THE THERMISTOR PROBLEM WITH DEGENERATE THERMAL CONDUCTIVITY AND METALLIC CONDUCTION

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Abstract. The aim of this work is to establish the existence of a capacity solution to the thermistor problem supposing that the thermal and the electrical conductivities are not bounded below by a positive constant value. Furthermore, the thermal conductivity vanishes at points where the temperature is null. These assumptions on data include the case of practical interest of the Wiedemann–Franz law with metallic conduction and lead us to very complex mathematical situations.

1. **Introduction.** The heat produced by an electrical current passing through a conductor device is governed by the so–called thermistor problem. This problem consists in a coupled system of parabolic–elliptic equations, whose unknowns are the temperature inside the conductor, u, and the electrical potential, φ.

The conservation laws of both current and energy lead us to the equations that determine the thermistor problem, namely,

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \nabla\cdot(a(u)\nabla u) &= \sigma(u)|\nabla\varphi|^2 && \text{in } \Omega_T = \Omega\times(0,T),\\ \nabla\cdot(\sigma(u)\nabla\varphi) &= 0 && \text{in } \Omega_T,\\ u &= 0 && \text{on } \Gamma_T = \partial\Omega\times(0,T),\\ \varphi &= \varphi_0 && \text{on } \Gamma_T,\\ u(\cdot,0) &= u_0 && \text{in } \Omega, \end{aligned}\right\} \tag{1}$$

whereΩ , the domain occupied by the electrical device, is an open, bounded and smooth enough subset of $\mathbb{R}^N$, $N \geq 1$, $T > 0$ and a and σ are the thermal and electrical conductivities.

A great deal of attention has been paid by several authors in the study of the thermistor problem during the last two decades [1, 2, 14, 16, 21]. In all these works, and many others, it is assumed that $a \equiv a_0 \in \mathbb{R}$ or $a_0 \leq a(s) \leq a_1$, for all $s \in \mathbb{R}$. Also, σ is taken as a very smooth and bounded function. These hypotheses on the

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conductivities together with the regularity supposed on data yield to the existence of weak solutions, and even to the uniqueness of a regular enough weak solution.

In [7] the existence of weak solutions is established assuming that the thermal conductivity satisfies the Wiedemann-Franz law, that is, $a(u) = Lu\sigma(u)$, $L > 0$ being a constant value, and $\sigma \in C(\mathbb{R})$. This hypothesis on a leads to a very complex mathematical situation and it is precisely the main obstacle in the resolution of the problem, since $a(0) = 0$ and the parabolic equation becomes degenerate.

The Wiedemann–Franz law is also taken into account in [9], where two existence results weak solutions of the steady state of the thermistor problem are achieved. Firstly, a and σ are not bounded below far from zero, arising to a doubly non–uniformly elliptic system. Secondly, apart from the hypotheses above, it is assumed that the thermal conductivity blows up for a finite value of the temperature; in this way, the system becomes singular and non–uniformly.

The notion of capacity solution was introduced in [18], where the term $\nabla \cdot (a(u)\nabla u)$ is replaced by $\nabla \cdot a(\nabla u)$, $a : \mathbb{R}^N \to \mathbb{R}^N$ being a continuous operator such that $|a(\xi)| \leq C|\xi|$, for $|\xi| >> 1$, and $[a(\xi) - a(\eta)]\,(\xi - \eta) \geq \alpha|\xi - \eta|^2$, for all $\xi, \eta \in \mathbb{R}^N$, $\alpha > 0$. Also it is assumed that $0 < \sigma(s) \leq \bar{\sigma}$ for all $s \in \mathbb{R}$; it is just this hypothesis which hampers the resolution of the problem, because it enables that $\sigma(s) \to 0$ as $|s| \to \infty$. Then, if u is unbounded in Ω_T, the elliptic equation becomes degenerate when u is infinite and no a priori estimates of $\nabla\varphi$ are available and, therefore, φ may not belong to a Sobolev space. Then, instead of φ, $\Phi = \sigma(u)\nabla\varphi$ is considered as a single function and it is shown that $\Phi \in L^2(\Omega_T)^N$. This yields a new formulation of the original system and its solution is called capacity solution. Later on, this type of solution was used in [17, 19, 20].

