On the Use of a 3-D Fundamental Solution for Axisymmetric Steady-State Dynamic Problems

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1. Introductory remarks

The application of the Boundary Element Method (BEM) to dynamic problems is especially well suited for several situations arising in soil-structure interaction. In those problems, the interest is generally focused on the structure while the soil is only interesting because of the effects that it introduces to the structure. The first application of the direct BEM to 3-D problems was presented in 1978 [1], for 2-D problems in 1980 [2] and [3], and several successive applications have been presented since then [4], [5], [6], etc. Because of the additional reduction in dimensions due to the symmetry, it is tempting to treat axisymmetric problems with the BEM. In this way, three-dimensional problems can be treated with monodimensional elements. The most direct way to accomplish this is to use a fundamental solution composed of rings of unit loads. For static problems, this was the direction taken in references [7], [8], etc. Our experience with the fundamental axisymmetric solution is, nevertheless, a bit discouraging due to the difficulties of integration produced by the complication of the fundamental solution. On the other hand, several studies developed with the 3-D fundamental solution have allowed us to appreciate the advantages of robustness and simplicity involved in it. In this paper, we shall present some of the results obtained using the 3-D dynamic fundamental solution as a weighting function, while maintaining the axisymmetric approach to the representation of geometry and interpolating functions.

2. Statement of the problem

To the best of our knowledge, the first author applying the 3-D fundamental solution to dynamic problems was F. CHAPEL [9]. The idea is to change the basic BEM equation:

$$\frac{1}{2}\,\underset{\sim}{u}^{c}(P) + \int_{\partial\Omega} \underset{\sim}{T}^{c}\,\underset{\sim}{u}^{c}\,(Q) = \int \underset{\sim}{U}^{c}\,\underset{\sim}{t}^{c}\,(Q) \tag{1}$$

where $\underset{\sim}{T}$ and $\underset{\sim}{U}$ are the matrices representing the stresses and displacements produced by the fundamental solution applied in P and the superscript c means "cartesian", $\underset{\sim}{u}(P)$, $\underset{\sim}{u}(Q)$, and $\underset{\sim}{t}(Q)$ are - written in cylindrical coordinates by using rotation matrices - where the superscript "p" is for "polar":

$$\begin{aligned} \underset{\sim}{u}^{c}(Q) &= \Omega\underset{\sim}{u}^{p}(Q) \\ \underset{\sim}{t}^{c}(Q) &= \Omega\underset{\sim}{u}^{p}(Q) \\ \underset{\sim}{u}^{p} &= \Omega'\,\underset{\sim}{u}^{c}(P) \end{aligned} \;;\quad \underset{\sim}{\Omega} = \begin{pmatrix} \cos\theta & -\mathrm{sen}\theta & 0 \\ \mathrm{sen}\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \;;\quad \Omega' = \begin{pmatrix} \cos\theta' & \mathrm{sen}\theta' & 0 \\ -\mathrm{sen}\theta' & \cos\theta' & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

The new equation written in cylindrical coordinates is now:

$$\frac{1}{2}\,\underset{\sim}{u}^{p}(P) + \int_{\partial\Omega} (\underset{\sim}{\Omega}'\underset{\sim}{T}^{c}\underset{\sim}{\Omega})\,\underset{\sim}{u}^{p}(Q) = \int_{\partial\Omega} (\underset{\sim}{\Omega}'\underset{\sim}{U}^{c}\Omega)\,\underset{\sim}{t}^{p}(Q) \tag{3}$$

because Ω' is constant (that is, it depends only on the collocation point P) and can be introduced inside the integrals. As discused by CHAPEL, P can be selected in several positions. For $\theta'=0$, for instance:

$$\underset{\sim}{T}^{P}=\underset{\sim}{\Omega}'\underset{\sim}{T}^{C}\underset{\sim}{\Omega}=\left[\begin{array}{c|c|c} T_{11}\cos\theta+T_{12}\sin\theta & -T_{11}\sin\theta+T_{12}\cos\theta & T_{13} \\ \hline T_{21}\cos\theta+T_{22}\sin\theta & -T_{21}\sin\theta+T_{22}\cos\theta & T_{23} \\ \hline T_{31}\cos\theta+T_{32}\sin\theta & -T_{31}\sin\theta+T_{32}\cos\theta & T_{33} \end{array}\right] \tag{4}$$

