LEIBNIZ ALGEBRAS OF HEISENBERG TYPE

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ABSTRACT. We introduce and provide a classification theorem for the class of Heisenberg-Fock Leibniz algebras. This category of algebras is formed by those Leibniz algebras L whose corresponding Lie algebras are Heisenberg algebras H_n and whose H_n-modules I, where I denotes the ideal generated by the squares of elements of L, are isomorphic to Fock modules. We also consider the three-dimensional Heisenberg algebra H_3 and study three classes of Leibniz algebras with H_3 as corresponding Lie algebra, by taking certain generalizations of the Fock module. Moreover, we describe the class of Leibniz algebras with H_n as corresponding Lie algebra and such that the action $I \times H_n \to I$ gives rise to a minimal faithful representation of H_n. The classification of this family of Leibniz algebras for the case of $n = 3$ is given.

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1. INTRODUCTION

The term *Leibniz algebra* was introduced in the study of a non-antisymmetric analogue of Lie algebras by Loday [35], being so the class of Leibniz algebras an extension of the one of Lie algebras. However this kind of algebras was previously studied under the name of D-algebras by D. Bloh [10, 11, 12]. Since the 1993 Loday's work many researchers have been attracted by this category of algebras, being remarkable the great activity in this field developed in the last years. This activity has been mainly focussed in the frameworks of low dimensional algebras, nilpotence and physics applications (see [2, 5, 6, 14, 15, 16, 17, 21, 22, 23, 25, 28, 33, 40, 41]).

Definition 1. A *Leibniz algebra* L is a linear space over a base field $\mathbb{F}$ endowed with a bilinear product $[\cdot,\cdot]$ satisfying the *Leibniz identity*

$$[[y,z],x] = [[y,x],z] + [y,[z,x]],$$

for all $x, y, z \in L$.

In presence of anti-commutativity, Jacobi identity becomes Leibniz identity and therefore Lie algebras are examples of Leibniz algebras. Throughout this paper $\mathbb{F}$ will be algebraically closed and with zero characteristic.

Let L be a Leibniz algebra. The ideal I generated by the squares of elements of the algebra L, that is I is generated by the set $\{[x,x] : x \in L\}$, plays an important role in the theory since it determines the (possible) non-Lie character of L. From the Leibniz identity, this ideal satisfies

$$[L, I] = 0.$$

The quotient algebra L/I is a Lie algebra, called the *corresponding Lie algebra* of L, and the map $I \times L/I \to I$, $(i,[x]) \mapsto [i,x]$, endows I of a structure of L/I-module (see [4, 37]). Observe that we can write

$$L = V \oplus I \tag{1}$$

where V is a linear complement of I in L and V is isomorphic as linear space to L/I. From here, Leibniz algebras give us the opportunity of treating in an unifying way a Lie algebra together with a module over it.

On the other hand, we recall that Heisenberg (Lie) algebras play an important role in mathematical physics and geometry, in particular in Quantum Mechanics (see for instance [1, 8, 9, 19, 20, 24, 26, 27, 29, 30, 31, 32, 36, 42, 44]). Indeed, the Heisenberg Principle of Uncertainty implies the non-compatibility of position and momentum observables acting on fermions. This non-compatibility reduces to non-commutativity of the corresponding operators. If we represent by $\overline{x}$ the operator associated to position and by $\frac{\overline{\partial}}{\partial x}$ the one associated to momentum (acting for instance on a space V of differentiable functions of a single variable), then $[\overline{x}, \frac{\overline{\partial}}{\partial x}] = \overline{1}_V$ which is non-zero. Thus we can identify the subalgebra generated by $\overline{1}, \overline{x}$ and $\frac{\overline{\partial}}{\partial x}$ with the three-dimensional

Heisenberg algebra whose multiplication table in the basis $\{\overline{1}, \overline{x}, \frac{\overline{\partial}}{\partial x}\}$ has as unique non-zero product $[\overline{x}, \frac{\overline{\partial}}{\partial x}] = \overline{1}$.

For any non-negative integer k the *Heisenberg algebra* of dimension $n = 2k + 1$ (denoted further by H_n) is characterized by the existence of a basis

$$B = \{\overline{1}, \overline{x}_1, \frac{\overline{\delta}}{\delta x_1}, \dots, \overline{x}_k, \frac{\overline{\delta}}{\delta x_k}\} \tag{2}$$

in which the multiplicative non-zero relations are

$$[\overline{x}_i, \frac{\overline{\delta}}{\delta x_i}] = -[\frac{\overline{\delta}}{\delta x_i}, \overline{x}_i] = \overline{1}$$

for $1 \leq i \leq k$.

In the present paper we are focusing in introducing and studying several classes of Leibniz algebras whose corresponding Lie algebras are Heisenberg algebras H_n. Recall that there is a unique irreducible representation of the Heisenberg algebra (at least a unique one that can be exponentiated). This is why physicists are able to use the Heisenberg commutation relations to do calculations, without worry about what they are being represented on. This representation is called the Fock (or Bargmann-Fock) representation (see [3, 7, 34, 38, 39, 43]). Physically this representation corresponds to an harmonic oscillator, with the vector $\overline{1} \in \mathbb{C}[x]$ as the vacuum state and $\overline{x}$ the operator that adds one quantum to the vacuum state. This representation is also sometimes known as the oscillator representation. For a given Heisenberg algebra H_n, $n = 2k + 1$, this representation gives rise to the so-called *Fock module* on H_n, the linear space $\mathbb{F}[x_1, ..., x_k]$ with the action induced by

$$\begin{array}{lcl} (p(x_1, ..., x_k), \overline{1}) & \mapsto & p(x_1, ..., x_k) \\ (p(x_1, ..., x_k), \overline{x_i}) & \mapsto & x_i p(x_1, ..., x_k) \\ (p(x_1, ..., x_k), \frac{\overline{\delta}}{\delta x_i}) & \mapsto & \frac{\delta}{\delta x_i}(p(x_1, ..., x_k)) \end{array} \tag{3}$$

for any $p(x_1, ..., x_k) \in \mathbb{F}[x_1, ..., x_k]$ and $i = 1, ..., k$.

Taking now into account the above comments, we introduce in Section 2 the class of *Heisenberg-Fock Leibniz algebras* as those Leibniz algebras whose corresponding Lie algebras are H_n and whose H_n-modules I are isomorphic to Fock modules, and provide a classification theorem. Thus, we have the opportunity of considering Heisenberg Lie algebras together with their Fock representations in a unifying viewpoint. In this section we also consider a generalization of this class of algebras by means of a direct sum of Heisenberg algebras as corresponding Lie algebras, and provide also a classification theorem.

In Section 3, we center in the three-dimensional Heisenberg algebra H_3 and study three classes of Leibniz algebras with H_3 as corresponding Lie algebra by taking certain generalizations of the Fock module. We also note that Sections 2 and 3 allow us to introduce several new classes of infinite-dimensional Leibniz algebras.

Finally, in Section 4, we deal with the category of Leibniz algebras with H_n as corresponding algebra and such that the action $I \times H_n \to I$ gives rise to a minimal faithful representation of H_n. A description of this category of algebras is given and also a classification theorem when $n = 3$.

2. Classification of Heisenberg-Fock type Leibniz algebras

2.1. **Classification of HFL_n.** Consider a Heisenberg algebra H_n, with $n = 2k + 1$, and its Fock module $\mathbb{F}[x_1, ..., x_k]$ under the action (3). The *Heisenberg-Fock Leibniz algebra* HFL_n is defined as the Leibniz algebra with corresponding Lie algebra H_n and such that the action $I \times H_n \to I$ makes of I the Fock module. Since $\mathbb{F}[x_1, ..., x_k]$ is infinite-dimensional we get a family of infinite-dimensional Leibniz algebras.

