ON THE ISODIAMETRIC AND ISOMINWIDTH INEQUALITIES FOR PLANAR BISECTIONS

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Abstract. For a given planar convex compact set K, consider a bisection $\{A, B\}$ of K (i.e., $A \cup B = K$ and whose common boundary $A \cap B$ is an injective continuous curve connecting two boundary points of K) minimizing the corresponding maximum diameter (or maximum width) of the regions among all such bisections of K.

In this note we study some properties of these minimizing bisections and we provide analogous to the isodiametric (Bieberbach, 1915), the isominwidth (Pál, 1921), the reverse isodiametric (Behrend, 1937), and the reverse isominwidth (González Merino & Schymura, 2018) inequalities.

1. Introduction

The siblings Alice and Bob are deeply sad due to the loss of their uncle Charlie, who recently passed away. Soon, they will be awarded with his heritage, consisting of a countryside piece of ground. They have to divide this terrain into two connected pieces of ground, which must be *equal* according to some *even rule* or *fairness*. In this paper, we will try to solve their issues, when the rule is either that the diameter or the minimum width of each of the pieces of ground is as small as possible (and so, the largest distance in the two pieces is minimized, or the eventual use of an agrarian harvester is optimized).

Let $\mathcal{K}^2$ be the family of planar convex bodies (recall that, as usual, a convex body is a convex compact set). Throughout this paper, for a given compact set $A \subset \mathbb{R}^2$, we will denote its *area* (or 2-dimensional Lebesgue measure) by $\mathrm{A}(A)$, its *diameter* (largest Euclidean distance between two points in A) by $\mathrm{D}(A)$, and its *(minimum) width* (shortest distance between two parallel lines containing A between them) by $\mathrm{w}(A)$.

For a given $K \in \mathcal{K}^2$, a *bisection* of K will be a decomposition into two closed regions K_1, $K_2 \subset K$, such that $K = K_1 \cup K_2$ and $K_1 \cap K_2 = l([-1, 1])$, where $l : [-1, 1] \to K$ is an injective and continuous curve with endpoints $l(-1)$, $l(1)$ in the *boundary* $\mathrm{bd}(K)$ of K. Let $\mathcal{B}(K)$ be the set of all the bisections of K. Let us denote the *infimum of the maximum bisecting diameter* of $K \in \mathcal{K}^2$ by

$$\mathrm{D}_B(K) := \inf_{\{K_1, K_2\} \in \mathcal{B}(K)} \max\{\mathrm{D}(K_1), \mathrm{D}(K_2)\}. \tag{1}$$

In some sense, $\mathrm{D}_B(K)$ can be understood, for each $K \in \mathcal{K}^2$, as the optimal value for the diameter functional when considering bisections of K. In this work, we shall study the bisections which provide $\mathrm{D}_B(K)$, obtaining also an isodiametric-type inequality relating $\mathrm{D}_B(K)$ and $\mathrm{A}(K)$.

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Our motivation mainly emanates from a paper by Miori et al [MPS]. That paper focuses on bisections into two regions of *equal area* minimizing the maximum bisecting diameter in the setting of *centrally symmetric* planar convex bodies. Among other results, they prove that for every set of this family, there always exists a minimizing bisection determined by a line segment [MPS, Prop. 4], and describe in [MPS, Th. 5] the optimal set for this problem (that is, the set of fixed area with the minimum possible value for the maximum bisecting diameter). Moreover, for general planar convex bodies they also demonstrate that the minimum value for that functional when considering bisections by line segments is attained by a centrally symmetric set [MPS, Th. 6]. Then, Proposition 1 below follows from these results (although it is not explicitly stated in [MPS]): for a given $K \in \mathcal{K}^2$, consider

$$\widetilde{\mathrm{D}}_B(K) = \inf_{\{K_1,K_2\}\in\widetilde{\mathcal{B}}(K)} \max\{\mathrm{D}(K_1), \mathrm{D}(K_2)\},$$

where

$$\widetilde{\mathcal{B}}(K) = \{\{K_1, K_2\} \in \mathcal{B}(K) : K_1 \cap K_2 \text{ is a line segment, } \mathrm{A}(K_1) = \mathrm{A}(K_2)\}.$$

Notice that $\widetilde{\mathcal{B}}(K)$ contains the bisections of K determined by a line segment providing two equal-area regions. In [MPS] the authors consider the set

$$Q = \left\{(x_1, x_2) \in \mathbb{R}^2 : -\frac{1}{\sqrt{5}} \le x_1 \le \frac{1}{\sqrt{5}} \ \text{ and } \ \left(x_1 \pm \frac{1}{\sqrt{5}}\right)^2 + x_2^2 \le 1\right\},$$

proving that $\widetilde{\mathrm{D}}_B(Q)$ is given by the bisection of Q with subsets $Q \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \ge 0\}$ and $Q \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \le 0\}$ (an image of this optimal set can be seen in [MPS, pg. 469]).

Proposition 1. *Let* $K \in \mathcal{K}^2$. *Then*

$$\frac{\mathrm{A}(K)}{\widetilde{\mathrm{D}}_B(K)^2} \le 2\left(\arctan 2 - \arctan\frac{1}{2}\right), \tag{2}$$

with equality if $K = Q$.

Observe that the inequality (2) is an isodiametric-type inequality, in the sense of the classical *isodiametric inequality* of Bieberbach [Bi]: given $K \in \mathcal{K}^2$, we have that

$$\mathrm{A}(K) \le \frac{\pi}{4}\,\mathrm{D}(K)^2,$$

with equality if and only if K is an Euclidean disk.

Our Theorem 2 below is an extension of Proposition 1. On the one hand, we consider *arbitrary bisections*, determined by curves which are *not necessarily line segments*. And on the other hand, we allow the regions of the bisections to have *different areas*. In other words, we focus on $\mathcal{B}(K)$ instead of $\widetilde{\mathcal{B}}(\mathcal{K})$. This makes our approach completely general in this setting. In Section 3 we shall prove the following:

Theorem 2. *Let* $K \in \mathcal{K}^2$. *Then,*

$$\frac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2} \le 2\left(\arctan 2 - \arctan\frac{1}{2}\right), \tag{3}$$

with equality if and only if $K = Q$. *Moreover,* $\mathrm{D}_B(Q)$ *is given by the bisection of* Q *with subsets* $Q \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \ge 0\}$ *and* $Q \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \le 0\}$.

Surprisingly enough, the optimal set in the general situation, described in Theorem 2, is still the same set as in Proposition 1. This fact strengthens the idea that central symmetry is an inherent property for this optimization problem. On the other hand, we would like to emphasize that the argument exhibited in [MPS, Th. 6] cannot be applied under the general conditions of Theorem 2.

Finally, it is worth mentioning that the questions regarding the maximum bisecting diameter (treated firstly in [MPS]) have originated several works in the last years. In [CS] we can find some improvements for the centrally symmetric case, and some related problems for divisions into three or more regions have been studied in [CSS2, C]. Moreover, we also point out that these questions have been partially treated in surfaces of $\mathbb{R}^3$ [CMSS, CSS].

Apart from studying the diameter, we also consider in this work the analogous problem for the *width functional* (which is, in some sense, the geometric functional *reverse* to the diameter). Recall that by replacing the diameter with the width in the classical isodiametric inequality, Pál showed that

$$\mathrm{A}(K) \geq \frac{1}{\sqrt{3}}\,\mathrm{w}(K)^2, \tag{4}$$

with equality if and only if K is an equilateral triangle [Pal]. Our aim is obtaining a similar *isominwidth inequality* for bisections of a planar convex body. For this purpose, given $K \in \mathcal{K}^2$, we can define, analogously to $\mathrm{D}_B(K)$, the *infimum of the maximum bisecting width* by

$$\mathrm{w}_B(K) := \inf_{\{K_1,K_2\}\in\mathcal{B}(K)} \max\{\mathrm{w}(K_1), \mathrm{w}(K_2)\}.$$

We will prove in Section 4 the following inequality.

Theorem 3. *Let $K \in \mathcal{K}^2$. Then,*

$$\frac{\mathrm{A}(K)}{\mathrm{w}_B(K)^2} \geq \frac{4}{\sqrt{3}}, \tag{5}$$

with equality if and only if K is an equilateral triangle $\mathcal{T}$. Moreover, $\mathrm{w}_B(\mathcal{T})$ is attained by the bisection of $\mathcal{T}$ determined by a line segment passing through the midpoints of two edges of $\mathcal{T}$.

Remark 4. Notice that the quotients $\mathrm{A}(K)/\widetilde{\mathrm{D}}_B(K)^2$, $\mathrm{A}(K)/\mathrm{D}_B(K)^2$ and $\mathrm{A}(K)/\mathrm{w}_B(K)^2$ are invariant under dilations and rigid motions, due to the corresponding homogeneity of the area, the diameter and the width functionals and the invariance under rigid motions.

