_{1,3} y_3 + \dots + \alpha_{1,m} y_m, \\ [y_1, x_1] &= y_2, \\ [y_1, x_1] + [x_1, y_1] &= y_2 + \alpha_{1,3} y_3 + \dots + \alpha_{1,m} y_m \in \mathcal{R}(\mathcal{L}). \end{aligned}$$

By the similar arguments as in the previous case, we obtain $y_i \in \mathcal{R}(\mathcal{L})$ ($2 \leq i \leq m$). Applying the Leibniz superidentity for the elements $\{y_{j-1}, x_1, y_1\}$ we have

$$[y_j, y_1] = \sum_{i=2}^{n+1-j} \beta_{j,i} x_{i+j-1}, \quad 2 \leq j \leq m_1$$

which implies $\beta_{i,2} = 0$ for all $2 \leq i \leq m_1$.

Since $y_{m_1+\dots+m_{s-1}+1}$ ($2 \leq s \leq k$) is generated from the products $[x_t, y_1]$ ($1 \leq t \leq n$), then from the equality

$$[[x_t, y_1], y_1] = \frac{1}{2}[x_t, [y_1, y_1]] = [x_t, \sum_{l \geq 2} (*)x_l] = 0$$

and applying the Leibniz superidentity for the elements $\{y_{j-1}, x_1, y_1\}$ we get $\beta_{i,2} = 0$ for $2 \leq i \leq m$, but it is a contradiction to the assumption $x_2 \in \mathcal{L}^3$.

Thus, we have $x_2 \notin \mathcal{L}^3$, i.e.

$$\mathcal{L}^3 = \{x_3, x_4, \dots, x_n, y_2, y_3, \dots, y_m\}.$$

Since $\{y_2, y_{m_1}, y_{m_1+m_2}, \dots, y_{m_1+m_2+\dots+m_k}\} \in \mathcal{L}^3$, then there exist $i_1, i_2, \dots, i_k \geq 2$, such that

$$\begin{vmatrix} \alpha_{i_1,2} & \alpha_{i_1,m_1+1} & \alpha_{i_1,m_1+m_2+1} & \cdots & \alpha_{i_1,m_1+m_2+\dots+m_{k-1}+1} \\ \alpha_{i_2,2} & \alpha_{i_2,m_1+1} & \alpha_{i_2,m_1+m_2+1} & \cdots & \alpha_{i_2,m_1+m_2+\dots+m_{k-1}+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{i_k,2} & \alpha_{i_k,m_1+1} & \alpha_{i_k,m_1+m_2+1} & \cdots & \alpha_{i_k,m_1+m_2+\dots+m_{k-1}+1} \end{vmatrix} \neq 0. \quad (7)$$

Without loss of generality we can assume that $\alpha_{i_1,2} \neq 0$. Consider

$$[x_{i_1}, y_1] + [y_1, x_{i_1}] = \sum_{t=2}^m \alpha_{i_1,t} y_t \in \mathcal{R}(L).$$

Then multiplying sufficiently times from the right side to the element x_1 and taking into account the condition (7) we obtain $y_i \in \mathcal{R}(L)$ ($2 \leq i \leq m$).

Furthermore, proceeding with the bracket computing

$$\begin{aligned} [x_2, y_1] &= [[x_1, y_1], x_1] = \alpha_{1,2}y_3 + \cdots + \alpha_{1,m-1}y_m, \\ [x_3, y_1] &= [[x_2, y_1], x_1] = \alpha_{1,2}y_4 + \cdots + \alpha_{1,m-2}y_m, \\ &\vdots \\ [x_n, y_1] &= [[x_{n-1}, y_1], x_1] = \alpha_{1,2}y_{n+1} + \cdots + \alpha_{1,n-m+1}y_m. \end{aligned}$$

we obtain that $y_2 \notin \mathcal{L}^3$.

Thus, we get the contradictions in all considered cases, which leads that the superalgebra \mathcal{L} with characteristic sequence $C(\mathcal{L}) = (n|m_1, m_2, \dots, m_k)$ and $m_1 \leq m-2$ has a nilindex less than $n+m$. \square

Combining the assertion of Theorem 3.2 and the classifications of the papers [6], [9] we complete the classification of Leibniz superalgebras with even part zero-filiform Leibniz algebra and with nilindex equal to $n+m$.

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