Weak Solutions to a Nonuniformly Elliptic PDE System in the Harmonic Regime

María Teresa González Montesinos and Francisco Ortegón Gallego

Abstract We study the existence of weak solutions to a nonlinear strongly coupled parabolic–elliptic PDEs arising in the heating induction-conduction process of steel hardening. In this setting, our major concern is to consider the case when the electric conductivity is nonuniformly elliptic which, together with a right hand side in L^1 in the energy balance equation, yields to a difficult theoretical situation. The existence result gives a weak solution to a similar PDEs system where the energy balance equation has been perturbed by a measure term.

1 Introduction

The aim of this work is to analyze the existence of weak solutions to a nonlinear PDEs system arising in the heating induction-conduction process of a steel workpiece [7, 8, 10, 11, 13]. Since we are dealing with high oscillating sinusoidal in time for both electric potential and magnetic vector potential, we introduce a change of variables separating the two time scales. This leads us to a new PDEs system, the so-called harmonic regime, namely

$$-\nabla \cdot (\sigma(\theta)\nabla\varphi) = i\lambda\omega\nabla \cdot (\sigma(\theta)\boldsymbol{A}) + \nabla \cdot \left(\sigma(\theta)\nabla\varphi^0\right) \text{ in } \Omega_T = \Omega \times (0, T), \quad (1)$$

$$i\omega\sigma(\theta)\boldsymbol{A} + L(\boldsymbol{A}) = -\sigma(\theta)\nabla\varphi \text{ in } D_T = D \times (0, T), \quad (2)$$

$$\varphi = 0 \text{ on } \Gamma_0 \times (0, T), \quad \frac{\partial\varphi}{\partial n} = -i\lambda\omega\boldsymbol{A} \cdot n \text{ on } \Gamma_1 \times (0, T), \ + \text{b.c. on } \boldsymbol{A}, \quad (3)$$

M.T. González Montesinos (✉)
Departamento de Matemática Aplicada I, Universidad de Sevilla, Avda. Reina Mercedes s/n, 41012 Sevilla, Spain
e-mail: mategon@us.es

F. Ortegón Gallego
Facultad de Ciencias, Departamento de Matemáticas, Universidad de Cádiz, Campus del Río San Pedro, 11510 Puerto Real, Spain
e-mail: francisco.ortegon@uca.es

$$\rho c_\varepsilon \frac{\mathrm{d}\theta}{\mathrm{d}t} - \nabla \cdot (\kappa(\theta)\nabla\theta) = \frac{\sigma(\theta)}{2}|i\omega \boldsymbol{A} + \nabla\varphi|^2 + G \text{ in } \Omega_T, \tag{4}$$

$$\frac{\partial\theta}{\partial n} = 0 \text{ on } \partial\Omega \times (0, T), \quad \theta(\cdot, 0) = \theta_0 \text{ in } \Omega. \tag{5}$$

In this context, $\Omega,\ D \subset \mathbb{R}^3$ are open, bounded, connected and Lipschitz-continuous sets such that $\bar{\Omega} \subset D$, $\partial\Omega = \Gamma_0 \cup \Gamma_1$ is a smooth partition of the boundary of Ω. The unknowns are the electric potential, φ, the magnetic vector potential, $\boldsymbol{A}$, and the temperature, θ; σ and κ stand for the electric and thermal conductivities, respectively, ω is the frequency, θ_0 the initial temperature and $L \in \mathscr{L}(\mathbb{X}, \mathbb{X}')$ is an elliptic operator defined on a certain Hilbert space $\mathbb{X}$ with values on its dual space $\mathbb{X}'$. Also, ρ is the density and c_ε is the specific heat at constant pressure. Finally, $\varphi^0 \in L^2(H^1(\Omega))$ is a given function with zero flux gradient on Γ_1 and i is the imaginary unity.

In this work we have included in (1) the divergence term $i\lambda\omega\nabla \cdot (\sigma(\theta)\boldsymbol{A})$, where $\lambda \in \left[0, 1 - \frac{1}{\omega}\right)$ is a parameter. Usually, this term is not taken into account, that is $\lambda = 0$. Notice that in the original model we have $\lambda = 1$ (cf. [2, 3]).