Another interesting geometric question in this setting regards the *reverse inequalities* for these problems (see [Beh, B, CDT] and references therein). In the case of the isodiametric quotient, such inequality cannot be stated directly, since for an arbitrary planar convex body K with non-empty interior, the *isodiametric quotient* $\mathrm{A}(K)/\mathrm{D}(K)^2$ cannot be bounded from below by any constant different from 0 (it suffices to consider very thin rectangles with area approaching zero). However, Behrend treated this problem finding such lower bound for the family of sets in $\mathcal{K}^2$ that maximizes that quotient in their affine class. More precisely, we will say that $K \in \mathcal{K}^2$ is in *Behrend position* if

$$\frac{\mathrm{A}(K)}{\mathrm{D}(K)^2} = \sup_{\phi\in End(\mathbb{R}^2)} \frac{\mathrm{A}(\phi(K))}{\mathrm{D}(\phi(K))^2},$$

where $End(\mathbb{R}^2)$ denotes the set of affine endomorphisms of $\mathbb{R}^2$ [Beh]. Therefore, if K is in Behrend position, the above quotient achieves the maximum value among all the affine transformations of K. This approach allows to obtain an interesting reverse isodiametric inequality: for every $K \in \mathcal{K}^2$ in Behrend position, we have that

$$\mathrm{A}(K) \geq \frac{\sqrt{3}}{4}\mathrm{D}(K)^2, \tag{6}$$

with equality if and only if K is an equilateral triangle [Beh]. Moreover, if we restrict K to be centrally symmetric (that is, $K = x - K$ for some $x \in \mathbb{R}^2$), then

$$\mathrm{A}(K) \geq \frac{1}{2}\mathrm{D}(K)^2, \tag{7}$$

with equality if and only if K is a square ([Beh], see also [GMS]).

Following these ideas (also used by Ball for obtaining a reverse isoperimetric inequality [B]), we will establish an analogous inequality to (6) for the infimum of the maximum bisecting diameter. In order to do this, we will say that $K \in \mathcal{K}^2$ is in *Behrend-bisecting position* if

$$\frac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2} = \sup_{\phi \in End(\mathbb{R}^2)} \frac{\mathrm{A}(\phi(K))}{\mathrm{D}_B(\phi(K))^2}. \tag{8}$$

In Section 5 we give some necessary conditions for a set K to be in Behrend-bisecting position, and our Theorem 5 shows a reverse isodiametric inequality for minimizing bisections, which is not sharp in general.

Theorem 5. *Let $K \in \mathcal{K}^2$ be in Behrend-bisecting position. Then*

$$\frac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2} \geq \frac{\sqrt{3}}{4}. \tag{9}$$

Moreover, the restriction to centrally symmetric sets in Behrend-bisecting position allows to improve inequality (9), as shown in our Theorem 6.

Theorem 6. *Let $K \in \mathcal{K}^2$ be centrally symmetric and in Behrend-bisecting position. Then*

$$\frac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2} \geq \frac{\sqrt{3}}{2}. \tag{10}$$

The same spirit of the previous results leads us to study a *reverse isominwidth inequality* for minimizing bisections, of type $\mathrm{A}(K)/\mathrm{w}_B(K)^2 \leq \alpha$, for some $\alpha \in \mathbb{R}$. We will follow an approach similar to [GMS], considering again affine classes of sets in $\mathcal{K}^2$. In this sense, recall that $K \in \mathcal{K}^2$ is in *isominwidth optimal position* if

$$\frac{\mathrm{A}(K)}{\mathrm{w}(K)^2} = \inf_{\phi \in End(\mathbb{R}^2)} \frac{\mathrm{A}(\phi(K))}{\mathrm{w}(\phi(K))^2}.$$

The restriction to these suitable affine representatives of planar convex bodies yields, as in the case of the diameter functional, to the following result: for any set $K \in \mathcal{K}^2$ in isominwidth optimal position, it holds that

$$\mathrm{A}(K) \leq \mathrm{w}(K)^2, \tag{11}$$

with equality if and only if K is a square [GMS, Th. 5.4]. Our aim is obtaining an analogous inequality to (11) for the infimum of the maximum bisecting width for sets in a certain special position. Thus, given $K \in \mathcal{K}^2$, we shall say that K is in *isominwidth-bisecting position* if

$$\frac{\mathrm{A}(K)}{\mathrm{w}_B(K)^2} = \inf_{\phi \in End(\mathbb{R}^2)} \frac{\mathrm{A}(\phi(K))}{\mathrm{w}_B(\phi(K))^2}. \tag{12}$$

We will derive in Section 6 some necessary conditions for K to be in isominwidth-bisecting position, concluding with the following result.

Theorem 7. *Let $K \in \mathcal{K}^2$ be in isominwidth-bisecting position. Then*

$$\frac{\mathrm{A}(K)}{\mathrm{w}_B(K)^2} \leq 4, \tag{13}$$

with equality if and only if K is a square $\mathcal{C}$. Moreover, $\mathrm{w}_B(\mathcal{C})$ is given by the bisection determined by a segment parallel to an edge of $\mathcal{C}$ dividing $\mathcal{C}$ into two equal-area subsets.

We now establish some notation used throughout this paper. The *vectors of the canonical basis* of $\mathbb{R}^2$ will be $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Given two points $x, y \in \mathbb{R}^2$, $[x, y]$ will denote the *line segment* with endpoints x, y. For every $K \in \mathcal{K}^2$, $\mathrm{Ext}(K)$ will stand for the set of *extreme points* of K, i.e., if $x \in \mathrm{Ext}(K)$, then $x \in [y, z] \subset K$ implies $x = y$ or $x = z$. For any planar compact set A, we denote by $\mathrm{conv}(A)$ and $\mathrm{span}(A)$ the *convex hull* and the *linear hull* of A, respectively. Moreover, if A is a planar set, we denote by $A^\perp$ the *orthogonal complement* of A. For $K \in \mathcal{K}^2$ and $u \in \mathbb{R}^2 \setminus \{0\}$, the *Steiner symmetrization* $s_u(K)$ of K with respect to $\mathrm{span}(u)$ is defined as the only symmetric set with respect to $\mathrm{span}(u)$ such that each segment $(tu + u^\perp) \cap s_u(K)$ has the same length than $(tu + u^\perp) \cap K$, for every $t \in \mathbb{R}$ [BF, SY]. It is well known that $s_u(K) \in \mathcal{K}^2$ and

$$\mathrm{A}(s_u(K)) = \mathrm{A}(K), \quad \mathrm{D}(s_u(K)) \leq \mathrm{D}(K). \tag{14}$$

The paper is organized as follows. In Section 2 we obtain some general properties of the minimizing bisections for the maximum bisecting diameter and the maximum bisecting width. In particular, Lemma 8 shows that there always exists a minimizing bisection given by a line segment, which allows to focus only on this type of bisections along this work. In Section 3 we prove Theorem 2, determining the corresponding optimal set (of fixed area) for the maximum bisecting diameter by a constructive argument. Section 4 is devoted to show Theorem 3, which follows directly from Lemma 14. Finally, Sections 5 and 6 treat the reverse inequalities under the approach of affine representatives of planar convex bodies. In Section 5 we demonstrate Theorem 5, which requires a detailed study concerning the Behrend-bisecting position, and Section 6 contains the proof of Theorem 7.

2. Properties of minimizing bisections

In this section we will obtain some interesting properties for the minimizing bisections of the two functionals we are considering. Lemma 8 shows that there is always one of these bisections given by a line segment, extending [MPS, Prop. 4], and Lemma 10 proves that minimizing bisections always provide, in some sense, two regions which are *in equilibrium*. Besides, we also show in Lemma 9 that the infimum in (1) is actually a minimum.

Lemma 8. *Let $K \in \mathcal{K}^2$ and $\rho > 0$. For any bisection of K with maximum bisecting diameter (or width) equal to ρ, there exists another bisection of K given by a line segment with maximum bisecting diameter (or width) smaller than or equal to ρ.*

Proof. Consider $\{K_1, K_2\} \in \mathcal{B}(K)$ determined by an injective continuous curve $\gamma : [-1, 1] \to K$ with $\gamma(\pm 1) \in \mathrm{bd}(K)$. Suppose that $\max\{\mathrm{D}(K_1), \mathrm{D}(K_2)\} = \rho$ (or $\max\{\mathrm{w}(K_1), \mathrm{w}(K_2)\} = \rho$). Call $M_1 := \mathrm{bd}(K) \cap K_1$ and $M_2 := \mathrm{bd}(K) \cap K_2$. Since $M_i \subset K_i$, $i = 1, 2$, then $\mathrm{D}(M_i) \leq \mathrm{D}(K_i)$ and $\mathrm{w}(M_i) \leq \mathrm{w}(K_i)$, $i = 1, 2$.

Notice that the line segment $[\gamma(-1), \gamma(1)]$ will determine $\mathrm{conv}(M_i)$, $i = 1, 2$. We claim that $\mathrm{D}(M_i) = \mathrm{D}(\mathrm{conv}(M_i))$, $i = 1, 2$. On the one hand, $M_i \subset \mathrm{conv}(M_i)$ implies that $\mathrm{D}(M_i) \leq$

$\mathrm{D}(\mathrm{conv}(M_i))$. And on the other hand, it is not difficult to check that $\mathrm{Ext}(\mathrm{conv}(M_i)) \subset M_i$, because M_i is compact. Furthermore, since the diameter is always attained by extreme points, then

$$\mathrm{D}(\mathrm{conv}(M_i)) = \mathrm{D}(\mathrm{Ext}(\mathrm{conv}(M_i))) \leq \mathrm{D}(M_i).$$

On the other hand, we also have $\mathrm{w}(M_i) = \mathrm{w}(\mathrm{conv}(M_i))$, $i = 1, 2$, as a direct consequence of the fact that M_i is contained between two parallel lines if and only if $\mathrm{conv}(M_i)$ is contained between those lines.

