Cocyclic Hadamard matrices over Latin rectangles

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A B S T R A C T

In the literature, the theory of cocyclic Hadamard matrices has always been developed over finite groups. This paper introduces the natural generalization of this theory to be developed over Latin rectangles. In this regard, once we introduce the concept of binary cocycle over a given Latin rectangle, we expose examples of Hadamard matrices that are not cocyclic over finite groups but they are over Latin rectangles. Since it is also shown that not every Hadamard matrix is cocyclic over a Latin rectangle, we focus on answering both problems of existence of Hadamard matrices that are cocyclic over a given Latin rectangle and also its reciprocal, that is, the existence of Latin rectangles over which a given Hadamard matrix is cocyclic. We prove in particular that every Latin square over which a Hadamard matrix is cocyclic must be the multiplication table of a loop (not necessarily associative). Besides, we prove the existence of cocyclic Hadamard matrices over non-associative loops of order 2^{t+3}, for all positive integer $t > 0$.

1. Introduction

A *(binary) Hadamard matrix* H of order n is an $n \times n$ array with every entry either 1 or -1, which satisfies $HH^T = nI_n$, where I_n denotes the identity matrix of order n. It is well-known that n has to be necessarily $1,\ 2$ or a multiple of 4, but there is no certainty whether such a Hadamard matrix exists at every possible order. The *Hadamard Conjecture* asserts that there exists a Hadamard matrix of order $4t$ for every natural number t. Two Hadamard matrices are said to be *equivalent* if they are equal up to permutations or negation of their row and columns.

There exist so many different constructions for Hadamard matrices: Sylvester, Paley, Williamson, Ito, Goethals–Seidel, one and two circulant cores or cocyclic matrices, amongst others (see [15]). With respect to the latter, Horadam and de Launey introduced [16] the use of cocycles and cocyclic matrices within the theory of block designs and showed [17] that the cocyclic framework could provide a structural approach to resolve the Hadamard Conjecture. Since then, the theory of cocyclic Hadamard matrices has been widely developed, but always under the assumption of working on finite groups. Their distribution into equivalence classes is explicitly known for order $n < 40$ [8]. This paper generalizes this theory by introducing the concept of cocyclic Hadamard matrices on $r \times n$ Latin rectangles. Latin squares were already used in the classical theory of Hadamard matrices to construct symmetric Hadamard matrices of order 36 [6,7,13].

The paper is organized as follows. In Section 2, we expose some preliminary concepts and results on cocyclic Hadamard matrices, Latin rectangles and quasigroups that are used throughout our study. We introduce in Section 3 the concept of cocycle over a Latin rectangle and that of cocyclic Hadamard matrix over a Latin rectangle. Section 4 deals with the set of Hadamard matrices that are cocyclic over a given Latin rectangle, whereas Section 5 focuses on the reciprocal problem, that is, on the set of Latin rectangles over which a given Hadamard matrix is cocyclic. Finally, since this paper has a high dependence on notation, a glossary of symbols is shown in the Appendix.

2. Preliminaries

Let us review some basic concepts and results on cocyclic Hadamard matrices, Latin rectangles and quasigroups that are used throughout the paper. We refer the reader to [9,11,15,23] for more details about these topics.

2.1. Cocyclic Hadamard matrices

Let G be a finite group and let $\mathbb{Z}_2$ be the finite set $\{\pm 1\}$ considered as the cyclic group of order 2. Any map $\psi : G \times G \to \mathbb{Z}_2$ may be represented by the square matrix $M_\psi = (\psi(a, b))_{a,b\in G}$, whose rows and columns are indexed by the elements of the group G under some fixed ordering. If there exists a function $\phi : G \to \mathbb{Z}_2$ such that $\psi(a, b) = \phi(ab)$, for all $a, b \in G$, so that the matrix M_ψ is Hadamard, then the latter is said to be a *Hadamard matrix developed over* the group G. Turyn [29] proved that, if G is cyclic, then M_ψ is Hadamard only if the order of G is $4t^2$, for some odd positive integer $t \in \mathbb{N}$ that is not a prime power. The only known equivalence class of Hadamard matrices developed over a cyclic group $\mathbb{Z}_{4t^2}$ is represented by the following circulant matrix defined for $t = 1$

$$\begin{pmatrix} + & + & + & - \\ + & + & - & + \\ + & - & + & + \\ - & + & + & + \end{pmatrix}$$

where, from here on, the symbols $+$ and $-$ represent, respectively, the entries 1 and -1. The *Circulant Hadamard Conjecture* [27] asserts that there does not exist any Hadamard matrix over a cyclic group $\mathbb{Z}_{4t^2}$, for $t > 1$. It is known [5,22] that this is true for all positive integer $t < 11715$ and also that there exist 948 open cases for $t \leq 10^{13}$. In the literature, this fact has dissuaded the authors from searching for cocyclic Hadamard matrices over cyclic groups.

Horadam and de Launey [16,17] introduced the concept of (2-dimensional binary) *cocycle* over a finite group G as a map $\psi : G \times G \to \mathbb{Z}_2$ such that

$$\psi(ab, c) = \psi(a, b)\psi(b, c)\psi(a, bc), \text{ for all } a, b, c \in G. \tag{1}$$

Particularly, $\psi(1, 1) = \psi(a, 1) = \psi(1, a)$, for all $a \in G$. If $\psi(1, 1) = 1$, then the cocycle ψ is said to be *normalized*. The corresponding matrix M_ψ is called *(pure) cocyclic* over the group G. If this is Hadamard, then it is said to be a *(pure) cocyclic Hadamard matrix* over the group G.

Baliga and Horadam [2] dealt with the existence of cocyclic Hadamard matrices over the groups $\mathbb{Z}_t \times \mathbb{Z}_2^2$, for t odd. Shortly after, Flannery [12] proved constructively the existence of cocyclic Hadamard matrices over the *dihedral group*

$$D_{4t} := \langle\, a, b \mid a^{2t} = b^2 = (ab)^2 = 1 \,\rangle, \tag{2}$$

for all positive integer $t \leq 11$. He also established a series of conditions to ensure the existence of cocyclic Hadamard matrices over abelian, dicyclic, and dihedral groups over certain orders. Alternative constructions have recently been obtained by Álvarez et al. [1].

The Kronecker or direct product of two cocyclic Hadamard matrices is also cocyclic Hadamard over the direct product of the involved pair of groups [12]. More specifically, if ψ and ψ' are respective cocycles over two finite groups G and G', then, it is known [10] that the map $\psi \otimes \psi'$ defined as

$$\psi \otimes \psi'((a, a'), (b, b')) = \psi(a, b)\psi'(a', b'), \text{ for all } a, b \in G,\ a', b' \in G', \tag{3}$$

is a cocycle over the finite group $G \otimes G'$. If both cocyclic matrices M_ψ and $M_{\psi'}$ are Hadamard, then the cocyclic matrix $M_{\psi\otimes\psi'}$, which coincides indeed with their direct product, $M_\psi \otimes M_{\psi'}$, is also Hadamard.

2.2. Latin rectangles

Let r and n be two positive integers such that $r \leq n$. An $r \times n$ *Latin rectangle* is an $r \times n$ array $L = (l_{i,j})$ in which each cell contains one symbol of the set $[n] := \{1, \ldots, n\}$ and such that each symbol occurs exactly once in each row and at most once in each column. This is a *Latin square* of *order n* if $r = n$, in which case, each symbol of the set $[n]$ also occurs exactly once in each column. Hereafter, let $\mathcal{R}_{r,n}$ denote the set of $r \times n$ Latin rectangles.

The *direct product* of two Latin rectangles $L = (l_{i,j}) \in \mathcal{R}_{r,n}$ and $L' = (l'_{i',j'}) \in \mathcal{R}_{r',n'}$ is a Latin rectangle $L \otimes L' \in \mathcal{R}_{rr',nn'}$ such that, for each $(i, j, i', j') \in [r] \times [n] \times [r'] \times [n']$, its cell $((i-1)r' + i', (j-1)n' + j') \in [rr'] \times [nn']$ contains the symbol $(l_{i,j} - 1)n' + l'_{i',j'} \in [nn']$.

Example 1. Let us consider the Latin rectangles

$$L \equiv \begin{array}{|c|c|}\hline 2 & 1 \\ \hline\end{array} \in \mathcal{R}_{1,2} \qquad \text{and} \qquad L' \equiv \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline\end{array} \in \mathcal{R}_{2,3}.$$

Then,

$$L \otimes L' \equiv \begin{array}{|c|c|c|c|c|c|}\hline 4 & 5 & 6 & 1 & 2 & 3 \\ \hline 5 & 6 & 4 & 2 & 3 & 1 \\ \hline\end{array} \in \mathcal{R}_{2,6}.$$

Every Latin rectangle $L = (l_{i,j}) \in \mathcal{R}_{r,n}$ is uniquely determined by its *set of entries*

$$E(L) := \{(i, j, l_{i,j}) \mid\ i \leq r, j \leq n\}. \tag{4}$$

The Latin rectangle L is said to be *reduced* if its set of entries contains the triples $(1, j, j)$ and $(i, 1, i)$, for all $i \in [r]$ and $j \in [n]$; that is, if all the symbols of the set $[n]$ (respectively, $[r]$) are displayed in natural order within the first row (respectively, first column) of L. Thus, for instance, the Latin rectangle $L' \in \mathcal{R}_{2,3}$ in Example 1 is reduced. Its set of entries is $E(L') = \{(1, 1, 1), (1, 2, 2), (1, 3, 3), (2, 1, 2), (2, 2, 3), (2, 3, 1)\}$.

Let s be a positive integer such that $r \leq s \leq n$. It is said that a Latin rectangle $L \in \mathcal{R}_{r,n}$ can be *extended* to a Latin rectangle $L' \in \mathcal{R}_{s,n}$ if $E(L) \subseteq E(L')$. Marshall Hall [14] proved that every Latin rectangle in $\mathcal{R}_{r,n}$ can be extended to at least one Latin rectangle in $\mathcal{R}_{s,n}$.

Example 2. The following two Latin rectangles, $L \in \mathcal{R}_{1,8}$ and $L' \in \mathcal{R}_{2,8}$, satisfy that $E(L) \subseteq E(L')$.

$$L \equiv \begin{array}{|c|c|c|c|c|c|c|c|}\hline 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \\ \hline\end{array} \qquad L' \equiv \begin{array}{|c|c|c|c|c|c|c|c|}\hline 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \\ \hline 3 & 5 & 6 & 8 & 7 & 2 & 1 & 4 \\ \hline\end{array}$$

Let S_n denote the symmetric group on n elements. For each positive integer $i \leq r$, the ith row of a Latin rectangle $L = (l_{i,j}) \in \mathcal{R}_{r,n}$ can be interpreted as the permutation $\pi \in S_n$ such that $\pi(j) = l_{i,j}$, for all $j \leq n$. The disjoint cycles in which such a permutation decomposes determine the *cycles* of the row. Thus, for instance, the first row of the Latin rectangle L' in Example 2 constitutes the permutation $(1234)(5678) \in S_8$, whereas its second row determines the permutation $(136257)(48) \in S_8$.