Theorem 1. *The Heisenberg-Fock Leibniz algebra HFL_n admits a basis*

$$\{\overline{1}, \overline{x}_i, \frac{\overline{\delta}}{\delta x_i}, \ x_1^{t_1} x_2^{t_2} \dots x_k^{t_k} \mid t_i \in \mathbb{N} \cup \{0\}, \ 1 \leq i \leq k\}$$

in such a way that the multiplication table on this basis has the form:

$$\begin{array}{ll} [\overline{x_i}, \frac{\overline{\delta}}{\delta x_i}] = \overline{1}, & 1 \leq i \leq k, \\ [\frac{\overline{\delta}}{\delta x_i}, \overline{x_i}] = -\overline{1}, & 1 \leq i \leq k, \end{array}$$

$$\begin{array}{ll}[x_1^{t_1}x_2^{t_2}\dots x_k^{t_k},\overline{1}]=x_1^{t_1}x_2^{t_2}\dots x_k^{t_k}, & \\ [x_1^{t_1}x_2^{t_2}\dots x_k^{t_k},\overline{x}_i]=x_1^{t_1}\dots x_{i-1}^{t_{i-1}}x_i^{t_i+1}x_{i+1}^{t_{i+1}}\dots x_k^{t_k}, & 1\le i\le k,\\ [x_1^{t_1}x_2^{t_2}\dots x_k^{t_k},\frac{\overline{\delta}}{\delta x_i}]=t_ix_1^{t_1}\dots x_{i-1}^{t_{i-1}}x_i^{t_i-1}x_{i+1}^{t_{i+1}}\dots x_k^{t_k}, & 1\le i\le k,\end{array}$$

where the omitted products are equal to zero.

Proof. Taking into account Equations (1) and (3) we conclude that

$$\{\overline{1},\overline{x}_i,\frac{\overline{\delta}}{\delta x_i},\ x_1^{t_1}x_2^{t_2}\dots x_k^{t_k}\mid t_i\in\mathbb{N}\cup\{0\},\ 1\le i\le k\}$$

is a basis of HFL_n and

$$\begin{array}{l}[x_1^{t_1}x_2^{t_2}\dots x_k^{t_k},\overline{1}]=x_1^{t_1}x_2^{t_2}\dots x_k^{t_k},\\ [x_1^{t_1}x_2^{t_2}\dots x_k^{t_k},\overline{x}_i]=x_1^{t_1}\dots x_{i-1}^{t_{i-1}}x_i^{t_i+1}x_{i+1}^{t_{i+1}}\dots x_k^{t_k},\\ [x_1^{t_1}x_2^{t_2}\dots x_k^{t_k},\frac{\overline{\delta}}{\delta x_i}]=t_ix_1^{t_1}\dots x_{i-1}^{t_{i-1}}x_i^{t_i-1}x_{i+1}^{t_{i+1}}\dots x_k^{t_k},\end{array}$$

for $1\le i\le k$.

Observe that we can write

$$\begin{array}{ll}[\overline{x_i},\overline{1}]=p_i(x_1,x_2,\dots,x_k), & 1\le i\le k,\\ [\frac{\overline{\delta}}{\delta x_i},\overline{1}]=q_i(x_1,x_2,\dots,x_k), & 1\le i\le k,\\ [\overline{1},\overline{1}]=r(x_1,x_2,\dots,x_k), & \end{array}$$

where $p_i,q_i,r\in\mathbb{F}[x_1,...,x_k]$.

Taking the following change of basis,

$$\begin{array}{c}\overline{x_i}'=\overline{x_i}-p_i(x_1,x_2,\dots,x_k),\quad 1\le i\le k,\\ \frac{\overline{\delta}}{\delta x_i}'=\frac{\overline{\delta}}{\delta x_i}-q_i(x_1,x_2,\dots,x_k),\quad 1\le i\le k,\\ \overline{1}'=\overline{1}-r(x_1,x_2,\dots,x_k),\end{array}$$

we derive

$$[\overline{x_i},\overline{1}]=0,\quad [\frac{\overline{\delta}}{\delta x_i},\overline{1}]=0,\quad [\overline{1},\overline{1}]=0,\quad 1\le i\le k.$$

Now denote

$$\begin{array}{lll}[\overline{x_i},\overline{x_j}]=a_{i,j}(x_1,x_2,\dots,x_k), & [\frac{\overline{\delta}}{\delta x_i},\frac{\overline{\delta}}{\delta x_j}]=b_{i,j}(x_1,x_2,\dots,x_k), & 1\le i,j\le k,\\ [\frac{\overline{\delta}}{\delta x_i},\overline{x_j}]=c_{i,j}(x_1,x_2,\dots,x_k), & [\overline{x_i},\frac{\overline{\delta}}{\delta x_j}]=d_{i,j}(x_1,x_2,\dots,x_k), & 1\le i,j\le k,\ i\ne j,\\ [\overline{x_i},\frac{\overline{\delta}}{\delta x_i}]=\overline{1}+e_i(x_1,x_2,\dots,x_k), & [\frac{\overline{\delta}}{\delta x_i},\overline{x_i}]=-\overline{1}+f_i(x_1,x_2,\dots,x_k), & 1\le i\le k,\\ [\overline{1},\overline{x_i}]=h_i(x_1,x_2,\dots,x_k), & [\overline{1},\frac{\overline{\delta}}{\delta x_i}]=g_i(x_1,x_2,\dots,x_k), & 1\le i\le k.\end{array}$$

The Leibniz identity on the following triples imposes further constraints on the products.

Leibniz identity		Constraint
$\{\overline{x_i},\overline{x_j},\overline{1}\}$	$\Rightarrow$	$a_{i,j}(x_1,x_2,\dots,x_k)=0,\quad 1\le i,j\le k,$
$\{\frac{\overline{\delta}}{\delta x_i},\frac{\overline{\delta}}{\delta x_j},\overline{1}\}$	$\Rightarrow$	$b_{i,j}(x_1,x_2,\dots,x_k)=0,\quad 1\le i,j\le k,$
$\{\frac{\overline{\delta}}{\delta x_i},\overline{x_j},\overline{1}\}$	$\Rightarrow$	$c_{i,j}(x_1,x_2,\dots,x_k)=0,\quad 1\le i,j\le k,\ i\ne j,$
$\{\overline{x_i},\frac{\overline{\delta}}{\delta x_j},\overline{1}\}$	$\Rightarrow$	$d_{i,j}(x_1,x_2,\dots,x_k)=0,\quad 1\le i,j\le k,\ i\ne j,$
$\{\overline{x_i},\frac{\overline{\delta}}{\delta x_i},\overline{1}\}$	$\Rightarrow$	$e_i(x_1,x_2,\dots,x_k)=0,\quad 1\le i\le k,$
$\{\frac{\overline{\delta}}{\delta x_i},\overline{x_i},\overline{1}\}$	$\Rightarrow$	$f_i(x_1,x_2,\dots,x_k)=0,\quad 1\le i\le k,$
$\{\overline{1},\overline{x_i},\overline{1}\}$	$\Rightarrow$	$h_i(x_1,x_2,\dots,x_k)=0,\quad 1\le i\le k,$
$\{\overline{1},\frac{\overline{\delta}}{\delta x_i},\overline{1}\}$	$\Rightarrow$	$g_i(x_1,x_2,\dots,x_k)=0,\quad 1\le i\le k.$

The proof is complete. ∎

2.2. **Classification of generalized Heisenberg-Fock Leibniz algebras.** In this subsection we are interested in classifying the class of (infinite-dimensional) Leibniz algebras formed by those Leibniz algebras L satisfying that their corresponding Lie algebras are finite direct sums of Heisenberg algebras and that the actions on I are induced by Fock representations.

Since

$$L/I \cong H_{2k_1+1} \oplus H_{2k_2+1} \oplus H_{2k_3+1} \oplus \cdots \oplus H_{2k_s+1}, \tag{4}$$

we easily get

$$\mathcal{B}_i := \{\overline{1_i}, \overline{x_{1,i}}, \overline{x_{2,i}}, ..., \overline{x_{k_i,i}}, \frac{\overline{\delta}}{\delta x_{1,i}}, \frac{\overline{\delta}}{\delta x_{2,i}}, ..., \frac{\overline{\delta}}{\delta x_{k_i,i}}\} \tag{5}$$

for the standard basis of H_{2k_i+1}, $i \in \{1, 2, ..., s\}$.