Note that $\mathrm{conv}(M_1)$, $\mathrm{conv}(M_2)$ are two subsets of K providing a bisection of K, satisfying

$$\max\{\mathrm{D}(\mathrm{conv}(M_1)), \mathrm{D}(\mathrm{conv}(M_2))\} \leq \max\{\mathrm{D}(K_1), \mathrm{D}(K_2)\} = \rho,$$

as well as

$$\max\{\mathrm{w}(\mathrm{conv}(M_1)), \mathrm{w}(\mathrm{conv}(M_2))\} \leq \max\{\mathrm{w}(K_1), \mathrm{w}(K_2)\} = \rho.$$

Thus, we conclude that $\{\mathrm{conv}(M_1), \mathrm{conv}(M_2)\}$ is a bisection of K given by a line segment with maximum bisecting diameter (or width) smaller than or equal to ρ. □

Lemma 9. *Let $K \in \mathcal{K}^2$. Then*

$$\mathrm{D}_B(K) = \min_{\{K_1, K_2\} \in \mathcal{B}(K)} \max\{\mathrm{D}(K_1), \mathrm{D}(K_2)\},$$

and

$$\mathrm{w}_B(K) = \min_{\{K_1, K_2\} \in \mathcal{B}(K)} \max\{\mathrm{w}(K_1), \mathrm{w}(K_2)\}.$$

Proof. We will focus on $\mathrm{D}_B(K)$, since the case of $\mathrm{w}_B(K)$ is analogous. Note that Lemma 8 allows to consider only bisections by line segments in order to compute $\mathrm{D}_B(K)$. Then, in view of (1), let $\{[a_i, b_i]\}_{i \in \mathbb{N}} \subset K$ be a sequence of line segments providing bisections of K, each of them with subsets $\{K_{1,i},\ K_{2,i}\}$, such that

$$\mathrm{D}_B(K) = \lim_{i \to \infty} \max\{\mathrm{D}(K_{1,i}), \mathrm{D}(K_{2,i})\}.$$

Since $\{K_{1,i}\}_{i \in \mathbb{N}} \subset K$ is an absolutely bounded sequence, *Blaschke Selection Theorem* [Sch, Th. 1.8.7] implies the existence of a convergent subsequence (which we assume without loss of generality to be the sequence itself), so there exists $K_1 \in \mathcal{K}^2$ such that $K_1 \subset K$ and $\lim_{i \to \infty} K_{1,i} = K_1$ in Hausdorff metric. Analogously, there exists $K_2 \in \mathcal{K}^2$ such that $K_2 \subset K$ and $\lim_{i \to \infty} K_{2,i} = K_2$. In particular, we also obtain that $\lim_{i \to \infty} [a_i, b_i] = [a, b]$, for certain $a, b \in K$, with $[a, b] = K_1 \cap K_2$. Since the diameter is a continuous functional in Hausdorff metric, we have that $\lim_{i \to \infty} \mathrm{D}(K_{j,i}) = \mathrm{D}(K_j)$, $j = 1, 2$, thus concluding that $\mathrm{D}_B(K) = \max\{\mathrm{D}(K_1), \mathrm{D}(K_2)\}$, as stated. □

Lemma 10. *Let $K \in \mathcal{K}^2$. There exists a bisection $\{K_1, K_2\}$ of K minimizing the maximum bisecting diameter (or width) of K such that $\mathrm{D}_B(K) = \mathrm{D}(K_1) = \mathrm{D}(K_2)$ (or $\mathrm{w}_B(K) = \mathrm{w}(K_1) = \mathrm{w}(K_2)$).*

Proof. This is a consequence of the continuity of the diameter and the width functionals. Taking into account Lemmas 8 and 9, let $\{K_1, K_2\}$ be a bisection of K minimizing the maximum bisecting diameter (or width), determined by the line segment $L = K_1 \cap K_2$. Fix $u \in L^{\perp} \setminus \{0\}$, and let $t_1 < 0 < t_2$ be such that $K \cap (tu + L) \neq \emptyset$ when and only when $t \in [t_1, t_2]$. Moreover let $K_1^t = K \cap \{su + L : s \in [t_1, t]\}$ and $K_2^t = K \cap \{su + L : s \in [t, t_2]\}$ for every $t \in [t_1, t_2]$, so that $K_i^0 = K_i$, $i = 1, 2$. In particular, we have that $K \cap (t_i u + L) \subset \mathrm{bd}(K)$, $i = 1, 2$, and thus $K_1^{t_2} = K_2^{t_1} = K$.

For $i = 1, 2$, let $f_i, g_i : [t_1, t_2] \to [0, \mathrm{D}(K)]$ be such that $f_1(t) = \mathrm{D}(K_1^t)$, $f_2(t) = \mathrm{D}(K_2^t)$, and $g_1(t) = \mathrm{w}(K_1^t)$, $g_2(t) = \mathrm{w}(K_2^t)$. By direct inclusion of sets, we have that f_1 and g_1 are non-decreasing, whereas f_2 and g_2 are non-increasing. Moreover, these four functions are continuous, with $f_1(t_2) = \mathrm{D}(K) = f_2(t_1)$ and $g_1(t_2) = \mathrm{w}(K) = g_2(t_1)$.

If $f_1(0) = f_2(0)$ (resp., $g_1(0) = g_2(0)$), then $\{K_1, K_2\}$ is a minimizing bisection with $\mathrm{D}_B(K) = \mathrm{D}(K_1) = \mathrm{D}(K_2)$ (resp., $\mathrm{w}_B(K) = \mathrm{w}(K_1) = \mathrm{w}(K_2)$), as desired. Otherwise, let us suppose without loss of generality that $f_1(0) < f_2(0) = \mathrm{D}(K_2) = \mathrm{D}_B(K)$ (resp., $g_1(0) < g_2(0) = \mathrm{w}(K_2) = \mathrm{w}_B(K)$). Since

$$f_1(t_2) = \mathrm{D}(K) \geq \mathrm{D}(K_2^{t_2}) = f_2(t_2),$$

(resp., $g_1(t_2) \geq g_2(t_2)$), *Bolzano Theorem* implies that there exists $t_0 \in [0, t_2]$ such that $f_1(t_0) = f_2(t_0)$ (resp., $g_1(t_0) = g_2(t_0)$). By using the monotonicity of the functions, we have that

$$\mathrm{D}(K_1) = f_1(0) \leq f_1(t_0) = f_2(t_0) \leq f_2(0) = \mathrm{D}(K_2) = \mathrm{D}_B(K)$$

and

$$\mathrm{w}(K_1) = g_1(0) \leq g_1(t_0) = g_2(t_0) \leq g_2(0) = \mathrm{w}(K_2) = \mathrm{w}_B(K),$$

thus $\mathrm{D}(K_1^{t_0}) = \mathrm{D}(K_2^{t_0}) \leq \mathrm{D}_B(K)$ (resp., $\mathrm{w}(K_1^{t_0}) = \mathrm{w}(K_2^{t_0}) \leq \mathrm{w}_B(K)$), and hence $\{K_1^{t_0}, K_2^{t_0}\}$ is a minimizing bisection of K providing subsets of equal diameters (or widths), as desired. □

Remark 11. A minimizing bisection $\{K_1, K_2\}$ with subsets of equal *diameters* as in Lemma 10 might be degenerate, that is, K_1 or K_2 might be a line segment. For instance, let $\mathcal{T} \in \mathcal{K}^2$ be an equilateral triangle of vertices p_i, $i = 1, 2, 3$. Then $\{\mathcal{T}, [p_1, p_2]\}$ is a minimizing bisection with $\mathrm{D}_B(\mathcal{T}) = \mathrm{D}(\mathcal{T}) = \mathrm{D}([p_1, p_2])$. This is not the case for the minimizing bisections with subsets of equal width, which have to split any convex set into two non-degenerate subsets, since the width of a line segment is 0.

3. The isodiametric inequality

In this section we will prove our Theorem 2, providing an isodiametric inequality for the maximum bisecting diameter. As we will see, the proof is constructive, yielding the corresponding optimal set taking into account the previous Lemma 12.

Lemma 12. *There exists a maximizer $K_0 \in \mathcal{K}^2$ of the quotient $\dfrac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2}$, with $\mathrm{D}_B(K_0)$ provided by a line segment $[(-a, 0), (a, 0)]$, $a > 0$, such that K_0 is symmetric with respect to the orthogonal line $L = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$.*

Proof. By continuity, there exists a maximizer $\widetilde{K} \in \mathcal{K}^2$ of $\mathrm{A}(K)/\mathrm{D}_B(K)^2$. By Lemma 8, we can suppose without loss of generality that $\mathrm{D}_B(\widetilde{K})$ is given by a bisection $\{K_1, K_2\}$ of $\widetilde{K}$, with $K_1 = \widetilde{K} \cap H^+$, $K_2 = \widetilde{K} \cap H^-$, $K_1 \cap K_2 = [(-a, 0), (a, 0)]$, for some $a \in [0, \mathrm{D}(K)/2]$, where $H^+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$ and $H^- = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 0\}$.