The interpolation functions are chosen by developing in series in the classical Fourier approach for this type of problems. In the following discussion, we are going to describe in detail the problem for the typical soil-structure situation where a rigid foundation is given vertical, horizontal, rocking and yawing displacements in order to determine the stiffness of the foundation. It can be shown that vertical and yawing displacements can be treated simultaneously by the equations:

$$\frac{1}{4}\begin{bmatrix} u'_\rho(P) \\ u'_\theta(P) \\ u'_z(P) \end{bmatrix} + \oint_{\partial\Omega} \left[\begin{array}{c|c|c} T_{11}\cos\theta+T_{12}\operatorname{sen}\theta & 0 & T_{13} \\ \hline 0 & -T_{21}\operatorname{sen}\theta+T_{22}\cos\theta & 0 \\ \hline T_{31}\cos\theta+T_{32}\operatorname{sen}\theta & 0 & T_{33} \end{array}\right] \begin{bmatrix} u'_\rho(Q) \\ u'_\theta(Q) \\ u'_z(Q) \end{bmatrix} =$$

$$= \oint_{\partial\Omega} \left[\begin{array}{c|c|c} U_{11}\cos\theta+U_{12}\operatorname{sen}\theta & 0 & U_{13} \\ \hline 0 & -U_{21}\operatorname{sen}\theta+U_{22}\cos\theta & 0 \\ \hline U_{31}\cos\theta+U_{32}\operatorname{sen}\theta & 0 & U_{23} \end{array}\right] \begin{bmatrix} \tau'_\rho(Q) \\ \tau'_\theta(Q) \\ \sigma'_z(Q) \end{bmatrix} \tag{5}$$

(where the integrations are done in $0 \leqq \theta \leqq \pi$), while the horizontal and rocking displacements follow as interpolation laws:

$$
\begin{array}{lll|lll}
u_\rho & = & u'_\rho \cos\theta & \tau_\rho & = & \tau'_\rho \cos\theta \\
u_\theta & = & -u'_\theta \operatorname{sen}\theta & \tau_\theta & = & -\tau'_\theta \operatorname{sen}\theta \\
u_z & = & u'_z \cos\theta & \sigma_z & = & \sigma'_z \cos\theta
\end{array}
\qquad (6)
$$

producing the following system:

$$
\frac{1}{4}\begin{bmatrix} u'_\rho(P) \\ u'_\theta(P) \\ u'_z(P) \end{bmatrix} + \int_{\partial\Omega} \begin{bmatrix} T_{11}\ C^2+T_{12}\ SC & T_{11}\ S^2-T_{12}\ SC & T_{13}\ C \\ T_{11}{*}C^2+T_{12}{*}SC & T_{11}{*}S^2-T_{12}{*}SC & T_{13}{*}C \\ T_{31}\ C^2+T_{32}\ SC & T_{31}\ S^2-T_{32}\ SC & T_{33}\ C \end{bmatrix} \begin{bmatrix} u'_\rho \\ u'_\theta \\ u'_z \end{bmatrix} =
$$

$$
= \int_{\partial\Omega} \begin{bmatrix} U_{11}\ C^2+U_{12}\ SC & U_{11}\ S^2-U_{12}\ SC & U_{13}\ C \\ U_{11}{*}C^2+U_{12}{*}SC & U_{11}{*}S^2-U_{12}{*}SC & U_{12}{*}C \\ U_{31}\ C^2+U_{32}\ SC & U_{31}\ S^2-U_{32}\ SC & U_{33}\ C \end{bmatrix} \begin{bmatrix} \tau'_\rho \\ \tau'_\theta \\ \sigma'_z \end{bmatrix}
$$

where $S=\sin\theta$; $C=\cos\theta$ and the second equation has been established by locating P in $\theta = -\frac{\pi}{2}$.

The integrals are done numerically except when working in the - same annular area containing the collocation point. In those cases, the static solution is substracted from the numerical solution and an analytical computation of the singularity is added - "a posteriori".

For the numerical integration, we are using a 2x10 Gauss quadrature on every semiring, but in order to weigh the influence of the collocation point, a parabolic transformation has been superimposed to accumulate points near P, i.e., we are using the following rule.

$$\begin{cases} \theta = \frac{\pi}{2}\sqrt{(\xi-1)} \\ \rho = \frac{1}{2}\left[(x_2-x_1)\eta + x_1+x_2\right] \end{cases} \qquad \begin{matrix} 0 \leqq \xi \leqq 1 \\ 0 \leqq \eta \leqq 1 \end{matrix}$$

The transformation is essential in order to improve the accuracy wilthout increasing the computation cost.

3. Examples

Fig. 1 shows the solution obtained for the torsional response of an elastic homogeneous halfspace, compared to LUCO's, et al. [10] solution, while Fig. 2 represents the horizontal response as compared to the one by VELETSOS [11]. In both cases $\nu=1/3$, $a_o = \frac{\omega r}{C_s}$ where is the excitation frequency, r the radius of the footing and C_s the celerity of S waves in the medium. The discretization was of only 8 elements under the footing. Fig. 3 presents the same results for a hysteretically damped halfspace with $\zeta=0.15$ against the results by VELETSOS et al [12]. Finally, Fig. 4 treats a more complicated case previously studied by CHAPEL [9]. It can be seen in the figures that all results are excellent.