We put

$$I = \mathbb{F}[x_1, ..., x_n], \tag{6}$$

where $n = k_1 + k_2 + \cdots + k_s$.

The action

$$I \times L/I \to I$$

given by

$$\begin{array}{lcl} (p(x_1,\ldots,x_n), \overline{1_i}) & \mapsto & p(x_1,\ldots,x_n) \\ (p(x_1,\ldots,x_n), \overline{x_{j,i}}) & \mapsto & p(x_1,\ldots,x_n)x_{k_1+k_2+\cdots+k_{i-1}+j} \\ (p(x_1,\ldots,x_n), \frac{\overline{\delta}}{\delta x_{j,i}}) & \mapsto & \frac{\delta}{\delta x_{k_1+k_2+\cdots+k_{i-1}+j}} p(x_1,\ldots,x_n) \end{array}$$

for any $p(x_1,\ldots,x_n) \in \mathbb{F}[x_1, ..., x_n]$ and (i, j) with $i \in \{1, 2, ..., s\}$, $j \in \{1, ..., k_i\}$, endows I of a structure of L/I-module. Hence, we get a new family of Heisenberg-Fock type Leibniz algebras which generalize the previous ones considered in §2.1 (case $s = 1$), that we call *generalized Heisenberg-Fock Leibniz algebras*, by introducing the algebras $L = L/I \oplus I$ with L/I and I as in Equations (4) and (6). We will denote them as

$$HFL_{2k_1+1,2k_2+1,\ldots,2k_s+1}.$$

Our aim is to classify this class of Leibniz algebras.

By taking into account the previous arguments, it is clear that for any $i \in \{1, 2, ..., s\}$ we have $[H_{2k_i+1}, H_{2k_i+1}] \subset H_{2k_i+1}$ being the multiplication table among the elements in the basis $\mathcal{B}_i$ as in Theorem 1. Therefore, we only need to study the products $[H_{2k_i+1}, H_{2k_j+1}]$ with $i, j \in \{1, 2, ..., s\}$ and $i \neq j$.

Lemma 1. *Let $a \in \mathcal{B}_i$ and $b \in \mathcal{B}_j$, $i, j \in \{1, 2, ..., s\}$ with $i \neq j$. Then $[a, b] = 0$.*

Proof. For $i \neq j$ we have $[a, b] = p$ and $[b, \overline{1_i}] = q$ for some $p, q \in \mathbb{F}[x_1, ..., x_n]$. Taking now into account Theorem 1 we derive $[a, \overline{1_i}] = 0$ and so

$$p = [[a, b], \overline{1_i}] = [[a, \overline{1_i}], b] + [a, [b, \overline{1_i}]] = 0.$$

□

The next theorem is now consequence of Theorem 1 and Lemma 1.

Theorem 2. *The Leibniz algebra $HFL_{2k_1+1,2k_2+1,\ldots,2k_s+1}$ admits a basis (see Equations (5) and (6))*

$$\mathcal{B}_1 \dot{\cup} \mathcal{B}_2 \dot{\cup} \cdots \dot{\cup} \mathcal{B}_s \dot{\cup} \{x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n} \mid t_i \in \mathbb{N} \cup \{0\},\ 1 \leq i \leq n\},$$

where $n = k_1 + k_2 + \cdots + k_s$, and in such a way that the multiplication table on this basis has the form:

$$\begin{array}{l} [\overline{x_{j,i}}, \frac{\overline{\delta}}{\delta x_{j,i}}] = \overline{1}_i, \qquad [\frac{\overline{\delta}}{\delta x_{j,i}}, \overline{x_{j,i}}] = -\overline{1}_i, \\ [x_1^{t_1} x_2^{t_2} \ldots x_n^{t_n}, \overline{1_i}] = x_1^{t_1} x_2^{t_2} \ldots x_n^{t_n}, \\ [x_1^{t_1} x_2^{t_2} \ldots x_n^{t_n}, \overline{x}_{j,i}] = x_1^{t_1} \ldots x_{k_1+\cdots+k_{i-1}+j-1}^{t_{k_1+\cdots+k_{i-1}+j-1}} x_{k_1+\cdots+k_{i-1}+j}^{t_{k_1+\cdots+k_{i-1}+j}+1} x_{k_1+\cdots+k_{i-1}+j+1}^{t_{k_1+\cdots+k_{i-1}+j+1}} \ldots x_n^{t_n}, \\ [x_1^{t_1} x_2^{t_2} \ldots x_k^{t_k}, \frac{\overline{\delta}}{\delta x_{j,i}}] = t_{k_1+\cdots+k_{i-1}+j} x_1^{t_1} \ldots x_{k_1+\cdots+k_{i-1}+j-1}^{t_{k_1+\cdots+k_{i-1}+j-1}} x_{k_1+\cdots+k_{i-1}+j}^{t_{k_1+\cdots+k_{i-1}+j}-1} x_{k_1+\cdots+k_{i-1}+j+1}^{t_{k_1+\cdots+k_{i-1}+j+1}} \ldots x_n^{t_n}, \end{array}$$

for $1 \leq i \leq s$, $1 \leq j \leq k_i$ and where the omitted products are equal to zero.

3. SEVERAL DEGENERATIONS OF THE FOCK REPRESENTATION FOR THE 3-DIMENSIONAL HEISENBERG ALGEBRA

In this section we consider several degenerations of the Fock representation of the Heisenberg algebra H_3. First, we study when an extension of the Fock action $\mathbb{F}[x]\times H_3 \to \mathbb{F}[x]$, (see Equation (3)), by allowing arbitrary polynomials as results of the action of a fixed element in the basis $\{\overline{1}, \overline{x}, \overline{\frac{\delta}{\delta x}}\}$ of H_3 over the elements of $\mathbb{F}[x]$, makes of $\mathbb{F}[x]$ an H_3-module. Second, the new H_3-modules obtained in this way give rise to new classes of Leibniz algebras that will be described.

For any linear mapping $\Omega : \mathbb{F}[x] \to \mathbb{F}[x]$, consider the linear space $\mathbb{F}[x]$ with the action induced by the following applications:

$$\begin{array}{cccc} \psi_1: & \mathbb{F}[x]\times H_3 & \to & \mathbb{F}[x] \\ & (p(x),\overline{1}) & \mapsto & \Omega(p(x)) \\ & (p(x),\overline{x}) & \mapsto & xp(x) \\ & (p(x),\overline{\frac{\delta}{\delta x}}) & \mapsto & \frac{\delta}{\delta x}p(x). \end{array} \qquad \begin{array}{cccc} \psi_2: & \mathbb{F}[x]\times H_3 & \to & \mathbb{F}[x] \\ & (p(x),\overline{1}) & \mapsto & p(x) \\ & (p(x),\overline{x}) & \mapsto & \Omega(p(x)) \\ & (p(x),\overline{\frac{\delta}{\delta x}}) & \mapsto & \frac{\delta}{\delta x}p(x). \end{array}$$

$$\begin{array}{cccc} \psi_3: & \mathbb{F}[x]\times H_3 & \to & \mathbb{F}[x] \\ & (p(x),\overline{1}) & \mapsto & p(x) \\ & (p(x),\overline{x}) & \mapsto & xp(x) \\ & (p(x),\overline{\frac{\delta}{\delta x}}) & \mapsto & \Omega(p(x)) \end{array}$$

for any $p(x) \in \mathbb{F}[x]$.

From now on, let us denote by $\{x^i\}_{i\in\mathbb{N}\cup\{0\}}$ the canonical basis of $\mathbb{F}[x]$. By considering $\psi_1(p(x),[\overline{x},\overline{\frac{\delta}{\delta x}}])$, it is immediate to get that the first action ψ_1 makes of $\mathbb{F}[x]$ an H_3-module if and only if $\Omega = 1_{\mathbb{F}[x]}$. As consequence we have.