The existence of capacity solutions is proved in [12], where $a : Q \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Leray–Lions operator which includes the particular case of the p–Laplacian operator, $p \geq 2$, and $\sigma \in C(\mathbb{R})$ is such that $0 < \sigma(s) \leq \bar{\sigma}$, for all $s \in \mathbb{R}$. In most situations of practical interest, σ satisfies this property and also $\sigma(s) \to 0$ as $|s|$ tends to infinity.

This paper is organized as follows. In section 2 we show some useful results for the development of the proof of the main result of this paper.

In section 3 we introduce the assumptions on data, the notion of capacity solution adapted to our setting and give the existence result.

Section 4 develops the proof of the main result, which is divided into several stages. First, we introduce a sequence of approximate problems and derive a priori estimates for the approximate solutions, (u_n, φ_n). Finally, we pass to the limit and conclude.

2. **Notation and some useful results.** Throughout this paper, we will denote $V = H_0^1(\Omega)$, $V' = H^{-1}(\Omega)$, $H = L^2(\Omega)$ and $\mathcal{V} = L^2(0,T;V)$, $\mathcal{V}' = L^2(0,T;V')$ and $\mathcal{H} = L^2(0,T;H)$. Also, for the sake of simplify the notation, we will write $L^p(X)$ instead of $L^p(0,T;X)$, X being a Banach space.

Now we introduce an interpolation result (see Proposition 3.1 in [5] pp. 7 and 8):

Lemma 1. *Let $r, p \geq 1$, and $v \in L^\infty(L^r(\Omega)) \cap L^p(W_0^{1,p}(\Omega))$. Then $v \in L^q(\Omega_T)$, with $q = p\frac{N+r}{N}$, and there exists a constant value $C = C(N,p,r) > 0$ such that*

$$\int_{\Omega_T} |v|^q \leq C \left(\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega |v|^r \right)^{p/N} \int_{\Omega_T} |\nabla v|^p.$$

Secondly, the following compacity result ([15]) will be required:

Lemma 2. *Let X, B and Y be three Banach spaces so that $X \hookrightarrow B \hookrightarrow Y$, every embedding being continuous and the inclusion $X \hookrightarrow B$ compact. For $1 \le p, q \le \infty$, let $\mathbf{W}$ be the Banach space defined as $\mathbf{W} = \left\{ v \in L^p(X) \,/\, \frac{\mathrm{d}v}{\mathrm{d}t} \in L^q(Y) \right\}$. Then, if $1 \le p, q < \infty$, the inclusion $\mathbf{W} \hookrightarrow L^p(B)$ holds and is compact. On the other hand, if $p = \infty$ and $q > 1$ the inclusion $\mathbf{W} \hookrightarrow C([0,T]; B)$ holds and is compact.*

The next lemma ([12, 18]) is a fundamental tool in order to obtain the strong convergence in $L^1(\Omega_T)$ of a suitable subsequence of certain approximate solutions.

Lemma 3. *Let (v_n) be a bounded sequence in $\mathcal{V}$ and relatively compact in $\mathcal{H}$. Then there exists a subsequence $(v_{n(k)}) \subset (v_n)$ such that, for every $\varepsilon > 0$, there are a constant value $K = K(\varepsilon) > 0$ and a function $\psi \in L^1(W^{1,1}(\Omega))$ satisfying*

$$0 \le \psi \le 1, \quad \|\psi - 1\|_{L^1(\Omega_T)} + \|\nabla \psi\|_{L^1(\Omega_T)^N} \le \varepsilon, \tag{2}$$

$$|v|,\, |v_{n(k)}| \le K \text{ in } \{\psi > 0\}, \text{ for all } k \ge 1, \tag{3}$$

where $v \in \mathcal{V}$ is such that $v_{n(k)} \rightharpoonup v$ weakly in $\mathcal{V}$.