4. Conclusions

The use of a 3-D fundamental solution is some axisymmetric problems is straightforward. The resulting algorithms seem to work better than the usual ones (at least for static solutions) and for dynamic cases, than those presented in the previous paragraph. The robustness of the method allows the computations for very high and very low frequencies without any noticeable difficulty.

5. References

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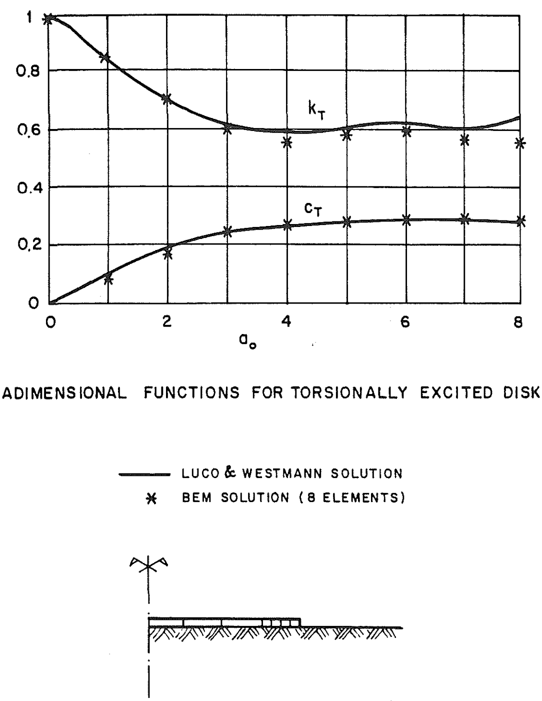
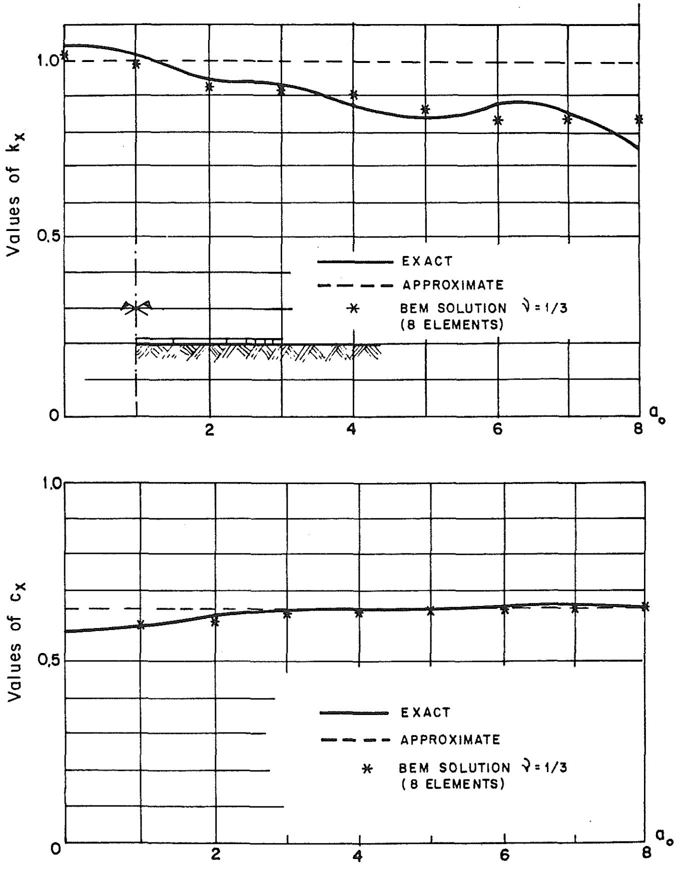