Proposition 1. *The Leibniz algebras obtained from the first action ψ_1 are the same as those obtained in Theorem 1.*

Consider now the second action $\psi_2 : \mathbb{F}[x] \times H_3 \to \mathbb{F}[x]$.

Proposition 2. *The action ψ_2 makes of $\mathbb{F}[x]$ an H_3-module if and only if*

$$\Omega(x^i) = x^{i+1} + \sum_{k=0}^{i} c_k \binom{i}{k} x^{i-k}, \tag{7}$$

where $\{c_k\}_{k\in\mathbb{N}\cup\{0\}}$ is a fixed sequence in $\mathbb{F}$ and $\binom{i}{k}$ are binomial coefficients.

Proof. Suppose $\mathbb{F}[x]$ is an H_3-module through the action ψ_2. Then we have

$$x^i = [x^i,\overline{1}] = [x^i,[\overline{x},\overline{\frac{\delta}{\delta x}}]] = [[x^i,\overline{x}],\overline{\frac{\delta}{\delta x}}] - [[x^i,\overline{\frac{\delta}{\delta x}}],\overline{x}] = [[x^i,\overline{x}],\overline{\frac{\delta}{\delta x}}] - [ix^{i-1},\overline{x}]$$

and so

$$[[x^i,\overline{x}],\overline{\frac{\delta}{\delta x}}] = x^i + [ix^{i-1},\overline{x}]. \tag{8}$$

Taking into account Equation (8), we can easily prove by induction (7). Indeed, for $i = 0$ we get from (8) that $[[1,\overline{x}],\overline{\frac{\delta}{\delta x}}] = 1$, which implies $[1,\overline{x}] = x + c_0 = \Omega(1)$. For $i = 1$ the same equation allows us to get $[[x,\overline{x}],\overline{\frac{\delta}{\delta x}}] = x + [1,\overline{x}] = 2x + c_0$ and so $[x,\overline{x}] = x^2 + c_0x + c_1 = \Omega(x)$.

Let the induction hypothesis true for $i = j$ and we will show it for $i = j + 1$. Taking into account (8) we have

$$\begin{aligned} [[x^{j+1},\overline{x}],\overline{\tfrac{\delta}{\delta x}}] &= x^{j+1} + [(j+1)x^j,\overline{x}] = x^{j+1} + (j+1)(x^{j+1} + \sum_{k=0}^{j} c_k \binom{j}{k} x^{j-k}) = \\ &= (j+2)x^{j+1} + \sum_{k=0}^{j} c_k (j+1)\binom{j}{k} x^{j-k} = \end{aligned}$$

$$\begin{aligned} &= (j+2)x^{j+1} + \sum_{k=0}^{j} c_k (j+1) \tfrac{j!}{k!(j-k)!} x^{j-k} = \\ &= (j+2)x^{j+1} + \sum_{k=0}^{j} c_k \tfrac{(j+1)!}{k!(j+1-k)!} (j+1-k) x^{j-k}. \end{aligned}$$

From here

$$[x^{j+1}, \overline{x}] = x^{j+2} + \sum_{k=0}^{j} c_k \frac{(j+1)!}{k!(j+1-k)!} x^{j+1-k} + c_{j+1} = x^{j+2} + \sum_{k=0}^{j+1} c_k \binom{j+1}{k} x^{j+1-k},$$

that is,

$$\Omega(x^{j+1}) = x^{j+2} + \sum_{k=0}^{j+1} c_k \binom{j+1}{k} x^{j+1-k}.$$

The converse is of immediate verification. □

Proposition 3. *Any Leibniz algebra obtained from the second action ψ_2 admits a basis*

$$\{\overline{1}, \overline{x}, \frac{\overline{\delta}}{\delta x}\} \dot{\cup} \{x^i : i \in \mathbb{N} \cup \{0\}\}$$

in such a way that the multiplication table on this basis has the form:

$$\begin{array}{lll} [x^i, \overline{1}] = x^i, & [x^i, \overline{x}] = \Omega(x^i), & [x^i, \frac{\overline{\delta}}{\delta x}] = i x^{i-1}, \\ [\overline{x}, \frac{\overline{\delta}}{\delta x}] = \overline{1}, & [\frac{\overline{\delta}}{\delta x}, \overline{x}] = -\overline{1}, & \end{array}$$

where the omitted products are equal to zero and $\Omega(x^i)$ satisfies Equation (7).

Proof. By Proposition 2 we have the restriction on $\Omega(x^i)$. On the other hand, we know

$$\begin{array}{lll} [x^i, \overline{1}] = x^i, & [x^i, \overline{x}] = \Omega(x^i), & [x^i, \frac{\overline{\delta}}{\delta x}] = i x^{i-1}, \\ [\overline{x}, \overline{1}] = p(x), & [\overline{x}, \frac{\overline{\delta}}{\delta x}] = \overline{1} + q(x), & [\overline{x}, \overline{x}] = a(x), \\ [\frac{\overline{\delta}}{\delta x}, \overline{1}] = r(x), & [\frac{\overline{\delta}}{\delta x}, \frac{\overline{\delta}}{\delta x}] = b(x), & [\frac{\overline{\delta}}{\delta x}, \overline{x}] = -\overline{1} + s(x), \\ [\overline{1}, \overline{x}] = c(x), & [\overline{1}, \overline{1}] = d(x), & [\overline{1}, \frac{\overline{\delta}}{\delta x}] = e(x). \end{array}$$

By making the change of basis $\overline{1}' = \overline{1} + q(x)$ we can suppose that $[\overline{x}, \frac{\overline{\delta}}{\delta x}] = \overline{1}$.
Now, from Leibniz identity we obtain the following equations:

Leibniz identity		Constraint
$\{\overline{1}, \overline{1}, \overline{1}\}$	$\Rightarrow$	$c(x) = [d(x), \overline{x}]$,
$\{\overline{1}, \overline{1}, \frac{\overline{\delta}}{\delta x}\}$	$\Rightarrow$	$e(x) = \frac{\delta}{\delta x}(d(x))$,
$\{\overline{1}, \overline{x}, \frac{\overline{\delta}}{\delta x}\}$	$\Rightarrow$	$[e(x), \overline{x}] = \frac{\delta}{\delta x}(c(x)) - d(x)$,
$\{\overline{x}, \overline{1}, \overline{x}\}$	$\Rightarrow$	$a(x) = [p(x), \overline{x}]$,
$\{\overline{x}, \overline{1}, \frac{\overline{\delta}}{\delta x}\}$	$\Rightarrow$	$d(x) = \frac{\delta}{\delta x}(p(x))$,
$\{\overline{x}, \overline{x}, \frac{\overline{\delta}}{\delta x}\}$	$\Rightarrow$	$p(x) + c(x) = \frac{\delta}{\delta x}(a(x))$,
$\{\frac{\overline{\delta}}{\delta x}, \overline{1}, \overline{x}\}$	$\Rightarrow$	$s(x) = d(x) + [r(x), \overline{x}]$,
$\{\frac{\overline{\delta}}{\delta x}, \overline{1}, \frac{\overline{\delta}}{\delta x}\}$	$\Rightarrow$	$b(x) = \frac{\delta}{\delta x}(a(x))$,
$\{\frac{\overline{\delta}}{\delta x}, \overline{x}, \frac{\overline{\delta}}{\delta x}\}$	$\Rightarrow$	$[b(x), \overline{x}] = -e(x) - r(x) + \frac{\delta}{\delta x}(a(x))$.