Permutations of rows, columns and symbols of Latin rectangles preserve the set $\mathcal{R}_{r,n}$. In this regard, two Latin rectangles $L = (l_{i,j})$ and $L' = (l'_{i,j})$ in $\mathcal{R}_{r,n}$ are said to be *isotopic* if there exists a triple $(f, g, h) \in S_r \times S_n \times S_n$ such that $l'_{f(i),g(j)} = h(l_{i,j})$, for all $i \leq r$ and $j \leq n$. The set of entries of the *isotope* L' is $E(L') = \{(f(i), g(j), h(l_{i,j})) \mid (i, j, l_{i,j}) \in E(L)\}$. The triple (f, g, h) is an *isotopism* between L and L'. This constitutes an *isomorphism* if $r = n$ and $f = g = h$. In this case, the Latin squares L and L' are said to be *isomorphic*. To be isotopic (respectively, isomorphic) are equivalence relations among Latin rectangles (respectively, Latin squares). Nowadays, the distribution of Latin squares into isotopism and isomorphism classes is known [18,19,25] for order up to 11, whereas that of $r \times n$ Latin rectangles into isotopism classes is known [24] for $r \leq n \leq 9$.

2.3. Quasigroups

A *quasigroup* [26] is a pair $Q = (S, \cdot)$ formed by a set S that is endowed with a binary operation $\cdot$, such that the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y in S, for all $a, b \in S$. Hence, Q has left- and right-division, which are respectively denoted $\backslash$ and $/$ (that is, $x = a\backslash b$ in the first equation, and $y = b/a$ in the second one). Every associative quasigroup constitutes a group. The quasigroup Q is a *loop* if there exists a unit element $e \in S$ such that $a \cdot e = e \cdot a = a$, for all $a \in S$. The quasigroup Q is *finite* if the set S is. In such a case, the *order* of the former is defined as the cardinality of the latter. The quasigroup Q is *monogenic* if it is generated by a single element $a \in S$. Every monogenic associative quasigroup is a cyclic group.

From here on, we denote by $\mathcal{Q}_n$ the set of finite quasigroups having $[n]$ as their finite set of symbols. Every Latin square $L = (l_{i,j}) \in \mathcal{R}_{n,n}$ constitutes, therefore, the multiplication table of a finite quasigroup $Q = ([n], \cdot) \in \mathcal{Q}_n$, where $i \cdot j = l_{i,j}$, for all $i, j \leq n$. Besides, any reduced Latin square constitutes the multiplication table of a loop. Transpositions and isotopisms of Latin squares preserve the set $\mathcal{Q}_n$. In this regard, the *transpose* of a quasigroup $Q = ([n], \cdot) \in \mathcal{Q}_n$ is the quasigroup $Q^t = ([n], \circ)$ that is defined so that $i \circ j = j \cdot i$, for all $i, j \leq n$. Further, two quasigroups $Q = ([n], \cdot)$ and $Q' = ([n], \circ)$ are said to be *isotopic* if there exist three permutations f, g and h in S_n such that $f(i) \circ g(j) = h(i \cdot j)$, for all $i, j \in [n]$. The triple (f, g, h) is an *isotopism* from Q to Q'. This constitutes an isomorphism if $f = g = h$, in which case the quasigroups are said to be *isomorphic*.

3. Cocycles over Latin rectangles

From here on, let $L = (l_{i,j})$ be a Latin rectangle in $\mathcal{R}_{r,n}$. We define the subset of symbols

$$S(L) := [r] \cup \{l_{i,j} \mid i, j \leq r\} \subseteq [n]. \tag{5}$$

As such, this set is composed partly of the row labels of the array L and partly of the symbols appearing in the left $r \times r$ subarray within the latter (see Example 3). Remark in particular that $S(L) = [n]$ whenever L is a Latin square of order n.

Example 3. Let $L \in \mathcal{R}_{1,8}$ and $L' \in \mathcal{R}_{2,8}$ be the Latin rectangles in Example 2. In order to determine the set $S(L)$, we focus on the left 1×1 subarray of L, which we highlight below.

2	3	4	1	6	7	8	5

According to (5), the set $S(L)$ is formed by the row label 1 and the symbol 2 appearing in the highlighted subarray. That is, $S(L) = \{1\} \cup \{2\} = \{1, 2\}$.

Similarly, in order to determine the set $S(L')$, we focus on the left 2×2 subarray of L', which we highlight below.

2	3	4	1	6	7	8	5
3	5	6	8	7	2	1	4

Thus, the set $S(L')$ is formed by the row labels 1 and 2, and the symbols 2, 3 and 5 appearing in the highlighted subarray. Hence, $S(L') = \{1, 2, 3, 5\}$.

Definition (5) enables us to introduce the concept of cocycle over a Latin rectangle. We say that a function $\psi\colon S(L)\times[n]\to\mathbb{Z}_2$ is a *cocycle* over the Latin rectangle L if

$$\psi(l_{i,j},k)=\psi(i,j)\psi(j,k)\psi(i,l_{j,k}),\ \text{ for all } i,j\le r \text{ and } k\le n. \tag{6}$$

In particular, if L is a Latin square of order n, which constitutes in turn the multiplication table of a quasigroup $Q=([n],\cdot)\in\mathcal{Q}_n$, then Condition (6) is equivalent to

$$\psi(i\cdot j,k)=\psi(i,j)\psi(j,k)\psi(i,j\cdot k),\ \text{ for all } i,j,k\le n. \tag{7}$$

In such a case, we also say that the function ψ is a *cocycle* over the quasigroup Q. This generalizes the classical notion of cocycle over a finite group, which arises when Q is associative.

Example 4. Let us consider the Latin rectangle

$$L=(l_{i,j})\equiv\boxed{\begin{array}{c|c}2&1\end{array}}\in\mathcal{R}_{1,2}.$$

In particular, $S(L)=\{1,2\}$. Then, the function $\psi:[2]\times[2]\to\mathbb{Z}_2$ defined so that $\psi(1,1)=1$ and $\psi(1,2)=\psi(2,1)=\psi(2,2)=-1$ is a cocycle over L, because

- $\psi(l_{1,1},1)=\psi(2,1)=-1=\psi(1,1)\psi(1,1)\psi(1,2)=\psi(1,1)\psi(1,1)\psi(1,l_{1,1})$, and
- $\psi(l_{1,1},2)=\psi(2,2)=-1=\psi(1,1)\psi(1,2)\psi(1,1)=\psi(1,1)\psi(1,2)\psi(1,l_{1,2})$.

From here on, we denote $\mathcal{Z}(L)$ the set of cocycles over the Latin rectangle L. The following result establishes lower and upper bounds for the size of such a set.

Lemma 5. *Let $L\in\mathcal{R}_{r,n}$. Then, $|\mathcal{Z}(L)|$ is an even positive integer such that*

$$2\le|\mathcal{Z}(L)|\le 2^{|S(L)|\cdot n}.$$

Proof. From (6), it is straightforwardly verified that a function ψ is a cocycle over L if and only if the function $-\psi$ is. Hence, $|\mathcal{Z}(L)|$ must be even. Further, its lower bound follows from the fact that both constant maps ψ and $-\psi$, which are defined so that $\psi(i,j)=1$, for all $(i,j)\in S(L)\times[n]$, are always cocycles over L. Finally, the upper bound holds readily from the domain and image of every cocycle over an $r\times n$ Latin rectangle. $\square$

The following example shows a family of reduced Latin rectangles whose set of cocycles fits the lower bound indicated in Lemma 5.

Example 6. Let $L=(l_{1,j})$ be the reduced Latin rectangle in the set $\mathcal{R}_{1,n}$. That is, $l_{1,j}=j$, for all $j\le n$. Let $\psi\in\mathcal{Z}(L)$. According to (6), for each $k\le n$, we have that

$$\psi(1,k)=\psi(l_{1,1},k)=\psi(1,1)\psi(1,k)\psi(1,l_{1,k})=\psi(1,1)\psi(1,k)\psi(1,k)=\psi(1,1).$$

As a consequence, ψ must be a constant map and hence, $|\mathcal{Z}(L)|=2$.

Unlike this lower bound, the upper bound in Lemma 5 is just fitted by the only Latin rectangle existing in the set $\mathcal{R}_{1,1}$. In order to prove it and also to determine the exact size of the set of cocycles of each Latin rectangle, we introduce here the concept of *cocyclic degree-of-freedom* of a Latin rectangle $L\in\mathcal{R}_{r,n}$ as the minimum number $d_f(L)$ of pairs $(i,j)\in S(L)\times[n]$ that are required to determine uniquely such a cocycle by keeping in mind to this end the set of constraints derived from Condition (6). The following example illustrates this definition.

Example 7. Let $L\in\mathcal{R}_{1,2}$ be the Latin rectangle in Example 4. Condition (6) implies that every cocycle $\phi\in\mathcal{Z}(L)$ satisfies that

- $\phi(2,1)=\phi(l_{1,1},1)=\phi(1,1)\phi(1,1)\phi(1,l_{1,1})=\phi(1,1)\phi(1,1)\phi(1,2)=\phi(1,2)$, and
- $\phi(2,2)=\phi(l_{1,1},2)=\phi(1,1)\phi(1,2)\phi(1,l_{1,2})=\phi(1,1)\phi(1,2)\phi(1,1)=\phi(1,2)$.

Hence, $\phi(1,2)=\phi(2,1)=\phi(2,2)$. Observe that none constraint concerning $\phi(1,1)$ derives from Condition (6). As a consequence, $d_f(L)=2$ and hence, keeping in mind the image $\mathbb{Z}_2$ of any cocycle over L, we have that $|\mathcal{Z}(L)|=2^2=4$. More specifically, we have that $\mathcal{Z}(L)=\{\phi_1,\phi_2,\phi_3,\phi_4\}$, where

- $\phi_1 = \psi$, the cocycle defined in Example 4.
- $\phi_2(i, j) = 1$, for all $(i, j) \in [2] \times [2]$.
- $\phi_3 = -\phi_1$.
- $\phi_4 = -\phi_2$.

This example enables us to ensure in particular that isotopisms of Latin rectangles do not preserve the size of their cocycles, because, from Example 6, the reduced Latin rectangle in the set $\mathcal{R}_{1,2}$ (which is isotopic to L) has only two cocycles.

As we have illustrated in Example 7, this notion of cocyclic degree-of-freedom is well-defined because it does not depend on any cocycle, but only on the entries of the Latin rectangle under consideration. Further, the following result holds straightforwardly from such a definition.