3. **Definition of capacity solution and main result.** Our purpose in this paper is to study system (1) under the following hypotheses:

(H.1) $u_0 \in L^2(\Omega)$ is such that $u_0 \ge 0$ almost everywhere in Ω.
(H.2) $\varphi_0 \in L^2(H^1(\Omega)) \cap L^\infty(\Omega_T)$.
(H.3) $\sigma \in C(\mathbb{R})$ and $0 < \sigma(s) \le \sigma_0$, for all $s \in \mathbb{R}$.
(H.4) $a \in C(\mathbb{R})$ and $0 < a(s) \le a_0$, for all $s \ne 0$, $a(0) = 0$.
(H.5) For every $\delta > 0$ there exists a constant value $a_\delta > 0$ so that

$$\operatorname*{ess\,inf}_{|s| > \delta} a(s) \ge a_\delta.$$

Remark 1. In view of (H.4) and (H.5) we have that $a_\delta \to 0$ as $\delta \downarrow 0$.

This situation was already analyzed in [7] under the same assumptions above, except for a small but crucial detail: now the function σ is not bounded below by a positive constant. Consequently, the parabolic and the elliptic equations of (1) are degenerate and non–uniformly elliptic, respectively, and the survey of the problem becomes much more complex. In fact, the existence of a weak solution is not assured and we have to deal with the notion of the capacity solution (see [12, 18]).

Notice that (H.3)–(H.5) includes the cases physically important of metallic conduction, that is, $\sigma(u) = O(u^{-1})$ when $u \to \infty$, whereas $a(u)$ may be given by the Wiedemann-Franz law.

Now let $A(s) = \int_0^s a(\tau)\,\mathrm{d}\tau$. It is clear that $A(0) = 0$, $A \in C^1(\mathbb{R})$, A is strictly increasing and Lipschitz–continuous. Furthermore, $\nabla A(\phi) = a(\phi)\nabla\phi$ for all $\phi \in L^2(H^1(\Omega))$. Then, instead of system (1), we consider the problem

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \Delta A(u) &= \sigma(u)|\nabla\varphi|^2 && \text{in } \Omega_T, \\ \nabla \cdot (\sigma(u)\nabla\varphi) &= 0 && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ \varphi &= \varphi_0 && \text{on } \Gamma_T, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega. \end{aligned}\right\} \tag{4}$$

Remark 2. If $\varphi \in L^2(H^1(\Omega))$ is a solution of the elliptic equation, then it is very easy to check that $\nabla \cdot (\sigma(u)\varphi\nabla\varphi) = \sigma(u)|\nabla\varphi|^2$ in $L^2((V \cap L^\infty(\Omega))')$.

Definition 1. A triplet (u, φ, Φ) is said to be a capacity solution to problem (1) (or (4)) if the following conditions are fulfilled:

(C.1) $u \in L^\infty(L^1(\Omega))$, $\frac{\mathrm{d}u}{\mathrm{d}t} \in \mathcal{V}'$, $A(u) \in \mathcal{V} \cap L^q(\Omega_T)$, for all $q < 2+2/N$, $\varphi \in L^\infty(\Omega_T)$ and $\Phi \in L^\infty(H^N)$.

(C.2) (u, φ, Φ) verifies the system of differential equations

$$\frac{\mathrm{d}u}{\mathrm{d}t} - \Delta A(u) = \nabla \cdot (\varphi \Phi) \text{ and } \nabla \cdot \Phi = 0 \text{ in } \mathcal{V}'. \tag{5}$$

(C.3) For every $S \in W^{1,\infty}(\mathbb{R})$ with $\operatorname{supp} S$ compact, $S(A(u))\varphi - S(0)\varphi_0 \in \mathcal{V}$ and

$$S(A(u))\Phi = \sigma(u)\left[\nabla(S(A(u))\varphi) - \varphi \nabla S(A(u))\right]. \tag{6}$$

(C.4) $u(\cdot, 0) = u_0$.

Remark 3. A capacity solution basically differs from a weak solution in that, in the first case, $\nabla\varphi$ is considered in the almost everywhere sense whereas in the second one, $\nabla\varphi$ is regarded in the sense of distributions. However, Definition 1 is somewhat different from the one given in other works (for instance [12, 18]). More precisely, instead of (6), a capacity solution should verify $S(u)\Phi = \sigma(u)\left[\nabla(S(u)\varphi) - \varphi\nabla S(u)\right]$. But this condition is pointless because, in general, $u \notin \mathcal{V}$. That is the reason why we use $A(u)$ in (6) instead of u.