By making the next change of basis:

$$\begin{array}{l} \overline{1}' = \overline{1} - \frac{\delta}{\delta x}(p(x)), \\ \overline{x}' = \overline{x} - p(x), \\ \frac{\overline{\delta}}{\delta x}' = \frac{\overline{\delta}}{\delta x} - r(x), \end{array}$$

we obtain the family of the proposition. □

Finally we consider the third action $\psi_3 : \mathbb{F}[x] \times H_3 \to \mathbb{F}[x]$, being then

$$\begin{aligned}&[x^i, \overline{1}] = x^i,\\ &[x^i, \overline{x}] = x^{i+1},\\ &[x^i, \tfrac{\overline{\delta}}{\delta x}] = \Omega(x^i), \quad i \in \mathbb{N} \cup \{0\}.\end{aligned}$$

By arguing in a similar way to Propositions 2 and 3 we can prove the next results.

Proposition 4. *The action ψ_3 makes of $\mathbb{F}[x]$ an H_3-module if and only if*

$$\Omega(x^i) = ix^{i-1} + x^i c(x). \tag{9}$$

for a fixed $c(x) \in \mathbb{F}[x]$ and $i \in \mathbb{N} \cup \{0\}$.

Proposition 5. *Any Leibniz algebra obtained from the third action ψ_3 admits a basis*

$$\{\overline{1}, \overline{x}, \frac{\overline{\delta}}{\delta x}\}\dot{\cup}\{x^i : i \in \mathbb{N} \cup \{0\}\}$$

in such a way that the multiplication table on this basis has the form:

$$\begin{array}{lll}[x^i, \overline{1}] = x^i, & [x^i, \overline{x}] = x^{i+1}, & [x^i, \frac{\overline{\delta}}{\delta x}] = \Omega(x^i),\\ [\overline{x}, \frac{\overline{\delta}}{\delta x}] = \overline{1}, & [\frac{\overline{\delta}}{\delta x}, \overline{x}] = -\overline{1}, & \end{array}$$

where the omitted products are equal to zero and $\Omega(x^i)$ satisfies Equation (9).

4. Leibniz algebras of minimal faithful representation-Heisenberg type

4.1. **General case.** Let H_{2m+1} be a Heisenberg algebra of dimension $2m+1$, then it is well-known that its minimal faithful representations have dimension $m+2$, (see [13]). From now on, for a more comfortable notation, we will denote by

$$\{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m, z\}$$

the standard basis of H_{2m+1}, (see Equation (2)), where the non-zero products are

$$[y_i, x_i] = -[x_i, y_i] = z.$$

By [18], we can take as minimal faithful representation the linear mapping

$$\varphi : H_{2m+1} \to \mathrm{End}(I),$$

where I is an $(m+2)$-dimensional linear space with a fixed basis $\{e_1, e_2, \ldots, e_{m+2}\}$, determined by

$$\begin{array}{ll}\varphi(x_i) = E_{1,i+1} & 1 \le i \le m,\\ \varphi(y_i) = E_{i+1,m+2} & 1 \le i \le m,\\ \varphi(z) = E_{1,m+2}. & \end{array}$$

Here $E_{i,j}$ denotes the elemental matrix with 1 in the (i,j) slot and 0 in the remaining places and we have $\varphi([x,y])(e) = \varphi(y)\big(\varphi(x)(e)\big) - \varphi(x)\big(\varphi(y)(e)\big)$ for any $x, y \in H_{2m+1}$ and $e \in I$. Observe that H_{2m+1} corresponds to the $(m+2) \times (m+2)$ matrices

$$\begin{pmatrix} 0 & a_2 & a_3 & \ldots & a_{m+1} & c\\ 0 & 0 & 0 & \ldots & 0 & b_2\\ 0 & 0 & 0 & \ldots & 0 & b_3\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 0 & b_{m+1}\\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

This representation makes of I an H_{2m+1}-module under the action

$$\begin{array}{llll}\phi : & I \times H_{2m+1} & \to & I\\ & (e_{i+1}, x_i) & \mapsto & e_1, \quad 1 \le i \le m,\\ & (e_{m+2}, y_i) & \mapsto & e_{i+1}, \quad 1 \le i \le m,\\ & (e_{m+2}, z) & \mapsto & e_1,\end{array} \tag{10}$$

being zero the remaining products among the bases elements in the action.

In this section we are going to study the Leibniz algebras $(L, [\cdot,\cdot])$ satisfying that $L/I \cong H_{2m+1}$ and where the H_{2m+1}-module I is isomorphic to the minimal faithful representation (I, ϕ). From the above, $\dim L = 3m+3$ and $\{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m, z, e_1, e_2, \dots, e_{m+2}\}$ is a basis of L. We also have

$$\begin{array}{ll} [e_{i+1}, x_i] = e_1, & 1 \le i \le m, \\ [e_{m+2}, y_i] = e_{i+1}, & 1 \le i \le m, \\ [e_{m+2}, z] = e_1. & \end{array}$$

Theorem 3. *Let L be a Leibniz algebra such that $L/I \cong H_{2m+1}$ ($m \neq 1$) and I is the L/I-module with the minimal faithful representation given by Equation (10). Then L admits a basis*

$$\{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m, z, e_1, e_2, \dots, e_{m+2}\}$$

in such a way that the multiplications table on this basis has the form

$$\begin{array}{ll} [e_{i+1}, x_i] = e_1, & [e_{m+2}, y_i] = e_{i+1}, \\ [e_{m+2}, z] = e_1, & [x_i, x_j] = \sum\limits_{s=1}^{m+1} \alpha_{i,j}^s e_s, \\ [x_i, y_j] = \gamma_{i,j} e_1, \ \ i \neq j, & [x_i, y_i] = -z + \delta_i e_1 + \tau e_2 + \sum\limits_{s=2}^{m} \nu_{1,s}^2 e_{s+1}, \\ [y_i, y_j] = \beta_{i,j} e_1, & [y_1, x_1] = z, \\ [y_i, x_j] = \sum\limits_{s=1}^{m+1} \nu_{i,j}^s e_s, \ \ i \neq j, & [y_i, x_i] = z + (\nu_{i,1}^{i+1} - \tau) e_2 + \varepsilon_i^{i+1} e_{i+1} + \sum\limits_{s=2}^{m}\sum\limits_{s \neq i} (\nu_{i,s}^{i+1} - \nu_{1,s}^2) e_{s+1}, \ i \neq 1, \\ [z, x_1] = \tau e_1, & [z, x_i] = \nu_{1,i}^2 e_1, \ \ i \neq 1, \end{array}$$

for $1 \le i, j \le m$, where any $\alpha_{p,q}^r, \gamma_{p,q}, \delta_p, \tau, \nu_{p,q}^r, \beta_{p,q}, \varepsilon_p^r \in \mathbb{F}$ and where the omitted products are equal to zero.

Proof. We consider the following products:

$$[y_i, x_i] = z + \sum_{k=1}^{m+2} \varepsilon_i^k e_k, \quad 1 \le i \le m.$$

Putting $z' = z + \sum\limits_{k=1}^{m+2} \varepsilon_1^k e_k$ we can assume $[y_1, x_1] = z$. Thus, we have

$$\begin{array}{lll} [e_{i+1}, x_i] = e_1, & [e_{m+2}, y_i] = e_{i+1}, & [e_{m+2}, z] = e_1, \\ [x_i, x_j] = \sum\limits_{k=1}^{m+2} \alpha_{i,j}^k e_k, & [x_i, y_j] = \sum\limits_{k=1}^{m+2} \gamma_{i,j}^k e_k, \ \ i \neq j & [x_i, y_i] = -z + \sum\limits_{k=1}^{m+2} \delta_i^k e_k, \\ [x_i, z] = \sum\limits_{k=1}^{m+2} \eta_i^k e_k, & [y_i, y_j] = \sum\limits_{k=1}^{m+2} \beta_{i,j}^k e_k, & [y_i, x_j] = \sum\limits_{k=1}^{m+2} \nu_{i,j}^k e_k, \ \ i \neq j, \\ [y_i, z] = \sum\limits_{k=1}^{m+2} \theta_i^k e_k, & [y_1, x_1] = z, & [y_i, x_i] = z + \sum\limits_{k=1}^{m+2} \varepsilon_i^k e_k, \ \ i \neq 1, \\ [z, x_i] = \sum\limits_{k=1}^{m+2} \tau_i^k e_k, & [z, y_i] = \sum\limits_{k=1}^{m+2} \lambda_i^k e_k, & [z, z] = \sum\limits_{k=1}^{m+2} \mu^k e_k, \end{array}$$

with $1 \le i, j \le m$.