By applying Steiner symmetrization s_{e_2} with respect to the vertical line $\mathrm{span}(e_2)$, we easily get that $s_{e_2}(\widetilde{K}) = s_{e_2}(K_1) \cup s_{e_2}(K_2)$ and $s_{e_2}(K_1) \cap s_{e_2}(K_2) = [(-a, 0), (a, 0)]$. Denoting by $K_0 := s_{e_2}(\widetilde{K})$ and $K_{0,i} := s_{e_2}(K_i)$, we have by (14) that $\mathrm{A}(K_{0,i}) = \mathrm{A}(K_i)$ and $\mathrm{D}(K_{0,i}) \leq \mathrm{D}(K_i)$, $i = 1, 2$, and so $\mathrm{A}(K_0) = \mathrm{A}(\widetilde{K})$ and $\mathrm{D}_B(K_0) \leq \mathrm{D}_B(\widetilde{K})$. Since $\widetilde{K}$ is a maximizer of $\mathrm{A}(K)/\mathrm{D}_B(K)^2$, then necessarily K_0 is also a maximizer, which possesses the desired symmetry by construction. □

Proof of Theorem 2. Let $\widetilde{K} \in \mathcal{K}^2$ be a maximizer of the isodiametric quotient $\mathrm{A}(K)/\mathrm{D}_B(K)^2$. We will prove that there exists another convex set K_0 whose quotient is strictly greater than $\mathrm{A}(\widetilde{K})/\mathrm{D}_B(\widetilde{K})^2$ whenever $\widetilde{K}$ is different from K_0 (up to dilations and rigid motions, see Remark 4), which implies that the maximizer must be precisely K_0.

Let us suppose without loss of generality that $\{K_1, K_2\}$ is a bisection of $\widetilde{K}$ providing $\mathrm{D}_B(\widetilde{K})$, with $K_1 = \widetilde{K} \cap H^+$, $K_2 = \widetilde{K} \cap H^-$, $K_1 \cap K_2 = [(-a,0),(a,0)]$, for some $a \in [0, \mathrm{D}_B(\widetilde{K})/2]$, where $H^+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$ and $H^- = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 \leq 0\}$, being $\widetilde{K}$ symmetric with respect to the vertical line $L = \{x \in \mathbb{R}^2 : x_1 = 0\}$, in view of Lemma 12. Moreover, we can also suppose by Lemma 10 that $\mathrm{D}(K_1) = \mathrm{D}(K_2) = \mathrm{D}_B(\widetilde{K})$.

Since $\widetilde{K}$ is convex and compact, and $(a,0) \in \mathrm{bd}(\widetilde{K})$, then there exists a supporting line M_+ to $\widetilde{K}$ at $(a,0)$. Due to the symmetry of $\widetilde{K}$, the symmetric line of M_+ with respect to L is also a supporting line at $(-a,0)$, namely M_-. By flipping the situation if necessary, we can suppose that the slope of M_+ is non-negative, and so $M_+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = m(x_1 - a)\}$, for some $m \geq 0$. Additionally, call $B_\pm = B((\pm a, 0), \mathrm{D}_B(\widetilde{K}))$ the closed balls of centers $(\pm a, 0)$ and radius $\mathrm{D}_B(\widetilde{K})$. Since $\mathrm{D}(K_i) = \mathrm{D}_B(\widetilde{K})$ and $(\pm a, 0) \in K_i$, it follows that K_i is necessarily contained in the symmetric lens $B_+ \cap B_-$, for $i = 1, 2$.

Note that K_2 is always contained in the triangle T determined by M_+, M_-, and the horizontal line $\{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0\}$. Then $\mathrm{D}(T) = \max\{2a, d\} \geq \mathrm{D}(K_2) = \mathrm{D}_B(\widetilde{K})$, where $d = d((a,0),(0,-ma)) = a\sqrt{1+m^2}$. We will distinguish two possibilities.

If $2a > d$, then $2a = \mathrm{D}(T) \geq \mathrm{D}_B(\widetilde{K})$ (and so $\mathrm{D}_B(\widetilde{K}) = 2a$). In this case, it is straightforward checking that the area of $B_+ \cap B_-$ equals $\mathrm{D}_B(\widetilde{K})^2 \, \dfrac{4\pi - 3\sqrt{3}}{6}$, and so

$$\frac{\mathrm{A}(\widetilde{K})}{\mathrm{D}_B(\widetilde{K})^2} \leq \frac{\mathrm{A}(B_+ \cap B_-)}{\mathrm{D}_B(\widetilde{K})^2} = \frac{4\pi - 3\sqrt{3}}{6}. \tag{15}$$

On the other hand, if $2a \leq d$, then $d = \mathrm{D}(T) \geq \mathrm{D}_B(\widetilde{K})$, which implies that $m \geq \sqrt{\mathrm{D}_B(\widetilde{K})^2 - a^2}/a$. Let us estimate the isodiametric quotient of $\widetilde{K}$ in this case.

Let $R(a,m)$ be the planar region contained between M_+, M_-, B_+ and B_-, with the dependance on a and m explained above. Since $\widetilde{K} \subseteq R(a,m)$, then $\mathrm{A}(\widetilde{K}) \leq \mathrm{A}(R(a,m))$. Moreover, let $R(a,+\infty)$ be the planar region contained between B_+, B_- and the vertical lines passing through $(\pm a, 0)$. Let us check that $\mathrm{A}(R(a,m)) < \mathrm{A}(R(a,+\infty))$, for every $m \geq \sqrt{\mathrm{D}_B(\widetilde{K})^2 - a^2}/a$ (and $2a \leq d$). Due to the symmetry of these regions, we can focus on the corresponding areas contained in $\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0\}$. The only region R_1 (resp., R_2) contained in $R(a,m)$ (resp., $R(a,+\infty)$) which is not in $R(a,+\infty)$ (resp., $R(a,m)$) is the one contained between M_+, $(a,0)+L$, B_-, and $\{(x_1,x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$ (resp., $\{(x_1,x_2) \in \mathbb{R}^2 : x_2 \leq 0\}$). It can be checked that the condition $m \geq \sqrt{\mathrm{D}_B(\widetilde{K})^2 - a^2}/a$ implies that the rotation centered at $(a,0)$ of angle π maps strictly R_1 onto R_2, and so $\mathrm{A}(R(a,m)) < \mathrm{A}(R(a,+\infty))$. Note also that the construction of $R(a,+\infty)$ implies that $\mathrm{D}_B(R(a,+\infty)) = \mathrm{D}_B(\widetilde{K})$ (the bisection of $R(a,+\infty)$ given by the subsets $R_+ = R(a,+\infty) \cap H^+$ and $R_- = R(a,+\infty) \cap H^-$ satifies $\mathrm{D}(R_+) = \mathrm{D}(R_-) = \mathrm{D}_B(\widetilde{K})$, see [MPS]).

Let us now compute the maximum value for $\mathrm{A}(R(a,+\infty))/\mathrm{D}_B(\widetilde{K})^2$, when $a>0$. It is straightforward checking that

$$\begin{aligned}\mathrm{A}(a) := \mathrm{A}(R(a,+\infty)) &= 4\int_0^a \sqrt{\mathrm{D}_B(\widetilde{K})^2-(x+a)^2}\,dx\\ &= 2\Bigg(2a\sqrt{\mathrm{D}_B(\widetilde{K})^2-4a^2}-a\sqrt{\mathrm{D}_B(\widetilde{K})^2-a^2}+\mathrm{D}_B(\widetilde{K})^2\arctan\left(\frac{2a}{\sqrt{\mathrm{D}_B(\widetilde{K})^2-4a^2}}\right)\\ &\quad -\mathrm{D}_B(\widetilde{K})^2\arctan\left(\frac{a}{\sqrt{\mathrm{D}_B(\widetilde{K})^2-a^2}}\right)\Bigg).\end{aligned}$$

For simplicity, call $b=a/\mathrm{D}_B(\widetilde{K})$ (which corresponds to a normalization for having $\mathrm{D}_B(\widetilde{K})$ equal to 1 by an appropriate dilation). Then, well-known properties of dilations gives

$$\mathrm{A}(b)=2\left(2b\sqrt{1-4b^2}-b\sqrt{1-b^2}+\arctan\left(\frac{2b}{\sqrt{1-4b^2}}\right)-\arctan\left(\frac{b}{\sqrt{1-b^2}}\right)\right),$$

which attains its maximum value (as a function on b) only at $b=1/\sqrt{5}$, and so, for any $b>0$,

$$\mathrm{A}(b)\le \mathrm{A}(1/\sqrt{5})=2\left(\arctan 2-\arctan\frac{1}{2}\right).$$

Thus

$$\frac{\mathrm{A}(\widetilde{K})}{\mathrm{D}_B(\widetilde{K})^2}\le\frac{\mathrm{A}(R(a,+\infty))}{\mathrm{D}_B(\widetilde{K})^2}\le 2\left(\arctan 2-\arctan\frac{1}{2}\right),$$

which gives a bound greater than the one obtained in (15), yielding the desired inequality (3). Moreover, equality above only holds for $R(1/\sqrt{5},+\infty)$, which coincides with Q by definition. □

Remark 13. The reader will realize that the line segment $[(-a,0),(a,0)]$ does not give a minimizing bisection of $R(a,m)$ above *for some values of the parameters* a,m. Indeed, in every step of the proof of Theorem 2, we replace the set by another one with greater (or equal) area. This process starts with $\widetilde{K}$ and ends with $Q=R(1/\sqrt{5},+\infty)$, and the corresponding horizontal line segment provides a minimizing bisection *for both sets*, whereas in the middle of the process, that line segment *does not give* necessarily a minimizing bisection of $R(a,m)$ in general. For instance, for $K=R(a,\sqrt{3})$, with $\mathrm{D}_B(K)>2a$, the bisection determined by the line segment $[(-a,0),(a,0)]$ is not minimizing, since it can be improved by a different line segment (placed slightly above).