Lemma 8. *Let $L \in \mathcal{R}_{r,n}$. Then, $|\mathcal{Z}(L)| = 2^{d_f(L)}$, where $1 \leq d_f(L) \leq |S(L)| \cdot n$.*

The following result indicates the only case for which the upper bounds of both Lemmas 5 and 8 are tight.

Proposition 9. *Let $L \in \mathcal{R}_{r,n}$. Then, $d_f(L) = |S(L)| \cdot n$ if and only if $r = n = 1$.*

Proof. Under our assumptions, we already consider that $1 \leq r \leq n$. So, suppose $L = (l_{i,j}) \in \mathcal{R}_{r,n}$ to be such that $n \geq 2$. According to the definition of the cocyclic degree-of-freedom of L, we have that $d_f(L) = |S(L)| \cdot n$ if and only if no constraints derive from Condition (6) apart from the trivial ones $1 = 1$ and $-1 = -1$. Keeping in mind the image $\mathbb{Z}_2$ of any cocycle over L, this is possible if and only if, for each triple $(i, j, k) \in [r] \times [r] \times [n]$, one of the following three possibilities holds.

1. $(l_{i,j}, k) = (i, j)$ and $(j, k) = (i, l_{j,k})$.
2. $(l_{i,j}, k) = (i, l_{j,k})$ and $(i, j) = (j, k)$.
3. $(l_{i,j}, k) = (j, k)$ and $(i, j) = (i, l_{j,k})$.

Observe that each one of the first two possibilities implies that $i = j = k$. Suppose, however, that $i = j < k$. This is possible because we consider $n \geq 2$. Then, the third possibility that we have just enumerated implies that $l_{i,i} = i = l_{i,k}$, which is not possible because of the Latin rectangle condition. Hence, $n = 1$ and the result holds. $\square$

Let us introduce now the notion of cocyclic matrix over a Latin rectangle $L \in \mathcal{R}_{r,n}$ in a similar way that the classical theory over finite groups does. To this end, suppose the symbols of the set $S(L)$ to be relabeled by following the natural order as $e_1, \ldots, e_{|S(L)|}$. We define the *cocyclic matrix* M_ψ of a cocycle $\psi \in \mathcal{Z}(L)$ as the $|S(L)| \times n$ array whose (i, j)th entry is $\psi(e_i, j)$, for all $i \leq |S(L)|$ and $j \leq n$. Thus, for instance, the Latin rectangles $L \in \mathcal{R}_{1,8}$ and $L' \in \mathcal{R}_{2,8}$ in Example 2 are respectively related to the cocyclic matrices

$$\begin{pmatrix} + & - & - & + & + & + & - & - \\ - & + & - & + & + & - & + & - \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} + & - & - & + & + & + & - & - \\ - & + & - & + & + & - & + & - \\ - & - & + & + & + & - & - & + \\ + & + & + & - & + & - & - & - \end{pmatrix}.$$

Observe that the respective multiplication of both matrices with their corresponding transposes are eight times the identity matrix of order 2 and 4, respectively. Keeping this fact in mind, for each Latin rectangle $L \in \mathcal{R}_{r,n}$, we define the set

$$\mathcal{H}(L) := \{M_\psi \mid \psi \in \mathcal{Z}(L) \text{ and } M_\psi M_\psi^t = nI_{|S(L)|}\}. \tag{8}$$

The following result follows straightforwardly from this definition.

Lemma 10. *Let $L \in \mathcal{R}_{r,n}$ be such that $|S(L)| = n$. Then, every matrix in $\mathcal{H}(L)$ is Hadamard.*

Under the assumptions of Lemma 10, we call any of the matrices in the set $\mathcal{H}(L)$ a *cocyclic Hadamard matrix* over the Latin rectangle L.

Example 11. The matrix

$$\begin{pmatrix} + & + & - & - \\ + & - & + & - \\ - & + & + & - \\ - & - & - & - \end{pmatrix}$$

is cocyclic Hadamard over the Latin rectangle

$$L \equiv \begin{array}{|c|c|c|c|} \hline 2 & 3 & 4 & 1 \\ \hline 3 & 4 & 1 & 2 \\ \hline \end{array} \in \mathcal{R}_{2,4}.$$

Observe, in particular, that $S(L) = \{1, 2, 3, 4\}$ and hence, $|S(L)| = 4$.

The following questions constitute a pair of open problems about the set of cocyclic Hadamard matrices over Latin rectangles that we study separately throughout the next two sections.

Problem 12. Let L be a Latin rectangle in $\mathcal{R}_{r,n}$. Does there exist a cocycle $\psi \in \mathcal{Z}(L)$ such that the cocyclic matrix M_ψ is Hadamard? Equivalently, when can we ensure that $\mathcal{H}(L) \neq \emptyset$?

Problem 13. Let M be a Hadamard matrix of order n. Does there exist a Latin rectangle $L \in \mathcal{R}_{r,n}$ such that $M \in \mathcal{H}(L)$? If so, which is the minimum positive integer $r \in \mathbb{N}$ so that the mentioned Latin rectangle exists?

Before focusing on these questions, we expose some preliminary results on the topic.

Lemma 14. *Let $L \in \mathcal{R}_{r,n}$. If $M \in \mathcal{H}(L) \neq \emptyset$, then $-M \in \mathcal{H}(L)$.*

Proof. It is enough to consider the correspondence $\alpha : \mathcal{Z}(L) \to \mathcal{Z}(L)$ that is defined so that $\alpha(\psi) = -\psi$, for all $\psi \in \mathcal{Z}(L)$. Particularly, M_ψ is Hadamard if and only if $M_{-\psi} = -M_\psi$ is. □

Let $L \in \mathcal{R}_{r,n}$. The following lemma ensures, for each $i \leq r$, the existence of r different cells within a common column of every cocyclic matrix over L, having all of them the same sign (1 or -1). The location of such cells is uniquely determined by the entries of the ith column of L.

Lemma 15. *Let $M = (m_{i,j})$ be a cocyclic matrix over a Latin rectangle $L = (l_{i,j}) \in \mathcal{R}_{r,n}$. For each row i of L, let j be the column of L in which the symbol i occurs. That is, $l_{i,j} = i$. Then, $m_{l_{1,i},j} = \cdots = m_{l_{r,i},j} = m_{i,j}$.*

Proof. Let $\psi \in \mathcal{Z}(L)$ be such that $M = M_\psi$. Let $(i, j) \in [r] \times [n]$ be such that $l_{i,j} = i$. The definition of Latin rectangle involves the existence of the column j, for all row $i \leq r$. The result holds because, from Condition (6), we have that $\psi(l_{k,i}, j) = \psi(k, i)\psi(i, j)\psi(k, l_{i,j}) = \psi(k, i)\psi(i, j)\psi(k, i)$, for all $k \leq r$, and hence, $\psi(l_{k,i}, j) = \psi(i, j)$. □

Example 16. Let us consider the Latin rectangle

$$L = (l_{i,j}) \equiv \begin{array}{|c|c|c|c|} \hline 2 & 1 & 3 & 4 \\ \hline 3 & 4 & 2 & 1 \\ \hline \end{array} \in \mathcal{R}_{2,4}.$$

Since $l_{1,2} = 1$, Lemma 15 ensures the existence of two cells within the second column of every cocyclic matrix $M = (m_{i,j})$ over L, having both of them the value $m_{1,2}$ as sign. The location of both cells (more specifically, the rows in which they appear) is uniquely determined by the entries of the first column of L, that is, by the symbols 2 and 3. In particular, it must be $m_{1,2} = m_{2,2} = m_{3,2}$.

Similarly, since $l_{2,3} = 2$ and the second column of L contains the symbols 1 and 4, Lemma 15 implies that $m_{1,3} = m_{2,3} = m_{4,3}$.

The following cocyclic matrix over L illustrates these facts.

$$\begin{pmatrix} + & - & + & - \\ - & - & + & + \\ + & - & - & + \\ + & + & + & + \end{pmatrix}$$

Now, we prove that the existence of a symbol $j \leq r$ within the jth column of a Latin rectangle $L \in \mathcal{R}_{r,n}$ involves the existence of a normalized row within every cocyclic matrix M over L. Moreover, we indicate a necessary condition on the diagonal of L and M ensuring the existence of a row within L having all its cycles of even length.

Lemma 17. *Let $L = (l_{i,j}) \in \mathcal{R}_{r,n}$ be such that there exists a pair $(e, j) \in [r] \times [r]$ satisfying that $l_{e,j} = j$. Let $M = (m_{i,j})$ be a cocyclic matrix over L. Then,*

(a) $m_{e,k} = m_{e,j}$, for all $k \leq n$.
(b) If there exists a positive integer $i \in [r] \setminus \{e\}$ such that $l_{i,i} = e$ and $m_{i,i} = -m_{e,1}$, then all the cycles in the ith row of L have even length.

Proof. Let $\psi \in \mathcal{Z}(L)$ be such that $M = M_\psi$. Let $k \leq n$. From the definition of Latin rectangle, there exists $m \leq n$ such that $l_{j,m} = k$. Then, from Condition (6), we have that $\psi(j, m) = \psi(e, j)\psi(j, m)\psi(e, k)$ and hence, $\psi(e, k) = \psi(e, j)$.

Now, in order to prove the second assertion, observe that Condition (6), together with assertion (a), involves that $\psi(i, l_{i,k}) = \psi(e, 1)\ \psi(i, i)\psi(i, k)$, for all $k \leq n$. Thus, if $\psi(i, i) = -\psi(e, 1)$, then $\psi(i, l_{i,k}) = -\psi(i, k)$, for all $k \leq n$. As a consequence, every cycle of odd length in the ith row of L would give rise to a cycle of odd length with alternating signs (1 and -1) in the ith row of the cocyclic matrix M_ψ, which is not possible. □

Example 18. Let us consider the Latin rectangle

$$L = (l_{i,j}) \equiv \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 3 & 1 & 4 & 2 \\ \hline \end{array} \in \mathcal{R}_{2,4}.$$

Since the first column of L contains the symbol 1, Lemma 17.a ensures that every cocyclic matrix M over L has a first row of $1's$ or -$1's$. The following cocyclic matrix over L illustrates this fact.

$$M = (m_{i,j}) \equiv \begin{pmatrix} + & + & + & + \\ + & - & - & + \\ + & - & - & + \\ - & + & + & + \end{pmatrix}$$

Since $l_{2,2} = 1$ and $m_{2,2} = -m_{1,1}$, Lemma 17.b implies that all the cycles in the second row of L should have even length. In fact, this is true, because such a row determines the cycle (1342).

We focus now on the relationship among the entry set, the set of cocycles and the set of cocyclic Hadamard matrices that are associated to two given Latin rectangles. We deal in particular with the case in which both Latin rectangles are isotopic or transpose of each other.