We compute all Leibniz identities using the software Mathematica and we get the following restrictions:

Leibniz identity	Constraint	
$\{z,z,y_k\}$	$\Rightarrow\ \mu^{m+2}=\lambda_k^{m+2}=0,$	$1\le k\le m,$
$\{z,z,x_k\}$	$\Rightarrow\ \mu^{k+1}=\tau_k^{m+2}=0,$	$1\le k\le m,$
$\{z,y_j,x_k\}$	$\Rightarrow\ \lambda_j^{k+1}=0,\ \mu^1=\lambda_1^2=\lambda_j^{j+1},$	$1\le j,k\le m,\ j\neq k,$
$\{z,x_j,x_k\}$	$\Rightarrow\ \tau_j^{k+1}=\tau_k^{j+1},$	$1\le j,k\le m,$
$\{y_i,z,y_k\}$	$\Rightarrow\ \theta_i^{m+2}=\beta_{i,k}^{m+2}=0,$	$1\le i,j,k\le m,$
$\{y_i,z,x_k\}$	$\Rightarrow\ \nu_{i,k}^{m+2}=\theta_i^{k+1},$	$1\le i,k\le m,\ i\neq k,$
	$\Rightarrow\ \theta_i^{i+1}-\mu^1=0,\ \mu^1=\theta_1^2,$	$1\le i\le m,\ k=i,$
$\{y_i,y_j,x_k\}$	$\Rightarrow\ \beta_{i,j}^{k+1}=\nu_{i,k}^{m+2}=0,$	$1\le i,j,k\le m,\ j\neq k\neq i,$
	$\Rightarrow\ \theta_i^1=\beta_{i,j}^{j+1},\ \theta_i^s=0,$	$1\le i,j\le m,\ k=j,\ i\neq j,\ 2\le s\le m+1,$
	$\Rightarrow\ \beta_{i,j}^{i+1}=\lambda_j^1,\ -\lambda_j^{j+1}-\varepsilon_i^{m+2}=0,$	$1\le i,j\le m,\ i\neq 1,$
	$\Rightarrow\ \lambda_j^{j+1}=0,$	$i=1,\ j\neq 1,$
	$\Rightarrow\ \theta_1^s=0,$	$3\le s\le m+1,\ i=j=k=1,$
$\{y_i,x_i,y_i\}$	$\Rightarrow\ \theta_i^1=\beta_{i,i}^{i+1},$	$1\le i\le m,$
$\{y_i,x_j,x_k\}$	$\Rightarrow\ \nu_{i,j}^{k+1}=\nu_{i,k}^{j+1},$	$1\le i,j,k\le m,\ j\neq i\neq k,$
	$\Rightarrow\ \tau_k^s=0,\ \tau_k^1+\varepsilon_i^{k+1}-\nu_{i,k}^{i+1}=0,$	$2\le s\le m+1,\ 1\le i,k\le m,\ j=i\neq k,$
	$\Rightarrow\ \tau_j^1=\nu_{1,j}^2,$	$1\le j\le m,\ i=k=1,\ j\neq 1,$
$\{x_i,z,y_k\}$	$\Rightarrow\ \eta_i^{m+2}=\gamma_{i,k}^{m+2}=0,$	$1\le i,k\le m,\ i\neq k,$
	$\Rightarrow\ \delta_i^{m+2}=0,$	$1\le i\le m,\ i=k,$
$\{x_i,z,x_k\}$	$\Rightarrow\ \eta_i^{k+1}=\alpha_{i,k}^{m+2},$	$1\le i,k\le m,$
$\{x_i,y_i,y_k\}$	$\Rightarrow\ \lambda_k^1=0,$	$1\le k\le m,$
$\{x_i,y_j,y_k\}$	$\Rightarrow\ \gamma_{i,j}^{k+1}=\alpha_{i,k}^{m+2}=0,$	$1\le i,j,k\le m,\ i\neq j\neq k$
	$\Rightarrow\ -\tau_k^1+\delta_i^{k+1}=0,$	$1\le i,k\le m,\ j=i\neq k,$
	$\Rightarrow\ \gamma_{i,j}^{i+1}=0,$	$1\le ij\le m,\ k=j\neq i,$
	$\Rightarrow\ \eta_i^1=-\tau_i^1+\delta_i^{i+1},\ \eta_i^s=0,$	$2\le s\le m,\ 1\le i\le m,\ j=k=i,$
$\{x_i,x_j,y_k\}$	$\Rightarrow\ \gamma_{i,j}^k=0,$	$1\le i,j,k\le m,\ i\neq k\neq j,$
	$\Rightarrow\ \eta_i^1=\gamma_{i,j}^{j+1},$	$1\le i,j\le m,\ k=j\neq i,$
$\{x_i,x_j,x_k\}$	$\Rightarrow\ \alpha_{i,j}^{k+1}=\alpha_{i,k}^{j+1},$	$1\le i,j,k\le m.$

From here,

$$
\begin{aligned}
&[e_{i+1},x_i]=e_1, && 1\le i\le m,\\
&[e_{m+2},y_i]=e_{i+1}, && 1\le i\le m,\\
&[e_{m+2},z]=e_1, &&\\
&[x_i,x_j]=\sum_{s=1}^{m+1}\alpha_{i,j}^s e_s, && 1\le i,j\le m,\\
&[y_i,y_j]=\beta_{i,j}^1 e_1+\theta_i^1 e_{j+1}, && 1\le i,j\le m,
\end{aligned}
$$

$$
\begin{array}{ll}
[x_i, y_j] = \gamma_{i,j}^1 e_1 + \eta_i^1 e_{j+1}, & 1 \leq i, j \leq m,\ i \neq j \\
[x_1, y_1] = -z + \delta_1^1 e_1 + (\eta_1^1 + \tau_1^1) e_2 + \sum\limits_{s=2}^{m} \nu_{1,s}^2 e_{s+1}, & 1 \leq i \leq m, \\
[x_i, y_i] = -z + \delta_i^1 e_1 + \tau_1^1 e_2 + (\eta_i^1 + \nu_{1,i}^2) e_{i+1} + \sum\limits_{s=2\, s\neq i}^{m} \nu_{1,s}^2 e_{s+1}, & 2 \leq i \leq m, \\
[y_1, x_1] = z, & \\
[y_i, x_i] = z + \varepsilon_i^1 e_1 + (\nu_{i,1}^{i+1} - \tau_1^1) e_2 + \varepsilon_i^{i+1} e_{i+1} + \sum\limits_{s=2\, s\neq i}^{m} (\nu_{i,s}^{i+1} - \nu_{1,s}^2) e_{s+1}, & 2 \leq i \leq m, \\
[y_i, x_j] = \sum\limits_{s=1}^{m+1} \nu_{i,j}^s e_s, & 1 \leq i, j \leq m,\ i \neq j \\
[x_i, z] = \eta_i^1 e_1, & 1 \leq i \leq m, \\
[y_i, z] = \theta_i^1 e_1, & 1 \leq i \leq m, \\
[z, x_1] = \tau_1^1 e_1, & \\
[z, x_i] = \nu_{1,i}^2 e_1, & 2 \leq i \leq m,
\end{array}
$$

with the following restrictions

$$
\begin{array}{ll}
\alpha_{i,j}^{k+1} = \alpha_{i,k}^{j+1}, & 1 \leq i, j, k \leq m, \\
\nu_{i,j}^{k+1} = \nu_{i,k}^{j+1}, & 1 \leq i, j, k \leq m,\ j \neq i \neq k.
\end{array}
$$

Only rest to make the next change of basis

$$
\begin{cases}
x_i' = x_i - \eta_i^1 e_{m+2}, & 1 \leq i \leq m, \\
y_1' = y_1 - \theta_1^1 e_{m+2}, & \\
y_j' = y_j - \varepsilon_j^1 e_{j+1} - \theta_j^1 e_{m+2}, & 2 \leq j \leq m,
\end{cases}
$$

and we obtain the family of the theorem (renaming the parameters). □

4.2. **Particular case: Classification of Leibniz algebras when $m = 1$.** In this subsection we classify the Leibniz algebras such that $L/I \cong H_3$ and I is the L/I-module with the minimal faithful representation given by Equation (10). Let us fix $\{x, y, z, e_1, e_2, e_3\}$ as basis of L. All computations have been made by using the software $Mathematica$.