We remark that the definition of capacity solution given by Xu in [18] (or by the authors in [12]) and Definition 1 both lead to the following identification:

$$\Phi = \sigma(u)\nabla\varphi \text{ almost everywhere in } \Omega_T, \tag{7}$$

where $\nabla\varphi$ must be suitably defined pointwise almost everywhere in Ω (for more details, see [13]). This means that though a capacity solution involves three unknowns, namely, u, φ and Φ, there are in fact only two, since Φ is actually related to u and φ according to the identity (7).

The main result now follows.

Theorem 1. *Under the hypotheses (H.1)–(H.5), system (4) admits a capacity solution (u, φ, Φ) in the sense of Definition 1.*

Moreover, $u \geq 0$ almost everywhere in Ω_T, the gradient of u is defined almost everywhere in Ω_T and $\nabla u \chi_{\{u>\delta\}} \in L^2(\Omega_T)$ for all $\delta > 0$.

Finally, if $S \in L^1_{\mathrm{loc}}(\mathbb{R})$ is such that $S' \in L^\infty(\mathbb{R})$ and $\operatorname{supp} S \subset \mathbb{R} \setminus (-\delta_0, \delta_0)$ for some $\delta_0 > 0$, then $S(u) \in \mathcal{V}$ and $\nabla S(u) = S'(u)\nabla u$ in Ω_T.

4. **Proof of the main result.** The proof of Theorem 1 is divided in four steps: firstly, we consider a sequence of approximate problems. Then a priori estimates for the approximate solutions (u_n, φ_n) are derived. Finally, we pass to the limit and conclude. One of the major difficulties in this stage lies in showing the strong convergence of (φ_n) for a suitable subsequence.

4.1. **Approximate problems.** For every $n \in \mathbb{N}$ we define the regularized functions $a_n(s) = a(s) + \frac{1}{n}$ and $\sigma_n(s) = \sigma(s) + \frac{1}{n}$; also, set $A_n(s) = \int_0^s a_n(\tau)\mathrm{d}\tau = A(s) + \frac{s}{n}$. Then the approximate problem of (1) (or (4)) is defined as

$$\left.\begin{aligned} \frac{\partial u_n}{\partial t} - \nabla \cdot (a_n(u_n)\nabla u_n) &= \sigma_n(u_n)|\nabla\varphi_n|^2 && \text{in } \Omega_T, \\ \nabla \cdot (\sigma_n(u_n)\nabla\varphi_n) &= 0 && \text{in } \Omega_T, \\ u_n &= 0 && \text{on } \Gamma_T, \\ \varphi_n &= \varphi_0 && \text{on } \Gamma_T, \\ u_n(\cdot, 0) &= u_0 && \text{in } \Omega. \end{aligned}\right\} \tag{8}$$

By (H.3) and (H.4), respectively, $\frac{1}{n} \le \sigma_n(s) \le \sigma_0 + 1 = \sigma_1$ and $\frac{1}{n} \le a_n(s) \le a_0 + 1 = a_1$, for all $s \in \mathbb{R}$. Then, classic results ([1]) lead us to the existence of a weak solution (u_n, φ_n) to (8) such that

$$u_n \in \mathcal{V} \cap C^0([0,T]; L^2(\Omega)), \quad \frac{\mathrm{d}u_n}{\mathrm{d}t} \in \mathcal{V}', \quad \varphi_n - \varphi_0 \in L^\infty(V). \tag{9}$$

Furthermore, it is straightforward that

$$\|\varphi_n\|_{L^\infty(\Omega_T)} \le \|\varphi_0\|_{L^\infty(\Omega_T)}, \tag{10}$$

$$\int_\Omega \sigma_n(u_n)|\nabla \varphi_n|^2 \le C, \text{ a.e. } t \in (0,T). \tag{11}$$

On the other hand, in view of (H.1), it is easy to show that

$$u_n \ge 0 \text{ a.e. in } \Omega_T. \tag{12}$$

Moreover, it is shown that

$$\langle \sigma_n(u_n)|\nabla\varphi_n|^2, \phi\rangle_{\mathcal{V}',\mathcal{V}} = -\int_{\Omega_T} \sigma_n(u_n)\varphi_n \nabla\varphi_n \nabla\phi, \tag{13}$$

where $(\nabla \cdot (\sigma_n(u_n)\varphi_n\nabla\varphi_n))$ is bounded in $\mathcal{V}'$ thanks to (H.3), (10) and (11).