Lemma 19. *Let $L \in \mathcal{R}_{r,n}$ and $L' \in \mathcal{R}_{s,n}$ be such that $r \leq s \leq n$.*

(a) If $E(L) \subseteq E(L')$, then

  i. $S(L) \subseteq S(L')$.
  ii. If $S(L) = S(L')$, then $\mathcal{Z}(L') \subseteq \mathcal{Z}(L)$ and $\mathcal{H}(L') \subseteq \mathcal{H}(L)$.

(b) If $r = s$ and L is isotopic to L' by means of an isotopism $(f, g, g) \in S_r \times S_n \times S_n$ such that $f(i) = g(i)$, for all $i \leq r$, then there exists a 1-1 correspondence between the sets $\mathcal{Z}(L)$ and $\mathcal{Z}(L')$.
(c) If $r = s = n$ and L and L' are two Latin squares that are transpose of each other, then there exists a 1-1 correspondence between the sets $\mathcal{Z}(L)$ and $\mathcal{Z}(L')$.

Proof. The first assertion follows straightforwardly from (5) and (6). Now, in order to prove the second assertion, it is enough to define the correspondence $\alpha : \mathcal{Z}(L) \to \mathcal{Z}(L')$ such that $\alpha(\psi)(i, j) = \psi(g^{-1}(i), g^{-1}(j))$, for all $\psi \in \mathcal{Z}(L)$ and $i, j \in [n]$. The result follows straightforwardly from (6) and the fact that g is a relabeling of the symbols of the sets $[r]$ and $\{r + 1, \ldots, n\}$, in both cases by means of the same set of symbols under consideration.

Table 1
Cocyclic Hadamard matrices over $r \times n$ Latin rectangles, with $r \leq n \leq 4$.

r	n	L	$d_f(L)$	Cocyclic Hadamard matrices in $\mathcal{H}(L)$			
1	1	1	1	+			
	2	21	2	+−	−−		
2	2	12 21	2	++	+−		
	4	1243 3421	5	++++	+−−+	+−+−	++−−
				++++	−+−+	+−−+	++−−
				++++	−++−	+−+−	−−++
				++++	+−+−	+−−+	−−++
		2341 3412	4	++−−	+−+−	−++−	−−−−
				+−−+	−+−+	−−++	++++
		2134 3421	2	+−+−	−−++	+−−+	++++
		2314 3421	3	+−−+	−+−+	−−−−	++−−
		2314 4123	2	+−+−	−−++	−++−	++++
3	4	1234 2143 3412	5	++++	++−−	+−−+	+−+−
				++++	++−−	+−+−	+−−+
				++++	+−+−	+−−+	++−−
				++++	+−+−	++−−	+−−+
				++++	+−−+	+−+−	++−−
				++++	+−−+	++−−	+−+−
		1234 2143 3421	4	++++	++−−	+−−+	+−+−
				++++	++−−	+−+−	+−−+
		2341 3412 4123	4	++−−	+−+−	−++−	−−−−
				+−−+	−+−+	−−++	++++
4	4	1234 2143 3412 4321	5	++++	++−−	+−−+	+−+−
				++++	++−−	+−+−	+−−+
				++++	+−+−	+−−+	++−−
				++++	+−+−	++−−	+−−+
				++++	+−−+	+−+−	++−−
				++++	+−−+	++−−	+−+−
		1234 2143 3421 4312	4	++++	++−−	+−−+	+−+−
				++++	++−−	+−+−	+−−+

Finally, in order to prove the third assertion, it is enough to define the correspondence $\alpha : \mathcal{Z}(L) \rightarrow \mathcal{Z}(L')$ such that $\alpha(\psi)(i,j) = \psi(j,i)$, for all $\psi \in \mathcal{Z}(L)$ and $i, j \leq n$. To see it, suppose L and L' to be the respective multiplication tables of two quasigroups $Q = ([n], \cdot)$ and $Q' = ([n], \circ)$. Then, from (7), we have that $\alpha(\psi)(i \circ j, k)\alpha(\psi)(i,j)\alpha(\psi)(j,k)\alpha(\psi)(i, j \circ k) = \psi(k, j \cdot i)\psi(j,i)\psi(k,j)\psi(k \cdot j, i) = 1$. $\square$

In case of being $r = s = n$, the isotopism (f, g, g) that is described in Lemma 19.b constitutes an isomorphism of Latin squares. In this regard, from here on and by an abuse of notation, we call such an isotopism an *isomorphism* of $r \times n$ Latin rectangles and we say that any two Latin rectangles related by any such an isotopism are *isomorphic*. This constitutes an equivalence relation among $r \times n$ Latin rectangles. Thus, for instance, the Latin rectangle

3	4	1	2
4	1	2	3

is isomorphic to the Latin rectangle $L \in \mathcal{R}_{2,4}$ in Example 11 by means of the isomorphism $((12), (12)(34), (12)(34)) \in S_2 \times S_4 \times S_4$. The following result holds straightforwardly from Lemma 19.

Proposition 20. *Let $L \in \mathcal{R}_{r,n}$ and $L' \in \mathcal{R}_{s,n}$ be such that $r \leq s \leq n$. If $r = s$ and L is isomorphic to L', then there exists a* 1-1 *correspondence between the sets $\mathcal{H}(L)$ and $\mathcal{H}(L')$.*

From Proposition 20, the distribution of $r \times n$ Latin rectangles into isomorphism classes constitutes a first approximation to determine the set of cocyclic Hadamard matrices over the set $\mathcal{R}_{r,n}$. By means of an exhaustive search, Table 1 illustrates the case $r \leq n \leq 4$, where both Latin rectangles and Hadamard matrices are written row after row in a single line. Besides, keeping in mind Lemma 14, we only enumerate those cocyclic Hadamard matrices having its first entry equal to 1. We also indicate the cocyclic degree-of-freedom of each Latin rectangle under consideration.

Let us finish this section with a pair of results that enable us to deal with cocyclic Hadamard matrices over Latin rectangles of higher orders.

Proposition 21. *Let r and t be two positive integers such that $r \geq 2t+1$ and let L be the Latin rectangle in $\mathcal{R}_{r,4t}$ that is formed by the first r rows of the dihedral group D_{4t}. Then, $\mathcal{H}(D_{4t}) \subseteq \mathcal{H}(L)$.*

Proof. Let a and b be the generators of the dihedral group D_{4t} according to (2). Suppose that the ordered set $\{1, a, a^2, \ldots, a^{2t-1}, b, ab, a^2b, \ldots, a^{2t-1}b\}$, which is formed by the elements of D_{4t}, determines the ordering under which the rows and columns of its multiplication table are indexed. The result follows from Lemma 19.a once we observe that the upper left square array of order $2t$ within such a multiplication table is formed by the $2t$ elements of the subset $\{1, a, a^2, \ldots, a^{2t-1}\}$, whereas the first $2t$ elements of its $(2t+1)$th row coincide with those of the subset $\{b, ab, a^2b, \ldots, a^{2t-1}b\}$. $\square$

Lemma 22. *Let $L \in \mathcal{R}_{r,n}$ be a reduced Latin rectangle and let $L' \in \mathcal{R}_{n',n'}$ be a reduced Latin square. Let $\psi \in \mathcal{H}(L)$ and $\psi' \in \mathcal{H}(L')$. Then, the map $\psi \otimes \psi'$ defined as in (3) is a cocycle over the reduced Latin rectangle $L \otimes L' \in \mathcal{R}_{rn',nn'}$. Moreover, if the cocyclic matrices M_ψ and $M_{\psi'}$ are Hadamard, then the cocyclic matrix $M_{\psi\otimes\psi'}$ is also Hadamard.*

Proof. According to the definition exposed in the preliminary section, it is straightforwardly verified that the direct product $L \otimes L'$ is an $rn' \times nn'$ reduced Latin rectangle. Further, the matrix $M_{\psi\otimes\psi'} = M_\psi \otimes M_{\psi'}$ is well-defined, because of the mentioned definition and the fact that $|S(L \otimes L')| = |S(L)| \cdot |S(L')|$. Then, the result follows similarly to the known case of cocyclic (Hadamard) matrices over groups, for which associativity does not play any role in the corresponding proof. $\square$

Example 23. Let us consider the reduced Latin rectangles

$$L = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 1 & 4 & 3 \\ \hline 3 & 4 & 1 & 2 \\ \hline \end{array} \qquad \text{and} \qquad L' = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array}.$$

Then, Table 1 and Lemma 22 involves that the matrix

$$\begin{pmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \end{pmatrix}$$

is cocyclic Hadamard over the reduced Latin rectangle

$$L \otimes L' = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\ \hline 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \\ \hline 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\ \hline 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\ \hline 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \\ \hline \end{array}.$$

The following example shows that the assumption of considering reduced Latin rectangles in Lemma 22 is necessary to get a well-defined cocyclic matrix over the direct product $L \otimes L'$.

Example 24. Let $L \in \mathcal{R}_{1,2}$ and $L' \in \mathcal{R}_{2,3}$ be the Latin rectangles in Example 1. The following two matrices are cocyclic over L and L', respectively.

$$M \equiv \begin{pmatrix} + & + \\ + & - \end{pmatrix} \qquad\qquad M' \equiv \begin{pmatrix} + & + & + \\ + & + & - \\ + & - & - \end{pmatrix}$$

Nevertheless, the Kronecker product $M \otimes M'$ is not cocyclic over $L \otimes L'$. To see it, observe that, even if $|S(L)| = 2$ and $|S(L')| = 3$, we have that $|S(L \otimes L')| = 5$. More specifically, $S(L) = \{1, 2\}$ and $S(L') = \{1, 2, 3\}$, but $S(L \otimes L') = \{1, 2, 4, 5, 6\}$.

4. Hadamard matrices that are cocyclic over a Latin rectangle

This section deals with Problem 12. Firstly, we establish a relation among the number of rows and columns of any Latin rectangle over which a cocycle exists.

Lemma 25. *Let $L \in \mathcal{R}_{r,n}$. If $\mathcal{H}(L) \neq \emptyset$, then $r \leq n \leq r + r^2$.*

Proof. Under our assumptions, $r \leq n$ in any $r \times n$ Latin rectangle. The result follows because, from (5), the number of rows of any cocyclic matrix over L is $|S(L)| \leq r + r^2$, and $\mathcal{H}(L) \neq \emptyset$ only if $|S(L)| = n$. □

Table 1 shows that both bounds in Lemma 25 are achieved when $r = 1$. Even the lower bound is tight, for instance, for any cocyclic Hadamard matrix over a finite group, further research is necessary to determine whether the upper bound is tight for some $r > 1$.

Now, we focus on those Latin rectangles $L \in \mathcal{R}_{r,n}$ that have at least one triple of the form (i, j, j) in their set of entries $E(L)$.