We have the following products:

$$
\begin{array}{lll}
[e_2, x] = e_1, & [e_3, y] = e_2, & [e_3, z] = e_1, \\
[x, x] = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, & [x, y] = -z + \delta_1 e_1 + \delta_2 e_2 + \delta_3 e_3, & [x, z] = \eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3, \\
[y, y] = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3, & [y, x] = z, & [y, z] = \theta_1 e_1 + \theta_2 e_2 + \theta_3 e_3, \\
[z, x] = \tau_1 e_1 + \tau_2 e_2 + \tau_3 e_3, & [z, y] = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3, & [z, z] = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3.
\end{array}
$$

The Leibniz identity on the following triples imposes further constraints on the products.

Leibniz identity		Constraint
$\{x, x, y\}$	$\Rightarrow$	$-\eta_1 = \tau_1 - \delta_2,\ \ \alpha_3 - \eta_2 = \tau_2,\ \ -\eta_3 = \tau_3,$
$\{x, x, z\}$	$\Rightarrow$	$\alpha_3 = \eta_2,$
$\{x, y, z\}$	$\Rightarrow$	$\mu_1 = \delta_3,\ \ \mu_2 = -\eta_3,\ \ \mu_3 = 0,$
$\{y, y, z\}$	$\Rightarrow$	$\beta_3 = \theta_3 = 0,$
$\{y, x, y\}$	$\Rightarrow$	$-\theta_1 = \lambda_1 - \beta_2,\ \ -\theta_2 = \lambda_2,\ \ -\theta_3 = \lambda_3,$
$\{y, x, z\}$	$\Rightarrow$	$\mu_1 = \theta_2,\ \ \mu_2 = 0,$
$\{z, x, y\}$	$\Rightarrow$	$\mu_1 = \lambda_2,\ \ \mu_2 = \tau_3,$
$\{z, x, z\}$	$\Rightarrow$	$\mu_2 = \tau_3,$
$\{z, y, z\}$	$\Rightarrow$	$\lambda_3 = 0,$
$\{x, z, x\}$	$\Rightarrow$	$\eta_2 = \alpha_3,$
$\{x, z, y\}$	$\Rightarrow$	$\mu_1 = \delta_3,\ \ \mu_2 = -\eta_3.$

Thus, we get the following family of algebras, $L(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\delta_1,\delta_2,\eta_1,\theta_1)$:

$$\begin{cases} [e_2,x]=e_1, & [e_3,y]=e_2, & [e_3,z]=e_1,\\ [x,x]=\alpha_1e_1+\alpha_2e_2+\alpha_3e_3, & [x,y]=-z+\delta_1e_1+\delta_2e_2, & [x,z]=\eta_1e_1+\alpha_3e_2,\\ [y,y]=\beta_1e_1+\beta_2e_2, & [y,x]=z, & [y,z]=\theta_1e_1,\\ [z,x]=(\delta_2-\eta_1)e_1-2\alpha_3e_2, & [z,y]=(\beta_2-\theta_1)e_1. & \end{cases}$$

Theorem 4. *Let L be a Leibniz algebra such that $L/I\cong H_3$ and I is the L/I-module with the minimal faithful representation given by Equation (10). Then L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{array}{lll} L(0,1,0,1,0,0,0,1,\lambda),\ \lambda\in\mathbb{F}, & L(0,1,0,1,0,0,0,0,1), & L(0,1,0,1,0,0,0,0,0),\\ L(0,1,0,0,0,0,0,1,\lambda),\ \lambda\in\mathbb{F}, & L(0,1,0,0,0,0,0,0,1), & L(0,1,0,0,0,0,0,0,0),\\ L(0,0,0,1,0,0,0,1,1), & L(0,0,0,1,0,0,0,1,0), & L(0,0,0,1,0,0,0,0,1),\\ L(0,0,0,1,0,0,0,0,0), & L(0,0,0,0,0,0,0,1,1), & L(0,0,0,0,0,0,0,1,0),\\ L(0,0,0,0,0,0,0,0,1), & L(0,0,0,0,0,0,0,0,0), & L(0,0,1,1,0,0,0,1,\lambda),\ \lambda\in\mathbb{F},\\ L(0,0,1,1,0,0,0,0,1), & L(0,0,1,1,0,0,0,0,0), & L(0,0,1,0,0,0,0,1,1),\\ L(0,0,1,0,0,0,0,1,0), & L(0,0,1,0,0,0,0,0,1), & L(0,0,1,0,0,0,0,0,0). \end{array}$$

Proof. We can distinguish two cases:

Case 1: $e_3\in[L,L]$. Then $\alpha_3=0$.

Applying the general change of basis generators:

$$x'=A_1x+A_2y+A_3z+\sum_{k=1}^{3}P_ie_i,\ \ y'=B_1x+B_2y+B_3z+\sum_{k=1}^{3}Q_ie_i,\ \ e_3'=C_1x+C_2y+C_3z+\sum_{k=1}^{3}R_ie_i$$

we derive the expressions of the new parameters in the new basis:

$$\begin{array}{ll} \alpha_1'=\dfrac{\alpha_1A_1^2B_2-\alpha_2A_1^2B_3+\delta_2A_1A_3B_2+A_1B_2P_2+A_3B_2P_3}{A_1B_2^2}, & \alpha_2'=\dfrac{\alpha_2A_1^2}{B_2R_3},\\ \beta_1'=\dfrac{\beta_1B_2}{A_1R_3}, & \beta_2'=\dfrac{\beta_2B_2+Q_3}{R_3},\\ \delta_1'=\dfrac{\beta_2A_3B_2+\delta_1A_1B_2+A_1Q_2+A_3Q_3}{A_1B_2R_3}, & \delta_2'=\dfrac{\delta_2A_1+P_3}{R_3},\\ \eta_1'=\dfrac{\eta_1A_1+P_3}{R_3}, & \theta_1'=\dfrac{\theta_1B_2+Q_3}{R_3}, \end{array}$$

and the following restrictions:

$$\begin{cases} C_1=C_2=C_3=B_1=A_2=0,\\ R_5=-\dfrac{A_3R_3}{A_1},\\ A_1B_2R_3\neq0. \end{cases}$$

We set

$$\begin{array}{lll} P_3=-\delta_2A_1 & \Rightarrow & \delta_2'=0,\\ Q_3=-\beta_2B_2 & \Rightarrow & \beta_2'=0,\\ Q_2=-\delta_1B_2 & \Rightarrow & \delta_1'=0,\\ P_2=-\dfrac{(\alpha_1B_2-\alpha_2B_3)A_1}{B_2} & \Rightarrow & \alpha_1'=0, \end{array}$$

then we get

$$\begin{array}{lll} [e_2,x]=e_1, & [e_3,y]=e_2, & [e_3,z]=e_1,\\ [x,x]=\alpha_2'e_2, & [x,y]=-z, & [x,z]=\eta_1'e_1,\\ [y,y]=\beta_1'e_1, & [y,x]=z, & [y,z]=\theta_1'e_1,\\ [z,x]=-\eta_1'e_1, & [z,y]=-\theta_1'e_1, & \end{array}$$

where

$$\alpha_2'=\frac{\alpha_2A_1^2}{B_2R_3},\quad \beta_1'=\frac{\beta_1B_2}{A_1R_3},\quad \eta_1'=\frac{(\eta_1-\delta_2)A_1}{R_3},\quad \theta_1'=\frac{(\theta_1-\beta_2)B_2}{R_3}.$$

We observe that the nullities of $\alpha_2,\ \beta_1,\ \eta_1,\ \theta_1$ are invariant. Thus, we can distinguish the following non-isomorphic cases. An appropriate choice of the parameter values ($A_1,\ B_2$ and R_3) allows us to obtain the following algebras or families of algebras.