4. The isominwidth inequality

In this section we will consider the problem analogous to the one studied in Section 3, but for the width functional. We will start proving that $\mathrm{w}_B(K)=\mathrm{w}(K)/2$, for any $K\in\mathcal{K}^2$, by using the following celebrated result by Bang on Tarski's plank problem [Ba]: for $K\in\mathcal{K}^2$, and $p,q\in\mathrm{bd}(K)$, let $\{K_1,K_2\}$ be the bisection given by the line segment $[p,q]$. Then

$$\mathrm{w}(K_1)+\mathrm{w}(K_2)\ge\mathrm{w}(K). \tag{16}$$

Lemma 14. *Let* $K\in\mathcal{K}^2$. *Then* $\mathrm{w}_B(K)=\mathrm{w}(K)/2$.

Proof. Let L_1, L_2 be two parallel supporting lines of K such that $d(L_1,L_2)=\mathrm{w}(K)$, and let $u\in\mathbb{S}^1$ be an orthogonal vector to these lines. Consider $p,q\in\mathrm{bd}(K)$ such that $[p,q]=K\cap L$, where L is parallel to L_i and lies at distance $\mathrm{w}(K)/2$ from each line L_i, $i=1,2$. Moreover, let $\{K_1,K_2\}$ be the bisection determined by the line segment $[p,q]$. Note that L and L_i are supporting lines of K_i, for $i = 1, 2$, and so $\mathrm{w}(K_i) \leq \mathrm{w}(K)/2$. Thus $\max\{\mathrm{w}(K_1), \mathrm{w}(K_2)\} \leq \mathrm{w}(K)/2$, and hence $\mathrm{w}_B(K) \leq \mathrm{w}(K)/2$. On the other hand, in view of Lemmas 8, 9 and 10, let $\{\widetilde{K_1}, \widetilde{K_2}\}$ be a minimizing bisection for the maximum bisecting width, given by a line segment and satisfying $\mathrm{w}_B(K) = \mathrm{w}(\widetilde{K_1}) = \mathrm{w}(\widetilde{K_2})$. Then, (16) implies that $\mathrm{w}(K) \leq \mathrm{w}(\widetilde{K_1}) + \mathrm{w}(\widetilde{K_2}) = 2\,\mathrm{w}_B(K)$, and so $\mathrm{w}_B(K) \geq \mathrm{w}(K)/2$, yielding the desired equality. □

Now we are able to prove immediately the main result of this section, which is Theorem 3, providing a sharp upper bound for w_B.

Proof of Theorem 3. By Lemma 14 and Pal's inequality (4) we directly have that

$$\frac{\mathrm{A}(K)}{\mathrm{w}_B(K)^2} = 4\,\frac{\mathrm{A}(K)}{\mathrm{w}(K)^2} \geq \frac{4}{\sqrt{3}}.$$

Moreover, in order to have equality, we must have equality in (4), hence implying that K is an equilateral triangle. □

5. The Behrend-Bisecting position and the reverse isodiametric inequality

As commented in the Introduction, we will now focus on a reverse isodiametric inequality for the maximum bisecting diameter. The following definitions and results arise mainly from the ideas in [Beh]. For every $K \in \mathcal{K}^2$, let

$$D_K := \{u \in \mathbb{S}^1 : \exists\, x \in K \ \text{ such that } x + \mathrm{D}(K)[0, u] \subset K\}$$

be the set of *diametrical directions* of K (that is, the directions for which $\mathrm{D}(K)$ is attained). Moreover, we will say that $u \in \mathbb{S}^1$ is a *bisector of* K if u is the direction of a line segment providing a minimizing bisection $\{K_1, K_2\}$ of K with $\mathrm{D}(K_1) = \mathrm{D}(K_2)$. We will denote by B_K the set of bisectors of K. Note that B_K contains the directions which determine suitable minimizing bisections by line segments for D_B.

The next result establishes that the supremum in the definition of the Behrend-bisecting position (8) is actually a maximum.

Lemma 15. *Let $K \in \mathcal{K}^2$ with non-empty interior. Then there exists $\phi \in End(\mathbb{R}^2)$ such that $\phi(K)$ is in Behrend-bisecting position.*

Proof. After a suitable translation of K, we can suppose that $r\mathbb{B}_2^2 \subseteq K$ for some $r > 0$. Let $\rho > 0$ be such that

$$\rho = \sup_{\phi \in End(\mathbb{R}^2)} \frac{\mathrm{A}(\phi(K))}{\mathrm{D}_B(\phi(K))^2}.$$

Since A and D_B^2 are homogeneous functionals of degree two, we can suppose without loss of generality that $|\det(\phi)| = 1$ and

$$\inf_{\substack{\phi \in End(\mathbb{R}^2) \\ |\det(\phi)|=1}} \mathrm{D}_B(\phi(K)) = \frac{1}{\sqrt{\rho}}. \tag{17}$$

By definition of infimum, consider a sequence $\{\phi_i\}_{i\in\mathbb{N}} \subset End(\mathbb{R}^2)$ such that

$$\mathrm{D}_B(\phi_i(K)) \to \frac{1}{\sqrt{\rho}} \text{ when } i \to \infty.$$

In particular, there exists $C>0$ such that $\mathrm{D}_B(\phi_i(K))\leq C$ for every $i\in\mathbb{N}$. Since $(0,0)\in\phi_i(K)$ and $\mathrm{D}(\phi_i(K))\leq 2\,\mathrm{D}_B(\phi_i(K))\leq 2\,C$ for all $i\in\mathbb{N}$, then $\{\phi_i(K)\}_{i\in\mathbb{N}}$ is an absolutely bounded sequence (since $(0,0)\in\phi_i(K)$, we actually have that $\phi_i(K)\subseteq 2\,C\,\mathbb{B}_2^2$). Hence the *Blaschke Selection Theorem* [Sch, Th. 1.8.7] implies that there exists a subsequence (which will be denoted as the original one) such that $\phi_i(K)\to K_0$ when $i\to\infty$, for some $K_0\in\mathcal{K}^2$. Let us furthermore observe that if $\phi_i=(a^i_{jk})_{1\leq j,k\leq 2}\in\mathbb{R}^{2\times 2}$, since $r\mathbb{B}_2^2\subseteq K$ and $\phi_i(K)\subseteq 2\,C\,\mathbb{B}_2^2$, then it is not difficult to check that $|a^i_{jk}|\leq 2\,C/r$ for every $1\leq j,k\leq 2$ and $i\in\mathbb{N}$. Thus $\{\phi_i\}_{i\in\mathbb{N}}$ is bounded, and so there exists a subsequence (which will be denoted again as the original one) such that $\phi_i\to\phi_0$ when $i\to\infty$, for some $\phi_0\in End(\mathbb{R}^2)$. Moreover, $|\det(\phi_0)|=1$, with $\phi_i(K)\to K_0=\phi_0(K)$ when $i\to\infty$. We will now prove that $\mathrm{D}_B(\phi_0(K))=1/\sqrt{\rho}$, which will imply that $\phi_0(K)$ is in Behrend-bisecting position, as desired.

First of all, since each ϕ_i is linear and regular, we have that ϕ_i is bijective. Let $u_i\in B_{\phi_i(K)}$, and let $x_i\in K$, $\mu_i>0$ be such that the line segment $\phi_i(x_i)+\mu_i[0,u_i]\subset\phi_i(K)$ provides a minimizing bisection of $\phi_i(K)$, for each $i\in\mathbb{N}$. Let $\phi(K_1^i)$, $\phi(K_2^i)$ be the subsets of that bisection, satisfying $\mathrm{D}_B(\phi_i(K))=\mathrm{D}(\phi_i(K_1^i))=\mathrm{D}(\phi_i(K_2^i))$ for every $i\in\mathbb{N}$. Since ϕ_i is a bijection, we will have that $\{K_1^i,K_2^i\}$ is a bisection of K and moreover, we can consider $y_i\in K$ such that $\phi_i(y_i)=\phi_i(x_i)+\mu_i\,u_i$, for every $i\in\mathbb{N}$. Since $\{[x_i,y_i]\}_{i\in\mathbb{N}}\subset K$ is again absolutely bounded, we can suppose that $[x_i,y_i]\to[x_0,y_0]$, when $i\to\infty$. Let $\{K_1^0,K_2^0\}$ be the bisection given by $[x_0,y_0]$. Since $K_j^i\to K_j^0$ when $i\to\infty$, then $\phi_i(K_j^i)\to\phi_0(K_j^0)$ when $i\to\infty$, for $j=1,2$. Therefore $\mathrm{D}(\phi_0(K_j^0))=1/\sqrt{\rho}$, for $j=1,2$, and so $\mathrm{D}_B(K_0)\leq 1/\sqrt{\rho}$. But if this inequality is strict, we get a contradiction with (17), so equality must hold, which finishes the proof. □

The proof of the following characterization of the Behrend position for a convex set can be found in [GMS] (equivalence (ii) was already proved by Behrend [Beh]).