This work is organized as follows. In Sect. 2, we describe the notation used along this paper, introduce some functional spaces, enumerate the hypotheses on data and give the main result. In Sect. 3 we sketch the proof of the main result by introducing approximate problems, deriving the necessary a priori estimates and, finally, passing the limit.

2 Notation, Assumptions and Main Result

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^3$ be two open bounded, connected and Lipschitz-continuous sets such that $S = \bar{\Omega}_1 \cap \bar{\Omega}_2 \neq \emptyset$ is a smooth surface. We then consider the set of conductors $\Omega = \Omega_1 \cup \Omega_2 \cup \mathrm{int}(S)$ where $\mathrm{int}(S)$ means the interior of S within the induced topology. Ω_1 is the steel workpiece whereas Ω_2 is the copper inductor; since $S \neq \emptyset$, the workpiece and the inductor are put in contact so that Ω itself becomes the coil. Let $\Gamma_0 \subset \partial\Omega_2$ be a smooth surface.

For a normed linear space V, we put $\boldsymbol{V} = (V)^3$. Also, if X is a Banach space, we write $L^p(X) = L^p(0, T; X)$ and $W^{1,p}(X) = W^{1,p}(0, T; X)$, where p' is the conjugate exponent of p. Let V be the complex valued Hilbert space $V = \{\phi \in H^1(\Omega) \,/\, \phi = 0 \text{ on } \Gamma_0\}$ provided with the norm $\|\phi\|_V = \left(\int_\Omega |\nabla\phi|^2\right)^{1/2}$, which is equivalent to the standard norm in $H^1(\Omega)$ on V.

We also consider a complex valued Hilbert space $\mathbb{X}$ such that $\boldsymbol{H}_0^1(D) \subset \mathbb{X} \subset \boldsymbol{H}^1(D)$ where lies the magnetic vector potential $\boldsymbol{A}$. Obviously, the space $\mathbb{X}$ is related to the boundary conditions of $\boldsymbol{A}$. For instance, it may take the form

$$\mathbb{X} = \{\boldsymbol{v} \in \boldsymbol{H}^1(D) \,/\, \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial D\}, \text{ or}$$

$$\mathbb{X} = \{\boldsymbol{v} \in \boldsymbol{H}^1(D) \,/\, \nabla \cdot \boldsymbol{v} = \boldsymbol{0} \text{ in } D,\ \boldsymbol{v} \times n = 0 \text{ on } \partial D\} \text{ where } \partial D \in C^{1,1} \text{ in this case.}$$

On the other hand, the elliptic operator $L \in \mathscr{L}(\mathbb{X}, \mathbb{X}')$ is given by

$$L(\boldsymbol{v}) = \nabla \times \left(\frac{1}{\mu}\nabla \times \boldsymbol{v}\right) - \delta\nabla(\nabla \cdot \boldsymbol{v}),$$

where μ is the magnetic permeability (a positive bounded function) and $\delta > 0$ a constant value.

In the analysis of parabolic problems with right hand side in L^1 it is useful the next result (see [14])

Lemma 1 *Let X, B and Y be three Banach spaces such that $X \hookrightarrow B \hookrightarrow Y$, all embeddings being continuous and the injection $X \hookrightarrow B$ compact. For $1 \leq p, q < +\infty$ define $\mathscr{W}$ to be the Banach space $\mathscr{W} = \left\{v \in L^p(X) \,/\, \frac{\mathrm{d}v}{\mathrm{d}t} \in L^q(Y)\right\}$. Then, the embedding $\mathscr{W} \hookrightarrow L^p(B)$ holds and is compact.*

The assumptions on data now follows.

