THE BOUNDARY ELEMENT METHOD IN ELASTODYNAMICS

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Summary. The paper presents the application of B.E.M. to elastodynamic problems. Both the transient and steady state solutions are presented as well as some techniques to simplify problems with a free-stress boundary.

Introduction

As is well known B.E.M. is obtained as a mixture of the integral representation formula of classical elasticity and the discretization philosophy of the finite element method (F.E.M.).

Several books have been published on the technique((1)) ((2))(3) (4))whose main attractive feature is the reduction by one of the dimensionality of the problem. In this way complex tridimensional problems need to be discretized only at its two-dimensional boundary. Also interesting is the inclusion of the decay properties at infinity inside the fundamental solution used in the reciprocity theorem, which avoids the truncation problems that in dynamic cases are present when using the F.E.M.

The discretization is done by assuming polinomial interpolation inside each element for the geometry, displacements and tractions. Although the degree of these interpolations can be in principle independent it is usual to adopt a kind of "isoparametric" technique. Sometimes also it is better to use elements to simulate special characteristics. for instance singular behaviour near and end crack, and it is always interesting to choose a fundamental solution suited to the problem on hand. The generally adopted solution is the Kelvin-type load in the complete space which is a good alternative for instance when treating for deep buried cavities. A lot of problems in engineering are however best modelled in terms of a half-space geometry. In this are included the usual soil-structure interaction problems that appear in machine foundations or in nuclear power plants. In these cases it is desirable to use Mindlin-type solutions in order to discretize only the interface. Nevertheless this solution is difficult to incorporate in the integrals and some compromise alternatives have been proposed. The most obvious one is the truncation of the discretized area outside the interface((5,6))which produces surprisingly good results with a very rough discretization. Another technique is the use of the method of images to sweep out the most influencing terms from outside the interface.

In this paper we present an appraisal of the method with some applications of it to the steady-state case as well as a comparison between the truncation procedures outlined above.

Transient problems

The representation is obtained by using the response to an impulse P(t) acting in direction $\underline{a}$ at point ξ

$$\underline{f}(\underline{x},t) = \underline{a}\, P(t)\, \delta(\underline{x}-\underline{\xi}) \qquad (1)$$

and the Betti-Rayleigh dynamic reciprocal identity((9)

$$\int_{\partial\Omega} \underline{\overset{\nu}{T}} * \underline{u}' + \rho \int_{\Omega} [\underline{f} * \underline{u}' + \underline{\dot{u}}' \underline{\overset{\circ}{u}} + \underline{u}' \underline{\overset{\circ}{v}}] = \int_{\partial\Omega} \underline{\overset{\nu}{T}}' * \underline{u} + \rho \int_{\Omega} \underline{f}' * \underline{u} + \underline{\dot{u}}\,\underline{\overset{\circ}{u}}' + \underline{u}\,\underline{\overset{\circ}{v}}' \qquad (2)$$

Where $\underline{\overset{\nu}{T}}, \underline{f}, \underline{u}$ are respectively the traction vector, the body force vector and the displacement vector of a dynamic state whose initial conditions are

$$\underline{u}(\underline{x},0) = \underline{\overset{\circ}{u}} \qquad \underline{\dot{u}}(x,0^+) = \underline{\overset{\circ}{v}} \qquad (3)$$

The $*$ represents convolution products and the primes are assigned to a different state.

The fundamental solution can be written((9,10) as

$$4\pi u_i(\underline{x},t) = D_i^k[\underline{\xi}, P(t)] \qquad 4\pi \sigma_{ij}(\underline{x},t) = S_{ij}^k[\underline{\xi}, P(t)] \qquad (4)$$

when the load is directed along the X_k axis and its point of application is $\underline{\xi}$.

Substituting (4) in (2) the Wheeler and Sternberg formulae is obtained.

$$4\pi\rho \underline{u}(\underline{r},t) = \int_{t_0=0}^{\infty} \int_{\partial\Omega} [\underline{D}\,\underline{\overset{\nu}{T}}(\underline{\xi},t_0) - \underline{u}(\underline{\xi},t_0)\, \underline{S}\cdot\underline{\nu}]\, ds_0\, dt_0 + \int_{t_0=0}^{\infty} \int_{\Omega} \underline{D}\, \underline{f}(\underline{\xi},t_0)\, dv_0\, dt_0 + \int_{\Omega} \rho \left[\underline{D} \frac{\partial \underline{u}}{\partial t_0}(\underline{\xi},t_0) - \underline{u} \frac{\partial \underline{\xi}}{\partial t_0} \right]_{t_0=0} dv_0 \qquad (5)$$

valid for points inside the domain.