Proposition 26. *Let $L = (l_{i,j}) \in \mathcal{R}_{r,n}$ and $\psi \in \mathcal{Z}(L)$ be such that $M_\psi \in \mathcal{H}(L)$. If there exist two positive integers $e, j \leq r$ such that $l_{e,j} = j$, then*

- *(a) $l_{e,i} = i$, for all $i \leq r$.*
- *(b) Let $i \leq r$ be such that $i \neq e$. Then, $l_{i,k} \neq k$, for all $k \leq r$.*
- *(c) If the cardinality of the set $\{k \leq n \mid l_{e,k} = k\}$ is greater than $n/2$, then $l_{i,e} = i$, for all $i \leq r$. In such a case, $\psi(i, e) = \psi(e, k)$, for all $i \leq r$ and $k \leq n$.*
- *(d) If there exists a positive integer $i \in [r] \setminus \{e\}$ such that $l_{i,i} = e$ and $\psi(i, i) = \psi(e, 1)$, then the cycles of the ith row of L can be partitioned into two sets of cycles, for which the total sum of their respective lengths is $n/2$.*

Proof. Let us prove each assertion separately.

- (a) Let $i \leq r$. From Condition (6), we have that $\psi(l_{e,i}, k) = \psi(e, i)\psi(i, k)\psi(e, l_{i,k})$, for all $k \leq n$. Then, Lemma 17.a implies that $\psi(e, i) = \psi(e, l_{i,k})$ and hence, $\psi(l_{e,i}, k) = \psi(i, k)$, for all $k \leq n$. Since the cocyclic matrix M_ψ is Hadamard, this is possible only if $l_{e,i} = i$.
- (b) This follows from (a) and the fact that L is a Latin rectangle.
- (c) Let $i \leq r$. Since Lemma 17.a involves that $\psi(e, k) = \psi(e, j)$, for all $k \leq n$, we have that the sign of $\psi(i, e)\psi(e, k)$ coincides, for all $k \leq n$. Then, Condition (6) implies that the value of $\psi(l_{i,e}, k)\psi(i, l_{e,k})$ also coincides, for all $k \leq n$. In particular, from the hypothesis, the value of $\psi(l_{i,e}, k)\psi(i, k)$ coincides for more than $n/2$ values of k. Since the cocyclic matrix M_ψ is Hadamard, this is possible only if $l_{i,e} = i$. The last assertion follows from Lemmas 15 and 17.a, and the fact that $e \leq r$.
- (d) From Condition (6) and Lemma 17.a, we have that $\psi(e, 1) = \psi(i, i)\psi(i, k)\psi(i, l_{i,k})$, for all $k \leq n$. Thus, if $\psi(i, i) = \psi(e, 1)$, then $\psi(i, l_{i,k}) = \psi(i, k)$, for all $k \leq n$. This implies the coincidence of signs of all those entries of each set of cells in the ith row of the cocyclic matrix M_ψ corresponding to the cells of each cycle in the ith row of L. Hence, the non-existence of both sets of cycles indicated in the assertion would contradict the fact that M_ψ is a Hadamard matrix, because its ith and eth rows would not be orthogonal. □

Example 27. According to Proposition 26, no cocyclic Hadamard matrix exists over any of the following Latin rectangles in $\mathcal{R}_{3,8}$.

$L_1 \equiv$

1	3	5	4	7	8	2	6
3	5	8	2	1	6	7	4
4	7	6	1	8	3	2	5

$L_2 \equiv$

1	2	3	4	5	8	7	6
3	5	8	2	1	6	4	7
4	7	6	1	8	3	2	5

$L_3 \equiv$

2	1	4	5	3	7	8	6
1	2	7	4	5	6	3	8
5	6	8	2	1	3	4	7

Specifically, Proposition 26.a implies that $\mathcal{H}(L_1) = \emptyset$, because the set of entries $E(L_1)$ contains the triple $(1, 4, 4)$, but it does not contain, for instance, the triple $(1, 2, 2)$. Further, Proposition 26.c involves $\mathcal{H}(L_2) = \emptyset$, because the set of entries $E(L_2)$ contains five triples of the form $(1, i, i)$, but it does not contain, for instance, the triple $(2, 1, 2)$. Finally, suppose the existence of a cocycle $\psi \in \mathcal{H}(L_3)$. Lemma 17.b implies that $\psi(1, 1) = \psi(2, 1)$, because $\{(1, 1, 2), (2, 1, 2)\} \subset E(L_3)$ and the first row of L_3 constitutes the permutation $\pi = (21)(345)(678) \in S_8$, with two cycles of odd length. Nevertheless, this contradicts Proposition 26.d, because it is not possible to distribute the three cycles of π into two sets with total sum of their lengths equal to four. As a consequence, $\mathcal{H}(L_3) = \emptyset$. Further, in order to illustrate Proposition 26 in an affirmative sense, observe that the reduced Latin rectangle

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	5	6	7	8	1	2
4	3	7	8	1	2	6	5

is related to both cocyclic Hadamard matrices

$$\begin{pmatrix} + & + & + & + & + & + & + & + \\ + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - \\ + & + & + & + & - & - & - & - \\ + & + & - & - & + & + & - & - \\ + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & - & - & + & + \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} + & + & + & + & + & + & + & + \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & - & + & - & + & + & - \\ + & + & - & - & + & + & - & - \\ + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & - & - & + & + \end{pmatrix}.$$

Theorem 28. *Let $L \in \mathcal{R}_{n,n}$ be such that $\mathcal{H}(L) \neq \emptyset$. Then, L constitutes the multiplication table of a loop. Further, if $\psi \in \mathcal{H}(L)$ and e is the unit element of that loop, then $\psi(e, i) = \psi(j, e)$, for all $i, j \leq n$.*

Proof. Suppose $L = (l_{i,j})$. Since L is a Latin square, there exists a positive integer e such that $l_{e,1} = 1$. Proposition 26.a involves then that all the entries of the eth row of L appear in natural order and hence, Proposition 26.c enables us to ensure that the same occurs for all the entries of its eth column. Hence, L constitutes the multiplication table of a loop with unit element e. The last assertion also follows from Proposition 26.c. $\square$

The following result constitutes a generalization of the known *cocyclic Hadamard test* [17] for cocycles over finite groups.

Theorem 29. *Let L be a Latin square of order n that constitutes the multiplication table of a loop with unit element e and let $\psi \in \mathcal{Z}(L)$. Then, the cocyclic matrix M_ψ is Hadamard if and only if the following condition holds.*

$$\sum_{j \le n} \psi(i,j) = 0, \text{ for all } i \in [n] \setminus \{e\}.$$

Proof. The result follows straightforwardly from Theorem 28 and the fact that the ith and eth rows of L are orthogonal, for all $i \in [n] \setminus \{e\}$. □

Since every loop is isomorphic to a quasigroup having a reduced Latin square as its multiplication table, Proposition 21 and Theorem 28 enable us to focus on the study of cocyclic Hadamard matrices over the set $\mathcal{L}_{4t}$ of loops of order $4t$ having 1 as unit element. Moreover, since isomorphisms preserve associativity and the associative case corresponds to the classical study of cocyclic Hadamard matrices over groups, it is enough to focus on the isomorphism classes of non-associative loops. For $t = 1$, the set $\mathcal{L}_4$ is distributed into two isomorphism classes, which correspond to the groups $\mathbb{Z}_4$ and $\mathbb{Z}_2^2$. Both of them give rise to cocyclic Hadamard matrices (see Table 1). For $t = 2$, the set $\mathcal{L}_8$ is distributed into 106228849 isomorphism classes [18]. Only five of these classes constitute groups, from which only the elementary abelian group $\mathbb{Z}_2^3$, the abelian group $\mathbb{Z}_4 \times \mathbb{Z}_2$ and the dihedral group D_8 give rise to cocyclic Hadamard matrices. With respect to the rest of non-associative loops, an exhaustive search based on the previously exposed results enables us to ensure the following result.

Theorem 30. *There exists only one isomorphism class in the set $\mathcal{L}_8$ associated to non-associative loops over which a cocyclic Hadamard matrix exists. This class is represented by the monogenic loop whose multiplication table is*

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5	6	8	7	3	4	2	1
6	5	7	8	4	3	1	2
7	8	6	5	1	2	4	3
8	7	5	6	2	1	3	4

.

In order to see that the previous loop is non-associative, observe, for instance, that $(5 \cdot 5) \cdot 5 = 7 \neq 8 = 5 \cdot (5 \cdot 5)$. Besides, it is monogenic by means of any of the elements of the subset $\{5, 6, 7, 8\}$. A simple computation determines the four normalized cocyclic Hadamard matrices that are associated to this loop:

$$\begin{pmatrix}
+ & + & + & + & + & + & + & + \\
+ & + & + & + & - & - & - & - \\
+ & + & - & - & - & - & + & + \\
+ & + & - & - & + & + & - & - \\
+ & - & - & + & - & + & + & - \\
+ & - & - & + & + & - & - & + \\
+ & - & + & - & - & + & - & + \\
+ & - & + & - & + & - & + & -
\end{pmatrix}
\quad
\begin{pmatrix}
+ & + & + & + & + & + & + & + \\
+ & + & + & + & - & - & - & - \\
+ & + & - & - & - & - & + & + \\
+ & + & - & - & + & + & - & - \\
+ & - & - & + & + & - & - & + \\
+ & - & - & + & - & + & + & - \\
+ & - & + & - & + & - & + & - \\
+ & - & + & - & - & + & - & +
\end{pmatrix}$$

$$\begin{pmatrix}
+ & + & + & + & + & + & + & + \\
+ & + & + & + & - & - & - & - \\
+ & + & - & - & + & + & - & - \\
+ & + & - & - & - & - & + & + \\
+ & - & + & - & - & + & - & + \\
+ & - & + & - & + & - & + & - \\
+ & - & - & + & + & - & - & + \\
+ & - & - & + & - & + & + & -
\end{pmatrix}
\quad
\begin{pmatrix}
+ & + & + & + & + & + & + & + \\
+ & + & + & + & - & - & - & - \\
+ & + & - & - & + & + & - & - \\
+ & + & - & - & - & - & + & + \\
+ & - & + & - & + & - & + & - \\
+ & - & + & - & - & + & - & + \\
+ & - & - & + & - & + & + & - \\
+ & - & - & + & + & - & - & +
\end{pmatrix}$$

Table 2
$r \times n$ Latin rectangles related to each equivalence class of Hadamard matrices of order n.

n	Hadamard matrix M of order n	r	$L \in \mathcal{R}_{r,n}$ such that $M \in \mathcal{H}(L)$
1	1	1	1
2	2 0	1	21
		2	21 12
4	F 9 A C	2	1243 3421
		3	1234 2143 3421
		4	1234 2143 3421 4312
8	FF $F0$ $C3$ CC 96 99 $A5$ AA	3	12347856 25831674 47628315
		4	12345687 23851476 37526148 45187263
		5	12345678 21436587 34128765 43217856 56782143
		6	12345678 21436587 34128765 43217856 56782143 65871234
		7	12345678 21436587 34128765 43217856 56782143 65871234 78563412
		8	12345678 21436587 34128765 43217856 56782143 65871234 78563412 87654321

Theorem 30 also enables us to ensure that the Circulant Hadamard Conjecture, which was recalled in the preliminary section, cannot be generalized as such from cyclic groups to monogenic loops. Unlike the classical theory on finite groups, this fact encourages the search of cocyclic Hadamard matrices over monogenic non-associative loops.