(H.1) $\sigma : D \times \mathbb{R} \to \mathbb{R}$ is given by

$$\sigma(x, s) = \begin{cases} \sigma^{(1)}(s) & \text{if } x \in \Omega_1, s \in \mathbb{R} \\ \sigma^{(2)}(s) & \text{if } x \in \Omega_2, s \in \mathbb{R}, \\ 0 & \text{if } x \in D \setminus \bar{\Omega}, s \in \mathbb{R}, \end{cases}$$

where $\sigma^{(1)}, \sigma^{(2)} \in C(\mathbb{R})$, and there exist some constant $C_1,\ C_2,\ K_1,\ K_2 > 0$ and $0 < \alpha < 5/3$ such that for all $s \in \mathbb{R}$ we have

$$0 < \frac{C_1}{1 + |s|^\alpha} \leq \sigma^{(1)}(s) \leq C_2, \quad K_1 \leq \sigma^{(2)}(s) \leq K_2.$$

(H.2) $\rho = \rho_i$ and $c_\varepsilon = c_\varepsilon^i$ in Ω_i, $i = 1, 2$ where $\rho_1, \rho_2, c_\varepsilon^1, c_\varepsilon^2 \in \mathbb{R}$ are positive constant values.

(H.3) $\kappa : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function and there exist two constant values κ_1 and κ_2 such that, almost everywhere $x \in \Omega$ and for all $s \in \mathbb{R}$, we have $0 < \kappa_1 \leq \kappa(x, s) \leq \kappa_2$.

(H.4) $L \in \mathscr{L}(\mathbb{X}, \mathbb{X}')$ and there exists a constant value $\alpha > 0$ such that, for all $\boldsymbol{v} \in \mathbb{X}$,

$$\langle L(\boldsymbol{v}), \bar{\boldsymbol{v}}\rangle_{\mathbb{X}',\mathbb{X}} \geq \alpha\|\boldsymbol{v}\|_{\mathbb{X}}^2.$$

(H.5) $\lambda \in \left[0, 1 - \frac{1}{\omega}\right)$.

(H.6) $\varphi^0 \in L^2(H^1(\Omega))$ and $\frac{\partial\varphi^0}{\partial n} = 0$ on $\Gamma_1 \times (0, T)$.

(H.7) $G \in L^1(\Omega_T)$.

(H.8) $\theta_0 \in L^1(\Omega)$.

The main result of this paper is the next

Theorem 1 *Under the assumptions (H.1)–(H.8) there exist three measurable functions* φ, $\theta : \Omega_T \mapsto \mathbb{R}$, $\boldsymbol{A} : D_T \mapsto \mathbb{R}^3$, *and a Radon measure* $\boldsymbol{\mu} \in \mathscr{M}(\Omega_T)$ *such that*

$$\varphi \in L^r(W^{1,r}(\Omega)), \ \textit{for all } r \in [1, 10/(5+3\alpha));, \ \varphi = 0 \textit{ on } \Gamma_0, \tag{6}$$

$$\sigma(\theta)^{1/2}\nabla\varphi \in L^2(\boldsymbol{L}^2(\Omega)), \boldsymbol{A} \in L^2(\mathbb{X}), \tag{7}$$

$$\int_{\Omega_T} \sigma(\theta)\nabla\varphi \cdot \nabla\bar{\phi} = -i\omega\lambda \int_{\Omega_T} \sigma(\theta)\boldsymbol{A} \cdot \nabla\bar{\phi} + \int_{\Omega_T} \sigma(\theta)\nabla\varphi^0\nabla\bar{\phi}, \phi \in L^2(V), \tag{8}$$

$$i\omega \int_{\Omega_T} \sigma(\theta)\boldsymbol{A} \cdot \bar{\boldsymbol{v}} + \int_0^T \langle L(\boldsymbol{A}), \bar{\boldsymbol{v}}\rangle_{\mathbb{X}',\mathbb{X}} = -\int_{\Omega_T} \sigma(\theta)\nabla\varphi \cdot \bar{\boldsymbol{v}}, \ \boldsymbol{v} \in L^2(\mathbb{X}), \tag{9}$$

$$\theta \in L^p(W^{1,p}(\Omega)) \cap C([0,T]; (W^{1,p'}(\Omega))'), \textit{ for all } p \in [1, 5/4)\,, \tag{10}$$

$$\theta(\cdot, 0) = \theta_0 \textit{ in } \Omega, \tag{11}$$

$$\begin{aligned} -\int_{\Omega_T} \rho c_\varepsilon \theta \zeta_{,t} + \int_{\Omega_T} \kappa(\theta)\nabla\theta\nabla\zeta &= \int_{\Omega_T} \left[\frac{\sigma(\theta)}{2}|i\omega\boldsymbol{A} + \nabla\varphi|^2 + G\right]\zeta \\ &+ \int_{\Omega_T} \zeta \mathrm{d}\boldsymbol{\mu} + \int_\Omega \theta_0(x)\zeta(x,0), \\ &\textit{for all } \zeta \in \mathscr{D}(\bar{\Omega}_T) \textit{ such that } \zeta(\cdot, T) = 0 \textit{ in } \Omega. \end{aligned} \tag{12}$$