4.2. **Estimates of the approximate solutions.** Taking $A_n(u_n)$ as a test function in the parabolic equation of (8) and bearing in mind (13), we obtain

$$\int_0^t \int_\Omega |\nabla A_n(u_n)|^2 \le C, \text{ for all } t \in [0,T]. \tag{14}$$

Since the sequences $(\Delta A_n(u_n))$ and $(\nabla \cdot (\sigma_n(u_n)\varphi_n\nabla\varphi_n))$ are bounded in $\mathcal{V}'$,

$$\left(\frac{\mathrm{d}u_n}{\mathrm{d}t}\right) \text{ is bounded in } \mathcal{V}'. \tag{15}$$

Also, choosing as a test function $\frac{1}{\varepsilon}T_\varepsilon(u_n) \in \mathcal{V} \cap L^\infty(\Omega_T)$, with $\varepsilon > 0$, one has that

$$(u_n) \text{ is bounded in } L^\infty(0,T; L^1(\Omega)). \tag{16}$$

On the other hand, as $|\nabla A(u_n)| \le |\nabla A_n(u_n)|$, in view of (14) we have that

$$(A(u_n)) \text{ is bounded in } \mathcal{V}. \tag{17}$$

Moreover, $|A(s)| \le a_0|s|$ and $|A_n(s)| \le a_1|s|$, for all $s \in \mathbb{R}$, whereupon

$$(A(u_n)) \text{ and } (A_n(u_n)) \text{ are bounded in } L^\infty(L^1(\Omega)). \tag{18}$$

Thanks to (17) and (18), the above sequences satisfy Lema 1; then

$$(A(u_n)) \text{ and } (A_n(u_n)) \text{ are bounded in } L^{2+2/N}(\Omega_T). \tag{19}$$

For every $\delta > 0$, we define the function

$$g_\delta(s) = \begin{cases} s + \delta & \text{if } s < -\delta, \\ 0 & \text{if } |s| \le \delta, \\ s - \delta & \text{if } s > \delta. \end{cases}$$

Take $g_\delta(u_n) \in \mathcal{V}$ as a test function in the parabolic equation of (8); as $\nabla g_\delta(u_n) = g_\delta'(u_n)\nabla u_n = \chi_{\{|u_n|>\delta\}}\nabla u_n$, one has that $\nabla u_n \nabla g_\delta(u_n) = |\nabla g_\delta(u_n)|^2 = |\nabla u_n|^2$ in the set $\{|u_n| > \delta\}$. Hence, applying Young's inequality,

$$\int_{\Omega_T} |\nabla g_\delta(u_n)|^2 \le \frac{\|u_0\|^2_{L^2(\Omega)}}{2a_\delta} + \frac{1}{2}\int_{\Omega_T} |\nabla g_\delta(u_n)|^2 + \frac{\sigma_1\|\varphi_0\|^2_{L^\infty(\Omega_T)}}{2a_\delta^2}\int_{\Omega_T} \sigma_n(u_n)|\nabla\varphi_n|^2.$$

From (11) we have

$$\int_{\Omega_T} |\nabla g_\delta(u_n)|^2 \le C_\delta = a_\delta^{-1}\|u_0\|^2_{L^2(\Omega)} + a_\delta^{-2}\sigma_1\|\varphi_0\|_{L^\infty(\Omega_T)}CT. \tag{20}$$