The following results show how Theorem 30, together with Lemma 22, also ensures the existence of cocyclic Hadamard matrices over new examples of non-associative loops.

Theorem 31. *Let $G \in \mathcal{L}_n$ be a loop over which a cocyclic Hadamard matrix exists and let $Q_0 \in \mathcal{L}_8$ be the non-associative loop referred in Theorem 30. For each positive integer $t > 0$, the direct product $Q_t = G \otimes Q_{t-1}$ is a non-associative loop over which a cocyclic Hadamard matrix exists.*

Proof. According to Lemma 22, we have that, for each positive integer $t > 0$, the direct product of both reduced Latin squares associated to G and Q_{t-1} is a reduced Latin square of order 2^{t+3}, and hence, Q_t is a loop having 1 as unit element. Moreover, the upper left square array of order 8 of such a loop always coincides with Q_0, whatever the order t is, and thus, the same remark exposed just after Theorem 30 is enough to ensure the non-associativity of Q_t. Now, Table 2 and Theorem 30 ensure, respectively, the existence of a cocyclic Hadamard matrix over G and Q_0. Let ψ and ψ_0 be the respective cocycles associated to such matrices. In order to prove the existence of a cocyclic Hadamard matrix over Q_t, for each $t > 0$, it is enough to consider the cocycle $\psi_t = \psi \otimes \psi_{t-1}$ defined as in (3). Remark in this regard that associativity is not required for ψ_t to be a cocycle over Q_t. $\square$

We illustrate the previous theorem with the example developed in the following corollary.

Corollary 32. *For each positive integer $t > 0$, there exists a non-associative loop in the set $\mathcal{L}_{2^{t+3}}$ over which a cocyclic Hadamard matrix exists.*

Proof. The result is an immediate consequence of Theorem 31 once we observe that, according to Table 1, there exists a cocyclic Hadamard matrix over the cyclic group having as multiplication table the Latin square

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array}. \quad \square$$

Observe in Table 1 that a possibility for the cocyclic Hadamard matrix that is mentioned in the proof of Corollary 32 is the matrix

$$\begin{pmatrix} + & + \\ + & - \end{pmatrix}.$$

As such, the cocyclic Hadamard matrix over the non-associative loop in $\mathcal{L}_{2^{t+3}}$ can be chosen so that it constitutes indeed a *Sylvester Hadamard matrix* [28] derived from a cocyclic Hadamard matrix over the non-associative loop referred in Theorem 30.

Let us finish this section with a pair of results dealing with some necessary conditions under which two Hadamard matrices, which are equal up to negation of a subset of rows or a subset of columns, are cocyclic over the same Latin rectangle.

Lemma 33. *Let $L = (l_{i,j}) \in \mathcal{R}_{r,n}$ and $M = (m_{i,j}) \in \mathcal{H}(L)$. Let $\mathcal{N} \subseteq [n]$ be such that $\mathcal{N} \neq \emptyset$. We define the Hadamard matrix $M_{\mathcal{N}} = (m^{\mathcal{N}}_{i,j})$ so that, for each $i, j \leq n$, we have that $m^{\mathcal{N}}_{i,j} = -m_{i,j}$, if $i \in \mathcal{N}$, and $m^{\mathcal{N}}_{i,j} = m_{i,j}$, otherwise. If $M_{\mathcal{N}} \in \mathcal{H}(L)$, then the following results hold.*

(a) $\mathcal{N} \cap [r] \neq \emptyset$.
(b) If $j \in \mathcal{N} \cap [r]$, then $l_{i,j} \in \mathcal{N}$, for all $i \leq r$.
(c) If $i \in \mathcal{N} \cap [r]$, then $j \in \mathcal{N} \cap [r]$ if and only if $l_{i,j} \in \mathcal{N}$.

Proof. Let $\psi, \psi' \in \mathcal{Z}(L)$ be such that $M_\psi = M$ and $M_{\psi'} = M_{\mathcal{N}}$. We prove each assertion separately.

(a) Suppose that $\mathcal{N} \cap [r] = \emptyset$. Since $\mathcal{N} \neq \emptyset$, there exists a pair or positive integers $i, j \leq r$ such that $l_{i,j} \in \mathcal{N}$. In particular, $i, j \notin \mathcal{N}$. Then, for any given $k \leq n$, Condition (6) implies that

$$\psi'(i,j) = \psi(i,j) = \psi(l_{i,j},k)\psi(j,k)\psi(i,l_{j,k}) = -\psi'(l_{i,j},k)\psi'(j,k)\psi'(i,l_{j,k}) = -\psi'(i,j),$$

which is a contradiction.

(b) Let $j \in \mathcal{N} \cap [r]$ and let us consider two positive integers $i \leq r$ and $k \leq n$. Since $\psi'(i,j)\psi'(i,l_{j,k}) = \psi(i,j)\psi(i,l_{j,k})$ regardless of whether i belongs to the set $\mathcal{N}$ or not, Condition (6) implies that

$$\psi'(l_{i,j},k) = \psi'(i,j)\psi'(j,k)\psi'(i,l_{j,k}) = -\psi(i,j)\psi(j,k)\psi(i,l_{j,k}) = -\psi(l_{i,j},k),$$

and hence, $l_{i,j} \in \mathcal{N}$.

(c) Let $i \in \mathcal{N} \cap [r]$ and let us consider two positive integers $j \leq r$ and $k \leq n$. Then, Condition (6) implies that

$$\begin{aligned}\psi'(j,k) &= \psi'(l_{i,j},k)\psi'(i,j)\psi'(i,l_{j,k})\\ &= \psi'(l_{i,j},k)\psi(i,j)\psi(i,l_{j,k})\\ &= \psi'(l_{i,j},k)\psi(l_{i,j},k)\psi(j,k).\end{aligned}$$

Hence, $\psi'(j,k) = \psi(j,k)$ if and only if $\psi'(l_{i,j},k) = \psi(l_{i,j},k)$, and the result holds. □

Lemma 34. *Let $L = (l_{i,j}) \in \mathcal{R}_{r,n}$ and $M = (m_{i,j}) \in \mathcal{H}(L)$. Let $\mathcal{N} \subseteq [n]$ be such that $\mathcal{N} \neq \emptyset$. We define the Hadamard matrix $M_{\mathcal{N}} = (m^{\mathcal{N}}_{i,j})$ so that, for each $i, j \leq n$, we have that $m^{\mathcal{N}}_{i,j} = -m_{i,j}$, if $j \in \mathcal{N}$, and $m^{\mathcal{N}}_{i,j} = m_{i,j}$, otherwise. If $M_{\mathcal{N}} \in \mathcal{H}(L)$, then the following results hold.*

(a) If $\mathcal{N} \cap [r] \neq \emptyset$, then $\mathcal{N} = [n]$.
(b) If $j \in \mathcal{N}$, then $i \in \mathcal{N} \cap [r]$ if and only if $l_{i,j} \in \mathcal{N}$.

Proof. The proof of the second assertion follows similarly to that one of Lemma 33.c. Now, in order to prove the first statement, let $\psi, \psi' \in \mathcal{Z}(L)$ be such that $M_\psi = M$ and $M_{\psi'} = M_{\mathcal{N}}$. Let $j \in \mathcal{N} \cap [r]$ and let us consider two positive integers $i \leq r$ and $k \leq n$. Since $\psi'(l_{i,j},k)\psi'(j,k) = \psi(l_{i,j},k)\psi(j,k)$ regardless of whether k belongs to the set $\mathcal{N}$ or not, Condition (6) implies that

$$\psi'(i,l_{j,k}) = \psi'(l_{i,j},k)\psi'(i,j)\psi'(j,k) = -\psi(l_{i,j},k)\psi(i,j)\psi(j,k) = -\psi(i,l_{j,k}).$$

As a consequence, $l_{j,k} \in \mathcal{N}$, for all $k \leq n$. The result holds because every row of L contains all the symbols in the set $[n]$. □

Example 35. Let us consider the Latin rectangle

$$L \equiv \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 3\\ \hline 3 & 4 & 2 & 1\\ \hline\end{array} \in \mathcal{R}_{2,4}$$

over which the Hadamard matrix

$$H \equiv \begin{pmatrix} + & + & + & + \\ + & - & - & + \\ + & - & + & - \\ + & + & - & - \end{pmatrix}$$

is cocyclic. In what follows, we study the negation of either a proper subset of rows or a proper subset of columns within H in order to determine the existence of new Hadamard matrices that are also cocyclic over L.

Firstly, we study the possible negation of rows. According to Lemma 33.a, the negation of the third or fourth row of H requires also the negation of its first or second row. Further, Lemma 33.b implies that the negation of the first (respectively, second) row of H requires the negation of its third (respectively, fourth) row. Both possibilities are valid in this case. That is, both Hadamard matrices

$$\begin{pmatrix} - & - & - & - \\ + & - & - & + \\ - & + & - & + \\ + & + & - & - \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} + & + & + & + \\ - & + & + & - \\ + & - & + & - \\ - & - & + & + \end{pmatrix}.$$

are cocyclic over L.

Now, we analyze the possible negation of columns. From Lemma 34.a, the negation of the first or second columns of H involves the negation of all its columns. Now, from Lemma 34.b, the negation of both third and fourth columns of H implies the negation of the first column. This is because $l_{1,3} = 4$, or also, because $l_{1,4} = 3$. Then, again from Lemma 34.a, all the columns should be negated. As a consequence, the only candidates for proper subsets of columns in H to be negated are $\mathcal{N} = \{3\}$ and $\mathcal{N} = \{4\}$. In both cases, however, it can be checked that the resulting Hadamard matrix is not cocyclic over L.