Proposition 16. *Let $K\in\mathcal{K}^2$. The following statements are equivalent.*

- (i) *K is in Behrend position.*
- (ii) *For every $u\in\mathbb{S}^1$, there exists $v\in D_K$ such that $|u^Tv|\leq 1/\sqrt{2}$.*
- (ii') *For every $u\in\mathbb{S}^1$, there exists $v\in D_K$ such that $|u^Tv|\geq 1/\sqrt{2}$.*
- (iii) *There exist $u_i\in D_K$ and $\lambda_i\geq 0$, $i=1,2,3$, such that $\sum_{i=1}^{3}\lambda_i(u_i\,u_i^T)=\mathrm{I}_2$, where I_2 denotes the identity matrix of degree two.*

Next result establishes the analogous in Proposition 16 to (i) implies (ii) or (iii). We borrow most of the ideas from the proof of [GMS, Lemma 3.2].

Lemma 17. *Let $K\in\mathcal{K}^2$ be in Behrend-bisecting position. For every $u\in\mathbb{S}^1$ and every $w\in B_K$, being $\{K_1^w,K_2^w\}$ the corresponding minimizing bisection of K, we have that*

- (i) *there exists $v\in D_{K_1^w}\cup D_{K_2^w}$ such that $|u^Tv|\geq 1/\sqrt{2}$, and*
- (ii) *there exists $v\in D_{K_1^w}\cup D_{K_2^w}$ such that $|u^Tv|\leq 1/\sqrt{2}$.*

Proof. We start proving (i). Let us suppose that for every $v\in D_{K_1^w}\cup D_{K_2^w}$ then $|u^Tv|<1/\sqrt{2}$. Hence every $v\in D_{K_1^w}\cup D_{K_2^w}$ has an angle θ with the line $u^\perp$ satisfying

$$\theta=\frac{\pi}{2}-\arccos(u^Tv)=\arcsin(u^Tv)<\arcsin\frac{1}{\sqrt{2}}=\frac{\pi}{4},$$

and so $\cos^2\theta > 1/2$. More precisely, since K is compact (as well as K_i^w, for $i = 1, 2$), there exists $\delta > 0$ such that for every $v \in D_{K_1^w} \cup D_{K_2^w}$ making angle θ with respect to $u^\perp$, we have

$$\cos^2\theta > \frac{1}{2}(1+\delta). \tag{18}$$

After a suitable rotation of K, we can suppose that $u = e_1$. For small $\varepsilon > 0$, consider the endomorphism of $\mathbb{R}^2$ determined by the matrix

$$A_\varepsilon := \begin{pmatrix} 1 & 0 \\ 0 & 1-\varepsilon \end{pmatrix}.$$

Using elementary trigonometry and calculus, we can see that the length of any line segment ℓ, making angle θ with $u^\perp$, varies under A_ε according to the formula

$$||A_\varepsilon \ell|| = ||\ell||\sqrt{1 - 2\,\varepsilon\cos^2\theta + \varepsilon^2\cos^2\theta} = ||\ell||(1 - \varepsilon\cos^2\theta + O(\varepsilon^2)). \tag{19}$$

Let $K' = A_\varepsilon K$ and $(K_i^w)' = A_\varepsilon K_i^w$, for $i = 1, 2$ (since A_ε is bijective, then $\{(K_1^w)', (K_2^w)'\}$ is a bisection of K'). As A_ϵ is close to the identity matrix for small ε, and K, K_1^w, K_2^w are compact sets, for every $v' \in D_{(K_1^w)'} \cup D_{(K_2^w)'}$ with angle θ' with $u^\perp$ it is possible to choose $\delta' > 0$ small enough such that

$$\cos^2\theta' > \frac{1}{2}(1+\delta'). \tag{20}$$

Let $A_\varepsilon(\ell)$ be the line segment in K' with $||A_\varepsilon(\ell)|| = \max\{\mathrm{D}((K_1^w)'), \mathrm{D}((K_2^w)')\}$, being ℓ the corresponding line segment in K, with angle θ'' with $u^\perp$. Then, equation (20) implies that there exists δ'' such that

$$\cos^2\theta'' > \frac{1}{2}(1+\delta''),$$

since A_ε^{-1} is also close to the identity matrix.

Thus, taking into account (19) and the fact that $w \in B_K$, we have

$$\begin{aligned}
\mathrm{D}_B(K') &\le \max\{\mathrm{D}((K_1^w)'), \mathrm{D}((K_2^w)')\} = ||A_\varepsilon(\ell)|| \\
&= ||\ell||(1 - \varepsilon\cos^2\theta + O(\varepsilon^2)) \le \max\{\mathrm{D}(K_1^w), \mathrm{D}(K_2^w)\}\,(1 - \varepsilon\cos^2\theta + O(\varepsilon^2)) \\
&= \mathrm{D}_B(K)\,(1 - \varepsilon\cos^2\theta + O(\varepsilon^2)),
\end{aligned}$$

and so, since $\mathrm{A}(K') = \mathrm{A}(A_\varepsilon K) = (1-\varepsilon)\,\mathrm{A}(K)$, we conclude that

$$\begin{aligned}
\frac{\mathrm{A}(K')}{\mathrm{D}_B(K')^2} &\ge \frac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2}\frac{1-\varepsilon}{(1-\varepsilon\cos^2\theta'' + O(\varepsilon^2))^2} \\
&\ge \frac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2}\frac{1-\varepsilon}{1 - 2\,\varepsilon\cos^2\theta'' + O(\varepsilon^2)} \\
&> \frac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2}\frac{1-\varepsilon}{1-(1+\delta'')\varepsilon + O(\varepsilon^2)} > \frac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2},
\end{aligned}$$

for ε small enough, contradicting the fact that K is in Behrend-bisecting position.

On the other hand, (ii) follows directly from (i), since (ii) holds for $u \in \mathbb{S}^1$ if (i) holds for $u' \in \mathbb{S}^1 \cap u^\perp$ (and viceversa). □

Remark 18. In contrast with Proposition 16, the necessary condition in Lemma 17 for K to be in Behrend-bisecting position is not *sufficient*. In order to clarify this, we will show a two-fold counterexample. First, we will compute the minimizing bisection for the class of isosceles triangles whose different angle θ belongs to $[0,\pi/3]$. In particular, some of those triangles satisfy the thesis of Lemma 17, but they have different isodiametric quotient and hence that thesis is not sufficient for asserting that the set is in Behrend-bisecting position. Second, we will compute the isodiametric quotient for the isosceles triangles whose different angle θ belongs to $[\pi/3,\pi]$. From both examples, we find out that *surprisingly* the isosceles triangle with different angle equal to $\arccos(\sqrt{2/3})$ (see details below) is the only one maximizing the isodiametric quotient. Moreover, its isodiametric quotient equals $4/(3\sqrt{3})$. For the sake of completeness, we will also prove that the previous isosceles triangle is the *only* triangle in Behrend-bisecting position (and thus not even the equilateral triangle is in Behrend-bisecting position).

1. *Let $K^\theta \in \mathcal{K}^2$ be the isosceles triangle with different angle $\theta \in [0,\pi/3]$.* Let p_1 be the vertex of angle θ, and let p_2, p_3 be the other two vertices. Given any minimizing bisection $\{K_1^\theta, K_2^\theta\}$ of K^θ determined by a line segment, we can suppose that $p_1 \in K_1^\theta$ and $p_2, p_3 \in K_2^\theta$ (otherwise, the diameter of one of the subsets will be equal to $\mathrm{D}(K^\theta)$, and so the bisection will not be minimizing). By a suitable rescaling, we can suppose without loss of generality that $p_2 = (1,0)$, $p_3 = (-1,0)$, and $p_1 = (0, \tan((\pi-\theta)/2))$. The distance from $q_\lambda = (1-\lambda)\,p_1 + \lambda\,p_2$ to p_1 equals $\lambda\sqrt{1+\tan((\pi-\theta)/2)^2}$, whereas to p_3 equals $\sqrt{(1+\lambda)^2 + (1-\lambda)^2\tan((\pi-\theta)/2)^2}$. Since the bisection is minimizing, these two distances must coincide, and so the value of λ must be equal to

$$\lambda_m = \lambda_m(\theta) = \frac{1+\tan(\frac{\pi-\theta}{2})^2}{2\left(\tan(\frac{\pi-\theta}{2})^2 - 1\right)}.$$