Remark 1 Due to (H.1), the function σ is not uniformly elliptic. In particular, we cannot derive the regularity $\varphi \in L^2(V)$. This is also related with the "strange term" $\boldsymbol{\mu}$ appearing in the equation for the temperature.

3 Proof of the Main Result

In order to prove the Theorem 1 we first introduce a sequence of approximate problems then deduce some a priori estimates. The approximate problems regularize the solution in three different ways: (1) introduction of a time derivative term in the equations of φ and $\boldsymbol{A}$ to assure the measurability of both functions when passing to the limit; (2) modification of the electric conductivity in order to deal with uniformly elliptic operators; and (3) truncation of the L^1 terms in the energy equation.

3.1 Approximate Problems

For $k \in \mathbb{N}$ we introduce the approximate the function σ as follows

$$\sigma_k(x,s) = \begin{cases} \sigma^{(1)}(s) + \dfrac{1}{k} & \text{if } x \in \Omega_1, s \in \mathbb{R}, \\ \sigma^{(2)}(s) & \text{if } x \in \Omega_2, s \in \mathbb{R}, \\ 0 & \text{if } x \in D \setminus \bar{\Omega}, s \in \mathbb{R}. \end{cases}$$

We also use the truncation function T_k at height $k > 0$, that is

$$T_k(s) = \begin{cases} -k, \text{ if } s < -k, \\ \ \ s, \text{ if } |s| \leq k, \\ \ \ k, \text{ if } s > k. \end{cases}$$

The approximate problems of (1)–(5) are given by

$$\varphi_k \in L^2(V),\ \boldsymbol{A}_k \in L^2(\mathbb{X}),\ \theta_k \in L^2(H^1(\Omega)) \cap C\left([0,T]; L^2(\Omega)\right), \tag{13}$$

$$\begin{aligned} \frac{1}{k}\frac{\mathrm{d}\varphi_k}{\mathrm{d}t} - \nabla\cdot(\sigma_k(\theta_k)\nabla\varphi_k) &= i\lambda\omega\nabla\cdot(\sigma_k(\theta_k)\boldsymbol{A}_k) \\ &\quad + \nabla\cdot\left(\sigma_k(\theta_k)\nabla\varphi^0\right) \text{ in } \Omega_T = \Omega\times(0,T), \end{aligned} \tag{14}$$

$$\frac{1}{k}\frac{\mathrm{d}\boldsymbol{A}_k}{\mathrm{d}t} + \omega(i+\omega)\sigma_k(\theta_k)\boldsymbol{A}_k + (1-i\omega)L(\boldsymbol{A}_k) = -(1-i\omega)\sigma_k(\theta_k)\nabla\varphi_k \text{ in } D_T, \tag{15}$$

$$\varphi_k = 0 \text{ on } \Gamma_0\times(0,T)\ ,\ \frac{\partial\varphi_k}{\partial n} = -i\lambda\omega\boldsymbol{A}_k\cdot n \text{ on } \Gamma_1\times(0,T), \tag{16}$$

$$\boldsymbol{A}_k = 0 \text{ on } \partial D\times(0,T), \tag{17}$$

$$\varphi_k(\cdot,0) = 0 \text{ in } \Omega, \quad \boldsymbol{A}_k(\cdot,0) = 0 \text{ in } D, \tag{18}$$

$$\rho c_\varepsilon\frac{\mathrm{d}\theta_k}{\mathrm{d}t} - \nabla\cdot(\kappa(\theta_k)\nabla\theta_k) = F_k \text{ in } \Omega_T, \tag{19}$$

$$\frac{\partial\theta_k}{\partial n} = 0 \text{ on } \partial\Omega\times(0,T), \quad \theta_k(\cdot,0) = T_k(\theta_0) \text{ in } \Omega, \tag{20}$$

where $F_k = \dfrac{\sigma_k(\theta_k)}{2}T_k\left(|i\omega\boldsymbol{A}_k + \nabla\varphi_k|^2\right) + T_k(G)$ and $D_T = D\times(0,T)$.