Using a limiting process, well documented in((11)), it is possible to write a Somigliana-type formulae for points at the boundary

$$4\pi\rho \underline{c}\, \underline{u} = \int_{t_0=0}^{\infty} \int_{\partial\Omega} [\underline{D}\,\underline{\overset{\nu}{T}} - \underline{u}\, \underline{S}\,\underline{\nu}]\, ds_0\, dt_0 + \underline{R}(\underline{r},t) \qquad (6)$$

where $\underline{C}$ is a matrix depending on the local smoothness properties of the boundary and $\underline{R}$ collects the two last integrals in (5).

The discretization can be done((11))by taking J nodes on the boundary and a set $\{t_m = m\Delta t, m=1,\dots,N\}$ of equally spaced pieces of time and putting

$$\underline{u}(\underline{r},t) = \sum_j \sum_m U_j^m(\underline{r},t)\, u_j^m \qquad \underline{\overset{\nu}{T}}(\underline{r},t) = \sum_j \sum_m T_j^m(\underline{r},t)\, t_j^m \qquad (7)$$

where

$$\underline{u}(\underline{r}_i,t_m) = u_i^m \qquad \underline{\overset{\nu}{T}}(\underline{r}_i,t_m) = t_i^m \qquad (8)$$

Substitution in (6) produces the system

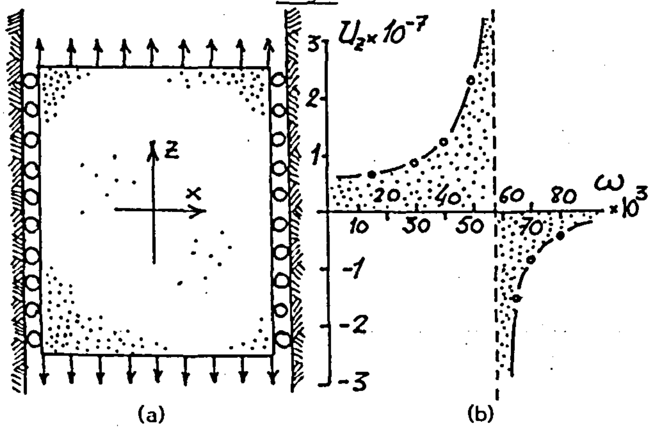
$$4\pi\rho c\, u_j^m = \sum_{m=1}^{N} \sum_{j=1}^{J} \{ G_{ji}^{nm} T_i^m + K_{ji}^{nm} u_i^m \} + F(r_j, t_n)$$

$$j = 1,\dots,J \qquad m = 1,\dots,N$$

$$G_{ji}^{nm} = \int_0^\infty\int_{\partial\Omega} D(\underline{r}_j, t_n; \underline{\xi}, t_0)\, \overset{\nu}{T}_i(\underline{\xi}, t_0)\, ds_0\, dt_0$$

$$K_{ji}^{nm} = \int_0^\infty\int_{\partial\Omega} \underline{n}\cdot\nabla_0 D(\underline{r}_j, t_n; \underline{\xi}, t_0)\, U_i^{M}(\underline{\xi}, t_0)\, ds_0\, dt_0 \qquad (9)$$

Fig.1



(a) (b)

By successively putting $n = 1, 2, \ldots$,(9) is solved for the unknown coefficients progressing along the time axis. Due to properties of symmetry only NJ^2 kernels have to be calculated.