We now introduce a regularized function of g_δ, namely, $\gamma_\delta \in C^\infty(\mathbb{R})$. To this end, let $\gamma \in C^\infty(\mathbb{R})$ be a function satisfying the following properties:

$$\gamma(s) = 0 \text{ if } s \in [0, 1/2],\ \gamma(s) = s - 1 \text{ if } s > 3/2,\ \gamma(s) \text{ is convex in } [0, +\infty),$$
$$\frac{\gamma(s)}{s} \text{ is increasing in } [1/2, 3/2] \text{ and } \gamma(-s) = -\gamma(s), \text{ for all } s \in \mathbb{R}.$$

Then for every $\delta \in (0, 1]$, $\gamma_\delta(s)$ is defined as $\gamma_\delta(s) = \delta\gamma(s/\delta)$. Notice that

$$0 \le \gamma_\delta(s) \le \gamma_{\delta'}(s), \text{ for all } s \ge 0 \text{ and } 0 < \delta \le \delta', \tag{21}$$

$$|\gamma'_\delta(s)| \le 1,\ |\gamma''_\delta(s)| \le \frac{K}{\delta}, \text{ for all } s \in \mathbb{R}. \tag{22}$$

For $\delta \in (0, 1]$ and $\phi \in \mathcal{D}(\Omega_T)$ we take $\gamma'_\delta(u_n)\phi$ as a test function in the parabolic equation of (8). Bearing in mind that $\nabla(\gamma'_\delta(u_n)\phi) = \gamma''_\delta(u_n)\nabla u_n\phi + \gamma'_\delta(u_n)\nabla\phi$ and after some calculations it yields

$$\left(\frac{\mathrm{d}\gamma_\delta(u_n)}{\mathrm{d}t}\right) \text{ is bounded in } L^1(\Omega_T) + \mathcal{V}', \text{ for every } \delta \in (0, 1]. \tag{23}$$

Furthermore, (20) and (22) imply that

$$(\gamma_\delta(u_n)) \text{ is bounded in } \mathcal{V}, \text{ for every } \delta \in (0, 1]. \tag{24}$$

Also, from (16) and the definition of γ_δ we obtain

$$\int_\Omega |\gamma_\delta(u_n)| \le \int_\Omega |u_n| \le C, \text{ a.e. } t \in (0, T), \text{ for all } n \in \mathbb{N} \text{ and } \delta \in (0, 1]. \tag{25}$$

We can then apply Lemma 1 and deduce

$$(\gamma_\delta(u_n)) \text{ is bounded in } L^{2+2/N}(\Omega_T), \text{ for every } \delta \in (0, 1]. \tag{26}$$

4.3. **Passing to the limit.** Applying Lemma 2 with $X = H_0^1(\Omega) \hookrightarrow B = H$, the inclusion being compact, and $Y = W^{-1,r'}(\Omega)$, for all $r' < \frac{N}{N-1}$ if $N \ge 2$, $r' < \infty$ if $N = 1$, one has that the space $\mathbf{V} = \left\{v \in \mathcal{V} \,/\, \frac{\mathrm{d}v}{\mathrm{d}t} \in L^1(W^{-1,r'}(\Omega))\right\}$ is such that the inclusion $\mathbf{V} \hookrightarrow L^2(\Omega_T)$ is compact. Thus, thanks to (23) and (24), $(\gamma_\delta(u_n))$ is relatively compact in $L^2(\Omega_T)$, for every $\delta \in (0, 1]$, wherefrom, for every $\delta \in (0, 1]$, there is a function $\Gamma_\delta \in L^2(\Omega_T)$ so that, up to a subsequence,

$$\gamma_\delta(u_n) \to \Gamma_\delta \text{ strongly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T. \tag{27}$$

Moreover, by (25) and (27), it is easy to check that

$$\int_\Omega |\Gamma_\delta| \le C, \text{ a.e. } t \in (0, T) \text{ and for all } \delta \in (0, 1]. \tag{28}$$

Also, (26) and (27) imply the convergences

$$\gamma_\delta(u_n) \rightharpoonup \Gamma_\delta \text{ weakly in } L^{2+2/N}(\Omega_T), \tag{29}$$

$$\gamma_\delta(u_n) \to \Gamma_\delta \text{ strongly in } L^q(\Omega_T), \text{ for all } q < 2 + \frac{2}{N}. \tag{30}$$