For the system (13)–(20) it can be shown the following existence result [12].

Lemma 2 *For every $k \geq 1$, there exists a weak solution $(\varphi_k, \boldsymbol{A}_k, \theta_k)$ to problem (13)–(20).*

Remark 2 Since we are dealing with complex valued function spaces, the key point is to define the right bilinear elliptic form related to the system for $(\varphi_k, \boldsymbol{A}_k)$ for a given θ_k. From that point on, the proof of Lemma 2 is a straightforward application of J. L. Lions' theorem together with Schauder's fixed point theorem.

3.2 A Priori Estimates

For the solution of (13)–(20) it is easy to obtain the following estimates

$$\int_{\Omega_T} \sigma_k(\theta_k)|\boldsymbol{A}_k|^2 \leq \frac{C_2}{\omega^2}\int_{\Omega_T} \sigma_k(\theta_k)|\nabla\varphi_k|^2. \tag{21}$$

$$\int_0^T \|\boldsymbol{A}_k\|_{\mathbb{X}}^2 \leq \frac{C_2}{\alpha\omega}\int_{\Omega_T} \sigma_k(\theta_k)|\nabla\varphi_k|^2. \tag{22}$$

$$\int_{\Omega_T} \sigma_k(\theta_k)|\nabla\varphi_k|^2 \leq C_\lambda\|\varphi^0\|^2_{L^2(H^1(\Omega))}, \tag{23}$$

where

$$\lim_{\lambda\to(1-1/\omega)^-} C_\lambda = +\infty.$$

From these estimates we deduce

$$\left(\sigma_k(\theta_k)^{1/2}\boldsymbol{A}_k\right) \text{ is bounded in } L^2(\boldsymbol{L}^2(\Omega)), \tag{24}$$

$$(\boldsymbol{A}_k) \text{ is bounded in } L^2(\mathbb{X}).$$

On the other hand, since $\mathbb{X} \hookrightarrow \boldsymbol{L}^2(D)$ there exists a constant $C > 0$ such that $\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leq \|\boldsymbol{v}\|_{\boldsymbol{L}^2(D)} \leq C\|\boldsymbol{v}\|_{\mathbb{X}}$, for all $\boldsymbol{v} \in \mathbb{X}$. Thus,

$$(\boldsymbol{A}_k) \text{ is bounded in } L^2(\boldsymbol{L}^2(\Omega)).$$

From (23) and (24) it yields

$$(F_k) \text{ is bounded in } L^1(\Omega_T),$$

and thus, owing to (H.7), we obtain

$$(\theta_k) \text{ is bounded in } L^p(W^{1,p}(\Omega)), \text{ for all } 1 \le p < 5/4, \tag{25}$$

Remark 3 In [4] it was shown that (25) holds true when dealing with homogeneous Dirichlet boundary conditions. In the case of homogeneous Neumann boundary conditions, this result was shown by Clain in [6].

According to (H.1) and (25) we obtain that $(\kappa(\theta_k)\nabla\theta_k)$ is bounded in $L^p(\boldsymbol{L}^p(\Omega))$. Therefore $(\nabla \cdot (\kappa(\theta_k)\nabla\theta_k))$ is bounded in $L^1((W^{1,p'}(\Omega))')$. Since $1 \le p < 5/4$, Sobolev's embedding implies in particular that

$$L^1(\Omega) \hookrightarrow (W^{1,p'}(\Omega))'$$

and, in conclusion,

$$\left(\frac{\mathrm{d}\theta_k}{\mathrm{d}t}\right) \text{ is bounded in } L^1((W^{1,p'}(\Omega))'), \text{ for all } 1 \le p < 5/4. \tag{26}$$