Case	Algebra
$\alpha_2 \neq 0,\ \beta_1 \neq 0,\ \eta_1 \neq 0,$	$L(0,1,0,1,0,0,0,1,\lambda),\ \lambda \in \mathbb{F},$
$\alpha_2 \neq 0,\ \beta_1 \neq 0,\ \eta_1 = 0,\ \theta_1 \neq 0,$	$L(0,1,0,1,0,0,0,0,1),$
$\alpha_2 \neq 0,\ \beta_1 \neq 0,\ \eta_1 = 0,\ \theta_1 = 0,$	$L(0,1,0,1,0,0,0,0,0),$
$\alpha_2 \neq 0,\ \beta_1 = 0,\ \eta_1 \neq 0,$	$L(0,1,0,0,0,0,0,1,\lambda),\ \lambda \in \mathbb{F},$
$\alpha_2 \neq 0,\ \beta_1 = 0,\ \eta_1 = 0,\ \theta_1 \neq 0,$	$L(0,1,0,0,0,0,0,0,1),$
$\alpha_2 \neq 0,\ \beta_1 = 0,\ \eta_1 = 0,\ \theta_1 = 0,$	$L(0,1,0,0,0,0,0,0,0),$
$\alpha_2 = 0,\ \beta_1 \neq 0,\ \eta_1 \neq 0,\ \theta_1 \neq 0,$	$L(0,0,0,1,0,0,0,1,1),$
$\alpha_2 = 0,\ \beta_1 \neq 0,\ \eta_1 \neq 0,\ \theta_1 = 0,$	$L(0,0,0,1,0,0,0,1,0),$
$\alpha_2 = 0,\ \beta_1 \neq 0,\ \eta_1 = 0,\ \theta_1 \neq 0,$	$L(0,0,0,1,0,0,0,0,1),$
$\alpha_2 = 0,\ \beta_1 \neq 0,\ \eta_1 = 0,\ \theta_1 = 0,$	$L(0,0,0,1,0,0,0,0,0),$
$\alpha_2 = 0,\ \beta_1 = 0,\ \eta_1 \neq 0,\ \theta_1 \neq 0,$	$L(0,0,0,0,0,0,0,1,1),$
$\alpha_2 = 0,\ \beta_1 = 0,\ \eta_1 \neq 0,\ \theta_1 = 0,$	$L(0,0,0,0,0,0,0,1,0),$
$\alpha_2 = 0,\ \beta_1 = 0,\ \eta_1 = 0,\ \theta_1 \neq 0,$	$L(0,0,0,0,0,0,0,0,1),$
$\alpha_2 = 0,\ \beta_1 = 0,\ \eta_1 = 0,\ \theta_1 = 0,$	$L(0,0,0,0,0,0,0,0,0).$

Case 2: $e_3 \notin [L,L]$. Then $\alpha_3 \neq 0$. Making the following change of basis in $L(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\delta_1,\delta_2,\eta_1,\theta_1)$

$$\begin{cases} e_3' = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \\ e_2' = \alpha_3 e_2, \\ e_1' = \alpha_3 e_1, \end{cases}$$

we obtain $L(0,0,1,\beta_1,\beta_2,\delta_1,\delta_2,\eta_1,\theta_1)$:

$$\begin{cases} [e_2,x] = e_1, & [e_3,y] = e_2, & [e_3,z] = e_1, \\ [x,x] = e_3, & [x,y] = -z + \delta_1 e_1 + \delta_2 e_2, & [x,z] = \eta_1 e_1 + e_2, \\ [y,y] = \beta_1 e_1 + \beta_2 e_2, & [y,x] = z, & [y,z] = \theta_1 e_1, \\ [z,x] = (\delta_2 - \eta_1)e_1 - 2e_2, & [z,y] = (\beta_2 - \theta_1)e_1. & \end{cases}$$

Analogously to the previous case, by making the general change of basis of generators

$$x' = A_1 x + A_2 y + A_3 z + \sum_{k=1}^{3} P_i e_i, \qquad y' = B_1 x + B_2 y + B_3 z + \sum_{k=1}^{3} Q_i e_i,$$

we derive the expressions of the new parameters in the new basis:

$$\begin{array}{ll} \beta_1' = \dfrac{\beta_1 B_2}{A_1^3}, & \beta_2' = \dfrac{\beta_2 B_2 + Q_3}{A_1^2}, \\ \delta_1' = \dfrac{\beta_2 A_3 B_2^2 + A_1 B_3^2 + \delta_1 A_1 B_2^2 + A_1 B_2 Q_2 + A_3 B_2 Q_3}{A_1^3 B_2^2}, & \delta_2' = \dfrac{-A_1 B_3 + \delta_2 A_1 B_2 + B_2 P_3}{A_1^2 B_2}, \\ \eta_1' = \dfrac{-A_1 B_3 + \eta_1 A_1 B_2 + B_2 P_3}{A_1^2 B_2}, & \theta_1' = \dfrac{\theta_1 B_2 + Q_3}{A_1^2}, \end{array}$$

with the restriction:

$$\begin{cases} A_2 = B_1 = 0, \\ A_1 B_2 \neq 0. \end{cases}$$

By putting

$$\begin{array}{ll} P_3 = \dfrac{A_1(B_3 - \delta_2 B_2)}{B_2} & \Rightarrow \quad \delta_2' = 0, \\ Q_3 = -\beta_2 B_2 & \Rightarrow \quad \beta_2' = 0, \\ Q_2 = -\dfrac{B_3^2 + \delta_1 B_2^2}{B_2} & \Rightarrow \quad \delta_1' = 0, \end{array}$$

we deduce

$$\begin{array}{lll} [e_2,x]=e_1, & [e_3,y]=e_2, & [e_3,z]=e_1, \\ [x,x]=e_3, & [x,y]=-z, & [x,z]=\eta_1' e_1+e_2, \\ [y,y]=\beta_1' e_1, & [y,x]=z, & [y,z]=\theta_1' e_1, \\ [z,x]=-\eta_1' e_1-2e_2, & [z,y]=-\theta_1' e_1, & \end{array}$$

where

$$\beta_1'=\frac{\beta_1 B_2}{A_1^3},\quad \eta_1'=\frac{\eta_1-\delta_2}{A_1},\quad \theta_1'=\frac{(\theta_1-\beta_2)B_2}{A_1^2}.$$

We observe that the nullities of $\beta_1,\ \eta_1,\ \theta_1$ are invariant. Thus, we can distinguish the following non-isomorphic cases. An appropriate choice of the parameter values (A_1 and B_2) allows us to obtain the following algebras or families of algebras.

Case	Algebra
$\beta_1\neq 0,\ \eta_1\neq 0,$	$L(0,0,1,1,0,0,0,1,\lambda),\ \lambda\in\mathbb{F},$
$\beta_1\neq 0,\ \eta_1=0,\ \theta\neq 0,$	$L(0,0,1,1,0,0,0,0,1),$
$\beta_1\neq 0,\ \eta_1=0,\ \theta=0,$	$L(0,0,1,1,0,0,0,0,0),$
$\beta_1=0,\ \eta_1\neq 0,\ \theta\neq 0,$	$L(0,0,1,0,0,0,0,1,1),$
$\beta_1=0,\ \eta_1\neq 0,\ \theta=0,$	$L(0,0,1,0,0,0,0,1,0),$
$\beta_1=0,\ \eta_1=0,\ \theta\neq 0,$	$L(0,0,1,0,0,0,0,0,1),$
$\beta_1=0,\ \eta_1=0,\ \theta=0,$	$L(0,0,1,0,0,0,0,0,0).$

The proof is complete.

□

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