An analogous reasoning for the points of the edge $\overline{p_1\,p_3}$ yields that the *only* minimizing bisection by a line segment is given by the horizontal segment

$$[(-\lambda_m, (1-\lambda_m)\tan(\frac{\pi-\theta}{2})), (\lambda_m, (1-\lambda_m)\tan(\frac{\pi-\theta}{2}))].$$

In this case,

$$\lambda_m\bigg(\pm 1, -\tan\left(\frac{\pi-\theta}{2}\right)\bigg) \in D_{K_1^\theta} \quad \text{and} \quad \bigg(\pm(\lambda_m+1), (1-\lambda_m)\tan\left(\frac{\pi-\theta}{2}\right)\bigg) \in D_{K_2^\theta}.$$

It can be checked that for $\theta \in [\pi/6, \pi/3]$, the triangles K^θ satisfy the thesis in Lemma 17, by a direct analysis of the positions of the vectors of $D_{K_1^\theta} \cup D_{K_2^\theta}$. However, not all of those triangles are in Behrend-bisecting position. Note that the isodiametric quotient

$$\frac{\mathrm{A}(K^\theta)}{\mathrm{D}_B(K^\theta)^2} = \frac{\tan(\frac{\pi-\theta}{2})}{\lambda_m^2(1+\tan(\frac{\pi-\theta}{2})^2)} = 2\cos^2(\theta)\sin(\theta)$$

attains its maximum value in the interval $[0,\pi/3]$ only when $\theta = \theta_M = \arccos(\sqrt{2/3})$ ($\approx 35.26°$) with maximum value

$$\frac{\mathrm{A}(K^{\theta_M})}{\mathrm{D}_B(K^{\theta_M})^2} = \frac{4}{3\sqrt{3}}.$$

2. *Let $K^\theta \in \mathcal{K}^2$ be the isosceles triangle with different (largest) angle $\theta \in [\pi/3,\pi]$.* Let p_1 be the vertex of angle θ, and let p_2, p_3 be the other two vertices. For any minimizing bisection $\{K_1^\theta, K_2^\theta\}$ of K^θ determined by a line segment, we can now suppose that $p_1, p_2 \in K_1^\theta$ and that $p_3 \in K_2^\theta$, and so $d(p_1,p_2) \leq \mathrm{D}_B(K^\theta)$. In particular, if we consider the bisection given by the line segment $[p_1, (1/2)(p_2+p_3)]$, then $\mathrm{D}(K_1^\theta) = \mathrm{D}(K_2^\theta) = d(p_1,p_2)$, and so $\mathrm{D}_B(K^\theta) = d(p_1,p_2)$. Call $a = d(p_1,p_2)$ and $b = d(p_2,p_3)$. Then, basic computations show that $b = 2\,a\sin(\theta/2)$ and

$$\frac{\mathrm{A}(K^\theta)}{\mathrm{D}_B(K^\theta)^2} = \frac{\frac{1}{2}(2\,a\sin(\frac{\theta}{2}))\sqrt{a^2 - a^2\sin(\frac{\theta}{2})^2}}{a^2} = \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{2},$$

and hence

$$\frac{\mathrm{A}(K^\theta)}{\mathrm{D}_B(K^\theta)^2} \leq \frac{\mathrm{A}(K^{\frac{\pi}{2}})}{\mathrm{D}_B(K^{\frac{\pi}{2}})^2} = \frac{1}{2},$$

which is smaller than the maximum value $4/(3\sqrt{3})$ from the previous case.

3. *Let $K \in \mathcal{K}^2$ be a general triangle.* We can assume that $K = \mathrm{conv}\{p_1, p_2, p_3\}$ for some $p_i \in \mathbb{R}^2$, $i = 1, 2, 3$, with $\mathrm{D}(K) = d(p_1, p_2)$. Let $\alpha_i > 0$ be the angle at vertex p_i, for $i = 1, 2, 3$, with $\alpha_1 \leq \alpha_2 \leq \alpha_3$. For any minimizing bisection $\{K_1, K_2\}$ of K, we can suppose that $p_1 \in K_1$ and that $p_2, p_3 \in K_2$ (otherwise, the bisection will not be minimizing). Call $q_\lambda = (1-\lambda)p_1 + \lambda p_3$, and let $\lambda_m \in [0, 1]$ be such that the distance d_1 from q_{λ_m} to p_1 is the same than to p_2. Analogously, consider $r_\mu := (1-\mu)p_1 + \mu p_2$, and let $\mu_m \in [0, 1]$ be such that the distance d_2 from r_{μ_m} to p_1 is the same than to p_3. In this case, and since the distance from p_1 to p_3 is not larger than to p_2, we clearly have that $d_1 \geq d_2$, and hence the line segment with endpoints q_{λ_m} and r_{μ_m} provides a minimizing bisection of K, with subsets $K_1 = \mathrm{conv}\{p_1, q_{\lambda_m}, r_{\mu_m}\}$ and $K_2 = \mathrm{conv}\{p_2, p_3, q_{\lambda_m}, r_{\mu_m}\}$ satisfying that $\mathrm{D}(K_1) = \mathrm{D}(K_2) = d_1$. Let p_3' be the point in the ray from p_1 to p_3 which is at the same distance from p_1 than p_2, and consider the isosceles triangle $K' = \mathrm{conv}\{p_1, p_2, p_3'\}$. Then we clearly have that $K \subseteq K'$. Moreover, the bisection minimizing the diameter of K' is given again by the line segment with endpoints q_{λ_m} and $(1-\lambda_m)p_1 + \lambda_m q_2$, with $\mathrm{D}_B(K') = \mathrm{D}_B(K) = d_1$. Hence

$$\frac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2} \leq \frac{\mathrm{A}(K')}{\mathrm{D}_B(K')^2},$$

which implies that the isodiametric quotient of K is always maximized by the isodiametric quotient of an isosceles triangle whose different angle is not larger than $\pi/3$ (because $\alpha_1 \leq \pi/3$). Taking into account the previous results (and the fact that any planar triangle can be obtained by applying an appropriate endomorphism to K), we conclude that the unique triangle in Behrend-bisecting position is the isosceles triangle with different angle equal to $\theta_M = \arccos(\sqrt{2/3})$.

In view of Remark 18, and taking into account the results from [Beh], it is natural to conjecture the following optimal reverse isodiametric bisecting inequality.

Conjecture 19. *Let $K \in \mathcal{K}^2$ be in Behrend-bisecting position. Then*

$$\frac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2} \geq \frac{4}{3\sqrt{3}},$$

with equality if and only if K is the isosceles triangle with different angle equal to $\arccos(\sqrt{2/3})$.

The following proof is strongly inspired in the original proof of Behrend [Beh] for showing (6).

Corollary 20. *Let $K \in \mathcal{K}^2$ be in Behrend-bisecting position. Given $w \in B_K$, let $\{K_1^w, K_2^w\}$ be the corresponding minimizing bisection of K. Then, there exist u_1, $u_2 \in D_{K_1^w} \cup D_{K_2^w}$ such that $|u_1^T u_2| \leq 1/2$.*

Proof. By applying a proper rotation, we can assume that $e_1 \in D_{K_1^w} \cup D_{K_2^w}$. Then, for $e_2 \in \mathbb{S}^1$, by Lemma 17 (i), there exists $u = (\cos\alpha, \sin\alpha) \in D_{K_1^w} \cup D_{K_2^w}$ such that $|e_2^T u| \geq 1/\sqrt{2}$, which implies that $\alpha \in [\pi/4, 3\pi/4]$. We can assume that $\alpha \in [\pi/4, \pi/2]$, by reflecting K with respect to $\mathrm{span}\{e_2\}$ if necessary. If $\alpha \geq \pi/3$, then $|e_1^T u| \leq 1/2$, which proves the statement for $u_1 = e_1$ and $u_2 = u$. So assume that $\alpha < \pi/3$, and note that, taking into account the previous argument, we can suppose that $(\cos\mu, \sin\mu) \notin D_{K_1^w} \cup D_{K_2^w}$ for $\mu \in [\pi/3, 2\pi/3]$. Consider the vector $\widetilde{u} = (\cos(\pi/3 + \pi/4), \sin(\pi/3+\pi/4)) \in \mathbb{S}^1$. Again by Lemma 17 (i), there exists $v = (\cos\beta, \sin\beta) \in D_{K_1^w} \cup D_{K_2^w}$ such that $|\widetilde{u}^T v| \geq 1/\sqrt{2}$. This necessarily implies that $2\pi/3 < \beta \leq \pi/3 + \pi/2 = 5\pi/6 < \pi$. In particular, the angle between u and v is at least $2\pi/3 - \pi/3 = \pi/3$ and at most $5\pi/6 - \pi/4 = 7\pi/12 < 2\pi/3$, and thus we have that $|u^T v| \leq 1/2$, as desired. □

We are now able to prove Theorem 5.

Proof of Theorem 5. Since K is in Behrend-bisecting position, for a given $w \in B_K$, there exist $u_1,\, u_2 \in D_{K_1^w} \cup D_{K_2^w}$ such that $|u_1^T u_2| \leq 1/2$, by Corollary 20, where $\{K_1^w, K_2^w\}$ is the corresponding minimizing bisection. Since $\mathrm{D}(K_1^w) = \mathrm{D}(K_2^w) = \mathrm{D}_B(K)$, there exist $x_1, x_2 \in K$ such that $x_1 + \mathrm{D}_B(K)[0, u_1],\ x_2 + \mathrm{D}_B(K)[0, u_2] \subset K$ (observe that each of these segments is contained in K_1^w *or* K_2^w).