Since the sequence $(\gamma_\delta(u_n))_\delta$ is increasing as $\delta \downarrow 0$, also we have that $\Gamma_\delta \le \Gamma_{\delta'}$ almost everywhere in Ω_T, for all $\delta' \le \delta$, that is, the sequence (Γ_δ) is increasing too as $\delta \downarrow 0$. Consequently, there exists a measurable function $u : \Omega_T \mapsto \mathbb{R}$ such that

$\lim_{\delta\downarrow 0}\Gamma_\delta = u$ almost everywhere in Ω_T; and by applying the monotone convergence theorem, using (28), we deduce that $u \in L^1(\Omega_T)$ and

$$\Gamma_\delta \to u \text{ strongly in } L^1(\Omega_T) \text{ and a.e. in } \Omega_T \tag{31}$$

up to a subsequence. Moreover, according to (28) and (31), we may conclude that

$$u \in L^\infty(L^1(\Omega)). \tag{32}$$

Also, taking into account the definition of γ_δ, we obtain

$$u_n \to u \text{ strongly in } L^1(\Omega_T) \text{ and a.e. in } \Omega_T, \tag{33}$$

which implies that $\gamma_\delta(u_n) \to \gamma_\delta(u)$ almost everywhere in Ω_T and, then, $\gamma_\delta(u) = \Gamma_\delta$.

Notice that (24) yields $u \geq 0$. Also (20) and (33) imply in particular that

$$g_\delta(u) \in \mathcal{V} \text{ for all } \delta \in (0,1]. \tag{34}$$

Accordance with the definition of g_δ and (34), we may define the gradient of u, at least in the almost everywhere sense. Indeed,

$$\nabla g_\delta(u) = \nabla u \chi_{\{u>\delta\}} \in L^2(\Omega_T) \text{ for all } \delta \in (0,1]. \tag{35}$$

Expression (35) allows us to define ∇u in $\{u > 0\}$; in the level set $\{u = 0\}$ we just put $\nabla u = 0$.

Estimates derived till now lead us to the following convergences for suitable subsequences:

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} \rightharpoonup \frac{\mathrm{d}u}{\mathrm{d}t} \text{ weakly in } \mathcal{V}', \tag{36}$$

$$A(u_n),\, A_n(u_n) \rightharpoonup A(u) \text{ weakly in } \mathcal{V} \text{ and a.e. in } \Omega_T, \tag{37}$$

$$A_n(u_n),\, A(u_n) \rightharpoonup A(u) \text{ weakly in } L^{2+2/N}(\Omega_T), \tag{38}$$

$$A_n(u_n),\, A(u_n) \to A(u) \text{ strongly in } L^q(\Omega_T), \text{ for all } q < 2 + \frac{2}{N}, \tag{39}$$

$$\varphi_n \rightharpoonup \varphi \text{ weakly--}* \text{ in } L^\infty(\Omega_T), \tag{40}$$

$$\sigma_n(u_n)\nabla\varphi_n \rightharpoonup \Phi \text{ weakly--}* \text{ in } L^\infty(H^N), \tag{41}$$

$$\sigma_n(u_n) \rightharpoonup \sigma(u) \text{ weakly--}* \text{ in } L^\infty(\Omega_T) \text{ and a.e. in } \Omega_T, \tag{42}$$

where φ and Φ are some functions belonging to the spaces $L^\infty(\Omega_T)$ and $L^\infty(H^N)$, respectively. All these convergences lead us directly to conditions (C.1) and (C.2) of Definition 1. Moreover; notice that we can not guarantee the convergence $\varphi_n \to \varphi$ almost everywhere in Ω_T, up to a subsequence. Nevertheless, the strong convergence in $L^1(\Omega_T)$ holds for a suitable subsequence.