Steady-state problems

The steady-state case is interesting, because some problems are best treated in the frequency domain. Using the Fourier transformation indicated by a $*$ the reciprocity relationship can be written as

$$\int_{\partial\Omega} \overset{\nu}{\underline{T}}^* \cdot \underline{u}'^* + \rho\int_{\Omega} \underline{f}^*\cdot \underline{u}'^* = \int_{\partial\Omega} \underline{t}'^* \underline{u}^* + \rho\int_{\Omega} \underline{f}'^* \underline{u}^* \qquad (10).$$

If the primes are reserved to a fundamental solution (10) can be written as

$$u_P = \int_{\partial\Omega} \underline{U}\,\underline{t} - \int_{\partial\Omega} \underline{T}\,\underline{u} \qquad P \in \Omega \qquad (11)$$

where U and T are the displacements and stresses produced by the fundamental load.

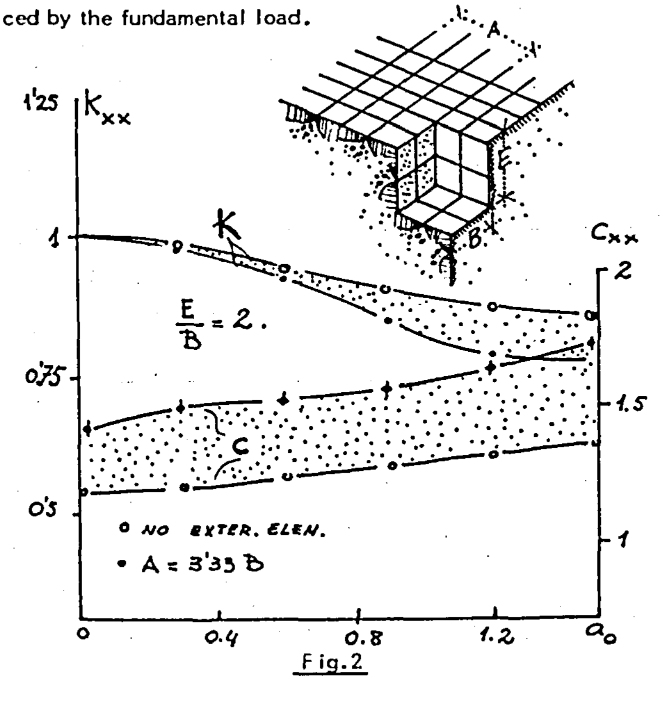


Fig.2

For instance the KELVIN type load.((1,10)).

If P is at the boundary the left-hand side of the equation (11) has to be modified in a way consistent with the boundary smocthness. Since then the discretization is done as in an elastostatic problem

$$c_K u_K + \Sigma\left[\int_{S_n}[\underline{T}\,\underline{N}^n|J|d\xi]\,\underline{u}^n = \Sigma\left[\int_{S_n} \underline{U} N^n |J| d\xi\right]\underline{t}^n \qquad (12)$$

where N are the interpolation function and $|J|$ the Jacobian of the transofrmation.

A simple axample is shown in fig.1. which shows a plate under uniform harmonic tractions on two opposite sides without transverse motions. Due to the boundary conditions only P waves are expected. In 1.b we can see the motion of the boundary versus frequency presenting resonance for a frequency well in agreement with the theoretical one

$$\omega = \frac{C_p \pi}{L} = \frac{(5397.17)\pi}{0.3} = 56519 \text{ rad/sec}$$

Elsewhere we have presented((5,6,10)) the application of the method to soil-structure interaction problems, following the lines established by Dominguez((12,13)). In them the surface

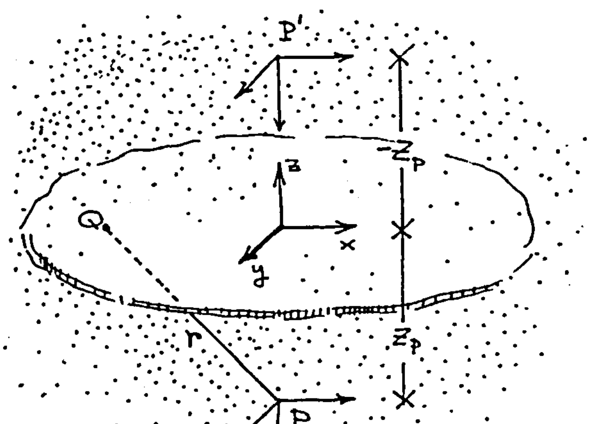


Fig.3

of the half-space was applied when calculating the dynamic impedances of embedded foundations in the form

$$K_{ij} = K_{ij}^0\,(k_{ij} + i\,a_0\,c_{ij}) \qquad (13)$$

where

$$a_0 = \omega B / C_S$$

B = half-width of the foundation

In fig.2.b we present the solution obtained with several lines of discretization and for different frequencies.