5. Latin rectangles over which a Hadamard matrix is cocyclic

This section deals with both questions exposed in Problem 13. With respect to the second question, observe that Lemma 25 enables us to ensure that the minimum possible positive integer $r \in \mathbb{N}$ to which that question referred must be such that $r \leq n \leq r+r^2$. This minimum is reached, for instance, for the following Hadamard matrix of order 8

$$\begin{pmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & - & - & + & + \\ + & + & - & - & + & + & - & - \\ + & - & - & + & - & + & + & - \\ + & - & - & + & + & - & - & + \\ + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \end{pmatrix}$$

which is cocyclic over the following two 3×8 Latin rectangles

$L_1 \equiv$

1	2	3	4	7	8	5	6
2	5	8	3	1	6	7	4
4	7	6	2	8	3	1	5

and

$L_2 \equiv$

1	2	3	4	7	8	5	6
2	5	8	3	1	6	7	4
4	7	6	1	8	3	2	5

These two Latin rectangles constitute indeed the only two distinct isomorphism classes of the set $\mathcal{R}_{3,8}$ that give rise to such a cocyclic matrix. Observe that both of them contain the eight symbols in their first three columns.

Let us expose some preliminary results dealing with the first question in Problem 13. Firstly, we establish how a normalized column within a cocyclic Hadamard matrix over a Latin rectangle connects signs of pair of entries within certain rows.

Lemma 36. *Let $M = (m_{i,j})$ be a cocyclic Hadamard matrix of order n over a Latin rectangle $L = (l_{i,j}) \in \mathcal{R}_{r,n}$ such that M has a normalized column e. That is, $m_{i,e} = m_{1,e}$, for all $i \leq n$. Then, for each row i in L, we have that $m_{i,j} = m_{i,l_{j,e}}$, for all $j \leq r$.*

Proof. Let $\psi \in \mathcal{Z}(L)$ be such that $M_\psi = M$. The result holds because, from Condition (6), we have that $\psi(l_{i,j}, e) = \psi(i,j)\psi(j,e)\psi(i, l_{j,e})$, for all $i, j \leq r$, and hence, $\psi(i,j) = \psi(i, l_{j,e})$. □

The following result characterizes the existence of a normalized row $e \leq r$ within a cocyclic Hadamard matrix over an $r \times n$ Latin rectangle.

Proposition 37. *A cocyclic Hadamard matrix over a Latin rectangle $L = (l_{i,j}) \in \mathcal{R}_{r,n}$ has a normalized row $e \leq r$ if and only if $l_{ej} = j$, for all $j \leq r$.*

Proof. The sufficiency holds from Lemma 17.a. In order to prove the necessary condition, let us suppose the existence of a positive integer $j \leq r$ such that $l_{e,j} \neq j$. Let $\psi \in \mathcal{Z}(L)$. Then, from Condition (6), we have that $\psi(l_{e,j}, k) = \psi(e,j)\psi(j,k)\psi(e, l_{j,k}) = \psi(j,k)$, for all $k \leq n$. This would imply the existence of two distinct rows in M with all their respective entries being equal, which contradicts the fact that M is a Hadamard matrix. □

Example 38. In order to illustrate the previous results, observe, for instance, that any Latin rectangle $L = (l_{i,j}) \in \mathcal{R}_{r,n}$, with $r \geq 3$, that is related to the Hadamard matrix indicated at the beginning of this section, satisfies that $l_{2,1} = 2$. To see it, let $M = (m_{i,j})$ denote that matrix. From Lemma 36, we have that $m_{2,2} = m_{2,l_{2,1}}$ and $m_{3,2} = m_{3,l_{2,1}}$. The former is possible only if $l_{2,1} \notin \{5,6,7,8\}$, whereas the latter is possible only if $l_{2,1} \notin \{3,4,5,6\}$. Hence, $l_{2,1} \in \{1,2\}$. But, from Proposition 37, it must be $l_{1,1} = 1$ and thus, since L is a Latin rectangle, it must be $l_{2,1} = 2$. A similar reasoning involves that $l_{3,1} \in \{3,4\}$.

The following example illustrates how Lemma 36 and Proposition 37 also enable us to ensure that not every Hadamard matrix of order n is cocyclic over an $r \times n$ Latin rectangle.

Example 39. Let us consider the Hadamard matrix

$$M = (m_{i,j}) = \begin{pmatrix} + & + & + & + \\ - & - & + & + \\ - & + & + & - \\ + & - & + & - \end{pmatrix}$$

Suppose $L = (l_{i,j}) \in \mathcal{R}_{r,4}$ to be such that $M \in \mathcal{H}(L)$. In particular, Lemma 25 involves that $2 \leq r \leq 4$. Now, from Proposition 37, it must be $l_{1,1} = 1$ and $l_{1,2} = 2$. As a consequence, $l_{1,3} \in \{3,4\}$ and hence, $m_{2,l_{1,3}} = 1$. Nevertheless, from Lemma 36, it should be $m_{2,l_{1,3}} = m_{2,1} = -1$. Therefore, there does not exist such a Latin rectangle L.

The just exposed examples suggest the question of which entries never appear in the possible related Latin rectangles over which a Hadamard matrix is cocyclic. In this regard, the following result holds.

Lemma 40. *Let $M = (m_{i,j})$ be a Hadamard matrix of order n over a Latin rectangle $L = (l_{i,j}) \in \mathcal{R}_{r,n}$. Then,*

(a) *For each $i \leq r$, the value of $l_{i,i}$ must be in the set*

$$V_M(i) := \{t \leq n \mid m_{i,t} = m_{t,i}\}. \tag{9}$$

(b) *Let $i, k \leq r$ and $j \leq n$ be such that $i \neq j$. Then, the value of $l_{i,j}$ must be in the set*

$$V_M^-(i,j;k) := \{t \leq n \mid m_{l_{k,i},j} m_{k,i} m_{i,j} m_{k,t} = 1\}. \tag{10}$$

As a consequence, it must be in the set

$$V_M^-(i,j) := \bigcap_{k \leq r} V_M^-(i,j;k). \tag{11}$$

(c) Let $i, j \leq r$ be such that $i \neq j$, and let $k \leq n$. Then, the value of $l_{i,j}$ must be in the set

$$V_M^+(i, j; k) := \{t \leq n \mid \; m_{t,k} m_{i,j} m_{j,k} m_{i,l_{j,k}} = 1\}. \tag{12}$$

As a consequence, it must be in the sets

$$V_M^+(i, j) := \bigcap_{k \leq n} V_M^+(i, j; k). \tag{13}$$

$$V_M(i, j) := V_M^-(i, j) \cap V_M^+(i, j). \tag{14}$$

Proof. The result follows straightforwardly from Condition (6). Thus, for instance, the first assertion follows from the fact that $\psi(l_{i,i}, i) = \psi(i, i)\psi(i, i)\psi(i, l_{i,i}) = \psi(l_{i,i}, i)\psi(i, l_{i,i})$, for all $i \leq n$. The other two assertions hold similarly. $\square$

Example 41. In order to illustrate Lemma 40, let us consider the Hadamard matrix

$$M = (m_{i,j}) = \begin{pmatrix} + & + & + & + \\ - & + & + & - \\ - & - & + & + \\ + & - & + & - \end{pmatrix}.$$

Suppose the existence of a Latin rectangle $L = (l_{i,j}) \in \mathcal{R}_{r,4}$, with $2 \leq r \leq 4$, such that $M \in \mathcal{H}(L)$. Similarly to Example 39, we have that $l_{1,1} = 1$ and $l_{1,2} = 2$. These conditions imply, respectively, that $V_M^+(2, 1; 1) = \{1, 4\}$ and $V_M^+(2, 1; 2) = \{3, 4\}$. Thus, $V_M^+(2, 1) = \{4\}$ and hence, from Lemma 40.c, we have that $l_{2,1} = 4$. As a consequence, since L is a Latin rectangle, it must be $l_{2,2} \not\in \{2, 4\}$, which contradicts that $l_{2,2} \in V_M(2) = \{2, 4\}$ according to Lemma 40.a. Therefore, there does not exist such a Latin rectangle L.

We have made use of all the previous results to show in Table 2 an $r \times n$ Latin rectangle that gives rise to each equivalent class of Hadamard matrices of order $n \leq 8$. The latter are written row after row, where each row is represented in hexadecimal form. To this end, after replacing each -1 by 0, the resulting row in binary form is translated to its equivalent hexadecimal form.

6. Conclusions and further work

In this paper, we have introduced the concept of cocyclic Hadamard matrices over Latin rectangles and we have exposed a series of preliminary results that enable us to determine the existence of such matrices and that of Latin rectangles over which a given Hadamard matrix is cocyclic. More specifically, we have proved that every Latin square related to a Hadamard matrix is the multiplication table of a loop. Particularly, we have proved the existence of cocyclic Hadamard matrices over non-associative loops of order 2^{t+3}, for all positive integer $t > 0$. Furthermore, we have exposed examples of Hadamard matrices that are not cocyclic over finite groups but they are over Latin rectangles.

In Section 5, we have also shown a pair of examples that enable us to ensure that not every Hadamard matrix of order n is cocyclic over an $r \times n$ Latin rectangle. According to this fact, we propose as further work to deal with the following question, which constitutes a natural generalization of Problem 13.

Problem 42. Let M be a Hadamard matrix of order n for which no Latin rectangle $L \in \mathcal{R}_{r,n}$ satisfies that $M \in \mathcal{H}(L)$. Does there exist, however, a Hadamard matrix M' in the same equivalence class of M for which one such a Latin rectangle can be found?

The following question also arises in a natural way as an open problem.

Problem 43. Let us consider an equivalence class of Hadamard matrices such that none of them are cocyclic over any finite group. Does there exist, however, a Hadamard matrix within such a class that is cocyclic over a Latin rectangle?

Even if the theory that we have presented in this paper derives from the classical notion of pure cocyclic Hadamard matrix over a finite group, let us remark the existence of a more general definition of cocyclic development over finite groups [10,16,17], whose natural generalization to Latin squares can be similarly studied. In this regard, we say that a Hadamard matrix $M = (m_{i,j})$ is *cocyclically developed over a Latin square* $L = (l_{i,j}) \in \mathcal{R}_{n,n}$ if there exists a cocycle $\psi : [n] \times [n] \to \mathbb{Z}_2$ over L and a map $\phi : [n] \to \mathbb{Z}_2$ so that

$$m_{i,j} = \psi(i,j) \cdot \phi(l_{i,j}), \text{ for all } i, j \leq n. \tag{15}$$

In such a case, we call ϕ a *development function over L*. Analogously to the cocyclic development over finite groups, if $\psi(i,j) = 1$, for all $i, j \leq n$, then we say that the Hadamard matrix M is *quasigroup developed over L*. This is *group developed* over L if the latter constitutes the multiplication table of a group. The following result ensures that the existence of quasigroup developed Hadamard matrices over a given Latin square only depends on the isotopism class of the latter, and not on its isomorphism class, as it happens with cocyclic Hadamard matrices over Latin squares (see Proposition 20).