3.3 Passing to the Limit

Choosing $1 \le q < p^* = 3p/(3-p)$, $X = W^{1,p}(\Omega)$, $B = L^q(\Omega)$ and $Y = (W^{1,p'}(\Omega))'$, and since the embeddings $X \hookrightarrow B$ and $B \hookrightarrow Y$ are continuous and compact, respectively, from Lemma 1 it yields that the space

$$\mathscr{W} = \left\{ v \in L^p(W^{1,p}(\Omega)) \,/\, \frac{\mathrm{d}v}{\mathrm{d}t} \in L^1((W^{1,p'}(\Omega))') \right\}$$

is compactly embedded in $L^p(L^q(\Omega))$. Moreover, since $1 \le p < 5/4$ and $1 \le q < 15/7$, and thanks to (25) and (26), we deduce that the sequence (θ_k) is relatively compact in $L^p(L^q(\Omega))$, for $1 \le p < \frac{5}{4}$ and $1 \le q < \frac{15}{7}$. Therefore, we may extract a subsequence, still denoted in the same way, such that $\theta_k \to \theta$ strongly in $L^p(L^q(\Omega))$ and almost everywhere in Ω_T. Consequently $\sigma_k(\theta_k) \rightharpoonup \sigma(\theta)$ in $L^\infty(\Omega_T)$-weak–$*$ and almost everywhere in Ω_T.

Since (θ_k) is bounded in $L^r(\Omega_T)$, for $1 \le r < 5/3$, and according to (H.1) it yields that $\big(\sigma_k(\theta_k)^{-1}\big)$ is bounded in $L^r(\Omega_T)$, for $1 \le r < 5/(3\alpha)$. Thus $(\nabla\varphi_k)$ is bounded in $L^r(\Omega_T)$, for $1 \le r < 10/(5+3\alpha)$, and, up to a subsequence, $\nabla\varphi_k \rightharpoonup \nabla\varphi$ in $L^r(\Omega_T)$, $\Phi = \sigma(\theta)^{1/2}\nabla\varphi$ in $L^2(\Omega_T)$.

As to $(\boldsymbol{A}_k)$, we deduce the existence of an element $\boldsymbol{A} \in L^2(\mathbb{X})$ such that, up to a subsequence, $\boldsymbol{A}_k \rightharpoonup \boldsymbol{A}$ weakly in $L^2(\boldsymbol{L}^2(\Omega))$, $\boldsymbol{A}_k \rightharpoonup \boldsymbol{A}$ weakly in $L^2(\mathbb{X})$, and thus $\sigma_k(\theta_k)^{1/2}\boldsymbol{A}_k \rightharpoonup \sigma(\theta)^{1/2}\boldsymbol{A}$ weakly in $L^2(\boldsymbol{L}^2(\Omega))$. Finally, by making $k \to \infty$ in (14) and (15) we obtain (8) and (9).

All the properties deduced up till now are not enough in order to assure the strong convergence of (F_k) in $L^1(\Omega_T)$. Nevertheless, there exists a Radon measure $\boldsymbol{\mu} \in \mathscr{M}(\Omega_T)$ such that $F_k \rightharpoonup \frac{1}{2}\sigma(\theta)|i\omega\mathbf{A} + \nabla\varphi|^2 + G + \boldsymbol{\mu}$ in $\mathscr{M}(\Omega_T)$-weak–$*$. We can pass to the limit in (19) to obtain (12).

Remark 4 Our future work consists in establishing under what conditions on σ can we assure that $\boldsymbol{\mu} = 0$ or, in other words, how can one derive the strong convergence $\sigma_k(\theta_k)^{1/2}\nabla\varphi_k \to \sigma(\theta)^{1/2}\nabla\varphi$ in $L^2(\Omega_T)$.

Remark 5 The analysis of the uniqueness of a solution to (6)–(12) is a very complex task even if we already know that $\boldsymbol{\mu} = 0$. This is related to the low regularity of the unknowns obtained in our existence result. Indeed, a system like (1)–(5) is a generalization of the so-called thermistor problem [1, 5, 9] which involves only two unknowns, namely, the electric potential and the temperature.

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