The previous experience has shown that discrepancies grow with the value of the frequency and this is why a use of the method of images has been proposed recently((7,8)).

The idea is to analyze matrix T in the light of the artisymmetric situation provoked by the images method. It is easy to see that for nodes on $z=0$, $\underline{T}$ has the form

$$\underline{T} = \begin{pmatrix} 0 & 0 & T_{13} \\ 0 & 0 & T_{23} \\ T_{31} & T_{32} & 0 \end{pmatrix} \qquad (14)$$

for every element on that plane, thus reducing the influence to contributions that, on physical grounds, are small. This allows the reduction of the discretization on the surface. A typical result can be seen in Fig.4 where the influence on the static $\overset{\circ}{K}_{ij}$ and dynamic parts k_{ij}, c_{ij} is reported. As can be seen the static part is strongly affected by the discretization when the single Kelvin point solution is used while the dynamic parts behave quite differently for the x-x displacement while the stiffness is well approximated the damping is underestimated.

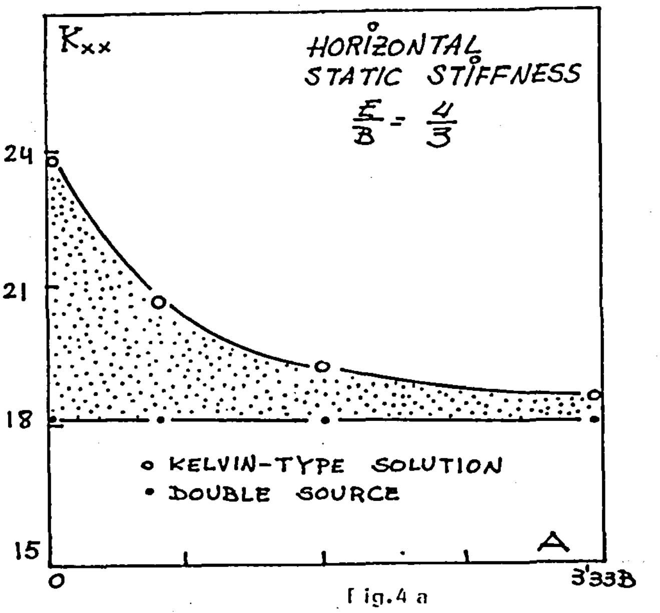


Fig.4 a

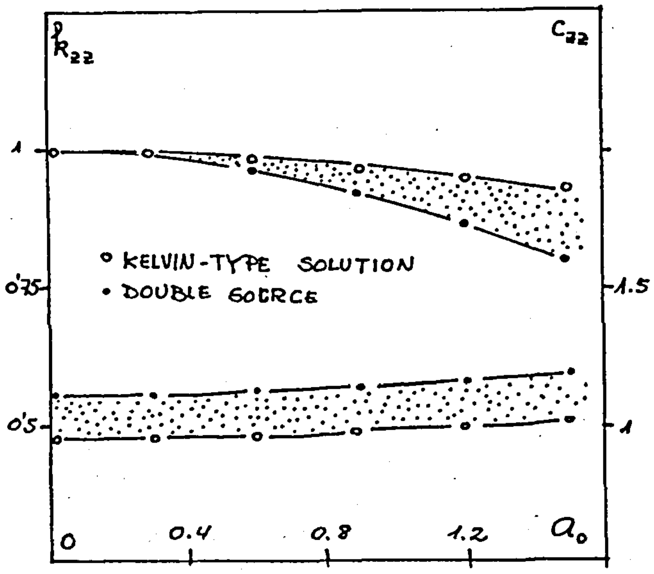


Fig.4 b

References

1. T.A.Cruse and F.J.Rizzo (Eds) (1975): Boundary-Integral equation method: Computational applications in applied mechanics. ASME, New York.