Lemma 44. *Quasigroup development is preserved by row and column permutations of Hadamard matrices, and by isotopisms of Latin squares.*

Proof. Let $M = (m_{i,j})$ be a Hadamard matrix of order n that is quasigroup developed over a Latin square $L = (l_{i,j})$ of the same order by means of a development function ϕ. For each triple of permutations $(f, g, h) \in S_n \times S_n \times S_n$, let us consider (a) the Hadamard matrix $M' = (m_{f(i),g(j)})$, (b) the Latin square $L' = (l'_{i,j})$ such that $l'_{f(i),g(j)} = h(l_{i,j})$, for all $i, j \leq n$, and (c) the function $\phi' = \phi \circ h^{-1} : [n] \to \mathbb{Z}_2$. Then, the Hadamard matrix M' is quasigroup developed over the Latin square L' by means of ϕ', because

$$m'_{i,j} = m_{f^{-1}(i),g^{-1}(j)} = \phi(l_{f^{-1}(i),g^{-1}(j)}) = \phi(h^{-1}(l'_{i,j})) = \phi'(l_{i,j}), \text{ for all } i, j \leq n. \quad \square$$

The interest of dealing with cocylic development over Latin squares as further work derives, therefore, from the existence of Latin squares over which no cocyclic Hadamard matrix exists, but over which cocyclically developed Hadamard matrices do. Thus, for instance, even if Theorem 28 implies that no cocyclic Hadamard matrix exists over the Latin square

$$L \equiv \begin{array}{|c|c|c|c|}\hline 1 & 3 & 2 & 4 \\ \hline 2 & 4 & 3 & 1 \\ \hline 3 & 1 & 4 & 2 \\ \hline 4 & 2 & 1 & 3 \\ \hline \end{array} \in \mathcal{R}_{4,4},$$

it is straightforwardly verified that the Hadamard matrix

$$\begin{pmatrix} + & + & + & - \\ + & - & + & + \\ + & + & - & + \\ - & + & + & + \end{pmatrix}$$

is quasigroup developed over L by means of the development function ϕ over L such that $\phi(1) = \phi(2) = \phi(3) = 1 = -\phi(4)$. Observe in particular that L constitutes the multiplication table of a non-associative quasigroup that is not a loop. As such, this example illustrates the fact that the cocyclic development is even less restrictive with respect to the assumptions that Latin squares have to satisfy. Thus, for instance, keeping in mind the previous example, the following result holds.

Proposition 45. *There exists a quasigroup developed Hadamard matrix over each Latin square of order four.*

Proof. Let $L = (l_{i,j}) \in \mathcal{R}_{4,4}$. For any given symbol s in L, let $M = (m_{i,j})$ be the matrix of order four such that $m_{i,j} = -1$, if $l_{i,j} = s$, and $m_{i,j} = 1$, otherwise. The Latin square condition of L implies that exactly one -1 exists within each row and each column of M. The dimension of the array makes this condition to be enough in order to ensure that M is Hadamard. Moreover, the matrix M is quasigroup developed over L by means of the development function ϕ over L such that $\phi(i) = 1$, for all $i \neq s$, and $\phi(s) = -1$. $\square$

There exist, however, some remarkable notions and results on group development theory that cannot be generalized as such to quasigroup development theory, because of their dependence on the associative property of the underlying structure. Thus, for instance, the concept of *coboundary* over a group G as a cocycle of the form $\psi(a,b) = \phi(a)\phi(b)\phi(ab)$, for all $a, b \in G$, and all normalized function $\phi : G \to \mathbb{Z}_2$, involves the function ϕ to satisfy that $\phi((ab)c) = \phi(a(bc))$, for all $a, b, c \in G$. A similar condition over a non-associative loop would involve ϕ to be the trivial normalized function and hence, the only well-defined coboundary over such a loop would be the trivial cocycle. As a consequence, every result in group development theory that uses coboundaries either in its statement or its proof cannot be generalized in a natural way to quasigroup development theory.

Such a dependence on the associative property occurs, for instance, in the majority of results concerning the interactions of group development theory with Hadamard equivalence and also with automorphism groups of Hadamard matrices [8]. Further work is, therefore, required to establish possible generalizations of both interactions. A possible approach in this regard consists of generalizing the notion of cocycle itself in order to adjust it to a *balanced equation* that regulates the quasigroup under consideration. Recall here that an equation that is satisfied by all the elements of a quasigroup is said to be *balanced* if each one of its variables appears precisely once on each side of the equation. Thus, for instance, the equation $(ab)c = a(bc)$ that regulates the associative property of every group is balanced.

Let us illustrate some aspects of this proposal by means of a specific balanced equation. Let $\mathcal{Q}_{\mathfrak{e}}$ be the set of quasigroups $Q = ([n], \cdot)$ verifying the balanced equation

$$\mathfrak{e}\colon\ (i \cdot j) \cdot k = (k \cdot j) \cdot i, \text{ for all } i, j, k \leq n. \tag{16}$$

This is indeed an abelian group whenever Q is a loop. We say that a function $\psi : [n] \times [n] \to \mathbb{Z}_2$ is an $\mathfrak{e}$-*cocycle* over a quasigroup $Q = ([n], \cdot) \in \mathcal{Q}_{\mathfrak{e}}$ if

$$\psi(i \cdot j, k) = \psi(i, j)\psi(k, j)\psi(k \cdot j, i), \text{ for all } i, j, k \leq n. \tag{17}$$

We say that an $n \times n$ matrix $M = (m_{i,j})$ is $\mathfrak{e}$-*cocyclic over* Q if there exists an $\mathfrak{e}$-cocycle ψ over Q such that $m_{i,j} = \psi(i, j)$, for all $i, j \leq n$.

Example 46. The Hadamard matrices

$$M_1 \equiv \begin{pmatrix} + & + & + & + \\ + & + & - & - \\ + & - & - & + \\ + & - & + & - \end{pmatrix} \qquad \text{and} \qquad M_2 \equiv \begin{pmatrix} + & + & + & + \\ + & + & - & - \\ - & + & - & + \\ - & + & + & - \end{pmatrix}$$

are $\mathfrak{e}$-cocyclic over the quasigroup Q having as multiplication table the Latin square

1	2	3	4
2	1	4	3
4	3	1	2
3	4	2	1

$\in \mathcal{R}_{4,4}$.

This quasigroup is not a loop. Moreover, it is non-abelian and non-associative.

We say that a function $\psi : [n] \times [n] \to \mathbb{Z}_2$ is an $\mathfrak{e}$-*coboundary* over a quasigroup $Q = ([n], \cdot) \in \mathcal{Q}_{\mathfrak{e}}$ if there exists a function $\phi : [n] \to \mathbb{Z}_2$ such that $\psi(i, j) = \phi(i)\phi(j)\phi(i \cdot j)$, for all $i, j \leq n$. In particular, ψ is an $\mathfrak{e}$-cocycle over Q, because Condition (17) holds if and only if $\phi((i \cdot j) \cdot k) = \phi((k \cdot j) \cdot i)$, for all $i, j, k \leq n$. At this point, we propose as further work a detailed analysis to adapt in the context of the set $\mathcal{Q}_{\mathfrak{e}}$ all those results in group development that deal with coboundaries.

Let us finish the exposition of this approach with an illustrative result that adjusts the necessary condition of Lemma 1 in [8] for quasigroups within the set $\mathcal{Q}_{\mathfrak{e}}$. Here, we say that a matrix $M = (m_{i,j})$ is *quasigroup developed* over a quasigroup $Q = ([n], \cdot) \in \mathcal{Q}_{\mathfrak{e}}$ if there exists a function $\phi : [n] \to \mathbb{Z}_2$ such that $m_{i,j} = \phi(i \cdot j)$, for all $i, j \leq n$.

Lemma 47. *Let $Q = ([n], \cdot) \in \mathcal{Q}_{\mathfrak{e}}$. For each positive integer $k \leq n$, we define the permutation matrix $T_k = (\delta_j^{i \cdot k})$, where $\delta_j^{i \cdot k}$ denotes the Kronecker delta. If an $\mathfrak{e}$-cocyclic matrix M over Q is quasigroup developed over Q, then $T_k M T_k = M^t$, for all $k \leq n$, where M^t denotes the transpose of M.*

Proof. Let ψ be a cocycle over Q such that $M = M_\psi$. Let $k \leq n$. Then,

- $T_k M T_k = M_{\psi_1}$, where $\psi_1(i, j) = \psi(i \cdot k, j/k)$, for all $i, j \leq n$.
- $M^t = M_{\psi_2}$, where $\psi_2(i, j) = \psi(j, i)$, for all $i, j \leq n$.

Now, let $\phi : [n] \to \mathbb{Z}_2$ be such that $\psi(i, j) = \phi(i \cdot j)$, for all $i, j \leq n$. Then,

$$\psi_1(i, j) = \psi(i \cdot k, j/k) = \phi((i \cdot k) \cdot j/k) = \phi((j/k \cdot k) \cdot i) = \phi(j \cdot i) = \psi(j, i) = \psi_2(i, j),$$

for all $i, j \leq n$. Hence, $T_k M T_k = M^t$. $\square$

Example 48. In Example 46, we have that

- $T_k M_1 T_k = M_1^t$ if and only if $k = 1$.
- $T_k M_2 T_k = M_2^t$ if and only if $k = 2$.

Hence, from Lemma 47, we can ensure that none of the $\mathfrak{e}$-cocyclic Hadamard matrices M_1 and M_2 are quasigroup developed over Q.

Of course, a much deeper analysis is required to deal with the different aspects derived from these new concepts. Moreover, similar studies related to other types of balanced equations different from $\mathfrak{e}$ are also necessary as further work. Particularly, the characterization of those equations that give rise to group isotopes could have special interest concerning the interaction of quasigroup development with Hadamard equivalence. We refer the interested reader to [3,4,20,21] for more details about the study of balanced equations on quasigroups.

From all the previous remarks and proposals, we realize that this paper only scratches the surface of a vast world where there is still much to explore. The results here presented constitute the starting point from which we can ensure that Latin rectangles and quasigroups arise as an interesting alternative to delve into the theory of cocyclic Hadamard matrices.

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Appendix. Glossary of symbols

$E(L)$	The set of entries of a Latin rectangle L.
$\mathcal{H}(L)$	The set of cocyclic Hadamard matrices over a Latin rectangle L.
$\mathcal{L}_n$	The set of loops of order n having 1 as unit element.
$[n]$	The set $\{1, \ldots, n\}$.
$\mathcal{Q}_n$	The set of finite quasigroups having $[n]$ as their finite set of symbols.
$\mathcal{R}_{r,n}$	The set of $r \times n$ Latin rectangles.
$S(L)$	The set $[r] \cup \{l_{i,j} \mid i, j \leq r\}$, where $L = (l_{i,j}) \in \mathcal{R}_{r,n}$.
$\mathcal{Z}(L)$	The set of cocycles over a Latin rectangle L.

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