Now we use an argument from the proof of [GMS, Th. 3.4]. Since K is convex, then $C := \mathrm{conv}\{x_1 + \mathrm{D}_B(K)[0, u_1],\ x_2 + \mathrm{D}_B(K)[0, u_2]\}$ is contained in K, and so $\mathrm{A}(C) \leq \mathrm{A}(K)$. In this situation, a result by Groemer [Gro] (see [BH, Th. 2]) states that $\mathrm{A}(C)$ is minimal if both segments have a common point, and thus, straightforward computations give

$$\begin{aligned}\mathrm{A}(K) &\geq \mathrm{A}(C) \geq \mathrm{A}(\mathrm{conv}\{\mathrm{D}_B(K)[0, u_1], \mathrm{D}_B(K)[0, u_2]\}) \\ &= \frac{\mathrm{D}_B(K)^2}{2}\sqrt{1 - (u_1^T u_2)^2} \geq \frac{\sqrt{3}}{4}\,\mathrm{D}_B(K)^2,\end{aligned}$$

which completes the proof. □

5.1. **The centrally symmetric case.** As in [Beh], we will also focus on the centrally symmetric case (considering always the origin as center of symmetry), pursuing an isodiametric inequality for bisections in this setting. The following result was proven in [MPS].

Lemma 21. *(*[MPS, Prop. 4]*) Let $K \in \mathcal{K}^2$ be centrally symmetric. Then there exists a minimizing bisection $\{K_1, K_2\}$ of K such that*

- *$K_1 \cap K_2 = [-p, p]$, for some $p \in \mathrm{bd}(K)$.*
- *$K_1 = -K_2$.*

The above Lemma 21 allows to obtain a necessary condition for a given centrally symmetric body to be in Behrend-bisecting position.

Lemma 22. *Let $K \in \mathcal{K}^2$ be centrally symmetric and in Behrend-bisecting position. For every $w \in B_K$ with $\{K_1^w, -K_1^w\}$ as the corresponding minimizing bisection of K, we have that K_1^w and $-K_1^w$ are in Behrend position.*

Proof. Since K is in Behrend-bisecting position and $w \in B_K$, Lemma 17 (ii) implies that for every $u \in \mathbb{S}^1$, there exists $v \in D_{K_1^w} \cup D_{-K_1^w} = D_{K_1^w} = D_{-K_1^w}$, such that $|u^T v| \leq 1/\sqrt{2}$. By Proposition 16, we obtain that K_1^w is in Behrend position, as well as $-K_1^w$. □

We can now prove Theorem 6, which establishes an isodiametric inequality for bisections in the centrally symmetric case.

Proof of Theorem 6. Let $\{K_1, K_2\}$ be a minimizing bisection of K. We can suppose by Lemma 21 that $K_2 = -K_1$. As K is centrally symmetric and in Behrend-bisecting position, Lemma 22 yields that K_1 (and also $K_2 = -K_1$) is in Behrend position. Thus (6) implies that

$$\frac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2} = \frac{\mathrm{A}(K_1)+\mathrm{A}(K_2)}{\mathrm{D}_B(K)^2} = \frac{\mathrm{A}(K_1)}{\mathrm{D}(K_1)^2} + \frac{\mathrm{A}(K_2)}{\mathrm{D}(K_2)^2} \geq \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}. \tag{21}$$

□

Remark 23. As we did in Remark 18, we will study the isodiametric quotient for the affine class of the square, i.e., the parallelograms, in order to determine which of them are in Behrend-bisecting position. We will find out that the *only* one in Behrend-bisecting position is the rectangle $[-1,1]\times[-2,2]$ (up to dilations and rigid motions, see Remark 4). This means that the parallelogram formed by two equilateral triangles touching in a common edge *is not* in Behrend-bisecting position, which implies that the *necessary* condition in Lemma 22 is *not sufficient* (recall that the equilateral triangles are in Behrend position). Moreover, this suggests that the inequality from Theorem 6 is not sharp.

Let $K \subset \mathbb{R}^2$ be a parallelogram (which is centrally symmetric), and let $[-p,p]$ be a line segment determining a minimizing bisection $\{K_1, K_2\}$ of K, for some $p \in \mathrm{bd}(K)$. If K is in Behrend-bisecting position, then K_1 (and $K_2 = -K_1$) is in Behrend position, by Lemma 22. We will distinguish two possibilities:

1. *Assume that p is a vertex of K.* Then K_1 and K_2 are triangles. Since the only triangle in Behrend position is the equilateral one, then the only candidate in this case is the parallelogram P formed by two congruent equilateral triangles joined by a common edge, with isodiametric quotient $\mathrm{A}(P)/\mathrm{D}_B(P)^2 = \sqrt{3}/2$, in view of (21).

2. *Assume that p is not a vertex of K.* Then K_1 is a quadrangle with two parallel edges that can be seen as $K_1 = \mathrm{conv}\{p_1,p_2,p_3,p_4\}$, where $p_i \in \mathbb{R}^2$, $i = 1,\ldots,4$, which is in Behrend position. Proposition 16 implies that there exist at least two different vectors $v_1, v_2 \in D_{K_1}$, and so K_1 contains at least two different diametrical segments. Since K_1 is a quadrangle with two parallel edges, then necessarily one of the diagonals of K_1, namely $[p_1,p_3]$, is a diametrical segment. Denote by h_1 (resp. h_2) the distance from p_2 (resp. p_4) to $[p_1,p_3]$. Then $h_1 + h_2 \leq d(p_2,p_4) \leq \mathrm{D}(K_1)$, and

$$\mathrm{A}(K_1) = \frac{1}{2}\,\mathrm{D}(K_1)\,(h_1+h_2) \leq \frac{\mathrm{D}(K_1)^2}{2}.$$

Since $K_2 = -K_1$, we will also have that $\mathrm{A}(K_2) \leq \mathrm{D}(K_2)^2/2$. Then

$$\frac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2} = \frac{\mathrm{A}(K_1)+\mathrm{A}(K_2)}{\mathrm{D}_B(K)^2} = \frac{\mathrm{A}(K_1)}{\mathrm{D}(K_1)^2} + \frac{\mathrm{A}(K_2)}{\mathrm{D}(K_2)^2} \leq \frac{1}{2} + \frac{1}{2} = 1.$$

Moreover, we have equality above if and only if $h_1 + h_2 = \mathrm{D}(K_1)$. This is equivalent to the fact that $[p_2,p_4]$ is orthogonal to $[p_1,p_3]$, i.e., when K_1 (and thus K_2) is a square. This implies that $K = K_1 \cup K_2$ is a rectangle of the form $[-1,1]\times[-2,2]$.

Finally, since the isodiametric quotient of the parallelogram P (consisting of two joined equilateral triangles) is equal to $\sqrt{3}/2$, whereas the corresponding one for the rectangle $[-1,1]\times[-2,2]$ equals 1, we conclude that the only parallelogram in Behrend-bisecting position is that rectangle.

The previous Remark 23 naturally leads us to the following conjecture.

Conjecture 24. *Let $K \in \mathcal{K}^2$ be centrally symmetric and in Behrend-bisecting position. Then*

$$\frac{\mathrm{A}(K)}{\mathrm{D}_B(K)^2} \geq 1,$$

with equality if and only if $K = [-1,1] \times [-2,2]$.

6. The isominwidth-bisecting position and the reverse isominwidth inequality

In this section we will establish a reverse isominwidth inequality, following the same scheme as in Section 5. In order to obtain such an inequality, we will focus on the planar convex bodies in isominwidth-bisecting position, defined by equality (12). Our first observation is that the infimum in (12) is actually a minimum, and so for any given $K \in \mathcal{K}^2$ there exists an affine representative in isominwidth-bisecting position (we will omit the proof of this fact since it is completely analogous to Lemma 15). Notice also that $\mathrm{w}_B(K) = \mathrm{w}(K)/2$ by Lemma 14, and so

$$\min_{\phi \in End(\mathbb{R}^2)} \frac{\mathrm{A}(\phi(K))}{\mathrm{w}_B(\phi(K))^2} = 4 \min_{\phi \in End(\mathbb{R}^2)} \frac{\mathrm{A}(\phi(K))}{\mathrm{w}(\phi(K))^2}.$$

This allows us to obtain the following characterization for the isominwidth-bisecting position (see [GMS, Th. 5.3] for other equivalences).

Corollary 25. *Let $K \in \mathcal{K}^2$. The following statements are equivalent:*

(i) *K is in isominwidth-bisecting position.*
(ii) *K is in isominwidth optimal position.*

Finally, we can prove Theorem 7.

Proof of Theorem 7. By Corollary 25, K is in isominwidth optimal position, and by using (11) we conclude that

$$\frac{\mathrm{A}(K)}{\mathrm{w}_B(K)^2} = 4 \, \frac{\mathrm{A}(K)}{\mathrm{w}(K)^2} \leq 4.$$

The equality case follows directly from the corresponding equality case in (11). □

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