2. M.A.Jaswon and G.T.Symm (1977): Integral equation methods in potential theory and elastostatics. Academic Press.

3. C.A.Brebbia (1978): The boundary element method for engineers. Pentech Press.

4. R.Butterfield and Banerjee (1981): The boundary element method in engineering science. McGraw Hill.

5. E.Alarcon and J.Dominguez (1978): Boundary methods in soil-structure interaction. 2nd Int.Conf.on Microzonation. S.Francisco.

6. E.Alarcon and J.Dominguez (1980): Impedance of foundations using the boundary integral equation method. Conf.on Innovative Numer. Anal. for Eng.Sci. Univ.Press Virginia.

7. J.Dominguez (1981): Use of the method of images in B.E.M. for elastodynamics. Communication at the NATO seminar on "Numerical Methods in Geomechanics" Vimeiro.

8. J.Dominguez and R.Abascar (1982): The use of a composed fundamental solution in the computation of foundation impedances with B.E.M. (in Spanish). Anales de Ingeniería mecánica 1.

9. J.D.Achenbach (1973): Wave propagation in elastic solids. North-Holland.

10. J.Dominguez and E.Alarcon (1981): Elastodynamics in "Progress in B.E.M." ed by Brebbia, Pentech Press.

11. D.M.Cole and D.D.Kosloff (1978): A numerical boundary integral equation for elastodynamics. Seism.Lab. Cal.Inst.of Tech. Pasadena.

12. J.Dominguez (1978): Dynamic stiffness of rectangular foundations. M.I.T. Res.Rep. R 78-20. Dept.Civ.Eng.

13. J.Dominguez (1978): Response of embedded foundations to travelling waves. M.I.T. Res.Rep. R 78-24. Dept. Civ.Eng.

APPENDIX

$$T_{13} = A\left[-2C\left(n_3 r_{,1} - 2 r_{,1} r_{,3} \frac{\partial r}{\partial n}\right) - 2D r_{,1} r_{,3} \frac{\partial r}{\partial n} + 2E r_{,1} n_3\right]$$

$$T_{23} = A\left[-2C\left(n_3 r_{,2} - 2 r_{,2} r_{,3} \frac{\partial r}{\partial n}\right) - 2D r_{,2} r_{,3} \frac{\partial r}{\partial n} + 2E r_{,2} n_3\right]$$

$$T_{31} = A\left[2B r_{,1} n_3 + (4C-2D) r_{,1} r_{,3} \frac{\partial r}{\partial n}\right]$$

$$T_{32} = A\left[2B r_{,2} n_3 + (4C-2D) r_{,2} r_{,3} \frac{\partial r}{\partial n}\right]$$

$$A = 1/4\pi$$

$$B = \left(-\frac{i\omega}{c_s} - \frac{3}{r}\right)\frac{e^{-\frac{i\omega r}{c_s}}}{r} + \left(\frac{c_s}{c_p}\right)^2\left(\frac{6c_p}{i\omega r^2} - \frac{6c_p^2}{\omega^2 r^3} + \frac{2}{r}\right)\frac{e^{-\frac{i\omega r}{c_p}}}{r}$$

$$C = 2\chi/r \qquad D = 2\, d\chi/dr$$

$$E = \left[\left(\frac{c_p}{c_s}\right)^2 - 2\right]\left[\frac{d\psi}{dr} - \frac{d\chi}{dr} - C\right]$$

$$\chi = \left(-\frac{3c_s^2}{\omega^2 r^2} + \frac{3c_s}{i\omega r} + 1\right)\frac{e^{-\frac{i\omega r}{c_s}}}{r} - \left(\frac{c_s}{c_p}\right)^2\left(-\frac{3c_p^2}{\omega^2 r^2} + \frac{3c_p}{i\omega r} + 1\right)\frac{e^{-\frac{i\omega r}{c_p}}}{r}$$

$$\psi = \left(1 - \frac{c_s^2}{\omega^2 r^2} + \frac{c_s}{i\omega r}\right)\frac{e^{-\frac{i\omega r}{c_s}}}{r} - \left(\frac{c_s}{c_p}\right)^2\left(-\frac{c_p^2}{\omega^2 r^2} + \frac{c_p}{i\omega r}\right)\frac{e^{-\frac{i\omega r}{c_p}}}{r}$$