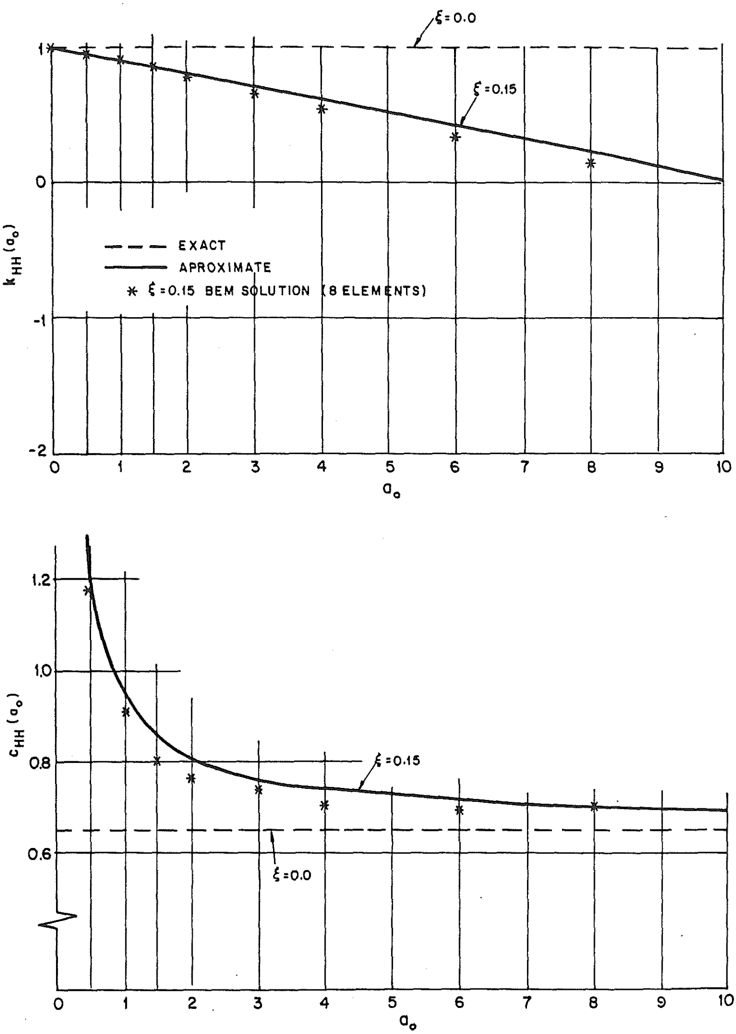
ADIMENSIONAL FUNCTIONS FOR TORSIONALLY EXCITED DISK

FIGURE 1



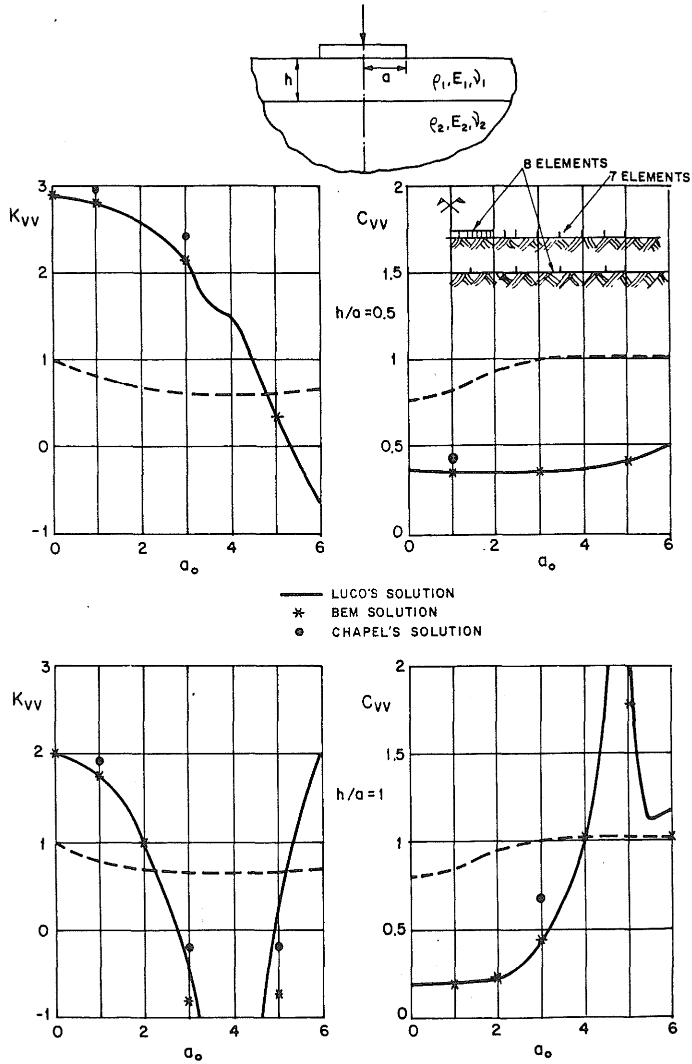
FUNCTIONS k_x AND c_x FOR HORIZONTALLY EXCITED DISK

FIGURE 2



HORIZONTAL IMPEDANCE FUNCTIONS FOR A HYSTERETICALLY DAMPED HALF-SPACE (a = 1/3)

FIGURE 3



FUNCTIONS K_{VV} AND C_{VV} FOR VERTICALLY EXCITED DISK ON LAYERED HALFSPACE ($V_{S1} : V_{S2} = 0.4$)

FIGURE 4