4.3.1. *Strong convergence of* (φ_n) *in* L^1. Along this section we show the existence of a subsequence $(\varphi_{n(k)}) \subset (\varphi_n)$ such that

$$\varphi_{n(k)} \to \varphi \text{ strongly in } L^1(\Omega_T). \tag{43}$$

To this end, let $S \in W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp} S$ is compact. Then, for a suitable subsequence, still denoted in the same way, one has

$$S(\gamma_\delta(u_n))\,\varphi_n \rightharpoonup S(\gamma_\delta(u))\,\varphi \text{ weakly in } L^2(H^1(\Omega)), \text{ for every } \delta \in (0,1]. \tag{44}$$

Now consider $S \in W^{1,\infty}(\mathbb{R})$ such that $\operatorname{supp} S$ is compact and $0 \leq S \leq 1$. Then, there exists a constant value $C_0 > 0$, independent of S, such that

$$\begin{aligned}\limsup_{n\to\infty} \int_{\Omega_T} \sigma_n(u_n) \left|\nabla\left[S\left(\gamma_\delta(u_n)\right)\varphi_n - S(\gamma_\delta(u))\varphi\right]\right|^2 \\ \leq C_0 C_{\delta/2}^{1/2} \|S'\|_\infty (1 + C_{\delta/2}^{1/2}\|S'\|_\infty), \text{ for every } \delta > 0. \end{aligned} \tag{45}$$

On the other hand, it is easy to check that, for every $\delta \in (0,1]$, the sequence $(\gamma_\delta(u_n))$ verifies the hypotheses of Lemma 3. Thus, taking into account (44) and (45), it can be shown that there exists a subsequence $(\varphi_{n(k)}) \subset (\varphi_n)$ such that

$$\lim_{k\to\infty} \int_{\Omega_T} |\varphi_{n(k)} - \varphi| = 0.$$

Remark 4. Since $|\varphi_{n(k)}| \leq \|\varphi_0\|_{L^\infty(\Omega_T)}$ for all $k \geq 1$, the strongly convergence in $L^1(\Omega_T)$, together with the almost everywhere convergence, imply that $\varphi_{n(k)} \to \varphi$ strongly in $L^p(\Omega_T)$ for all $p < \infty$.

4.3.2. *For all $S \in W^{1,\infty}(\mathbb{R})$ with* $\operatorname{supp} S$ *compact,* $S(A(u_{n(k)}))\varphi_{n(k)} \to S(A(u))\varphi$ *weakly in* $L^2(H^1(\Omega))$. We may assume that $\varphi_{n(k)} \to \varphi$ almost everywhere in Ω_T. Let $v_k = A(u_{n(k)}) \in \mathcal{V}$. Clearly, $S(v_k)\varphi_{n(k)} \in L^2(H^1(\Omega))$ and thanks to (33) we have $S(v_k)\varphi_{n(k)} \to S(A(u))\varphi$ almost everywhere in Ω_T. Thus it is enough to show that $S(v_k)\varphi_{n(k)}$ is bounded in $L^2(H^1(\Omega))$.

4.3.3. *Condition (C.3) of Definition 1.* Let $S \in W^{1,\infty}(\mathbb{R})$, $\operatorname{supp} S$ being compact, and consider the identity

$$\begin{aligned} S(A(u_{n(k)}))\sigma_{n(k)}(u_{n(k)})\nabla\varphi_{n(k)} \\ = \sigma_{n(k)}(u_{n(k)})\left\{\nabla\left[S(A(u_{n(k)}))\varphi_{n(k)}\right] - \varphi_{n(k)}\nabla S(A(u_{n(k)}))\right\}. \end{aligned} \tag{46}$$

Using (41), the strong convergence of $(\varphi_{n(k)})$ in $L^2(\Omega_T)$, the weakly convergence of $S(A(u_{n(k)}))$ and $S(A(u_{n(k)}))\varphi_{n(k)}$ in $L^2(H^1(\Omega))$, it is straightforward that we can pass to the limit in (46), which yields the desired identity (6).

4.3.4. *Condition* $u(0) = u_0$. Owing to (C.1), $u \in \mathbf{W} = \left\{v \in L^\infty(L^1(\Omega)) \,/\, \frac{\mathrm{d}v}{\mathrm{d}t} \in \mathcal{V}'\right\}$. Also we have $\mathbf{W} \subset \mathbf{W}_r \subset C([0,T]; W^{-1,r'}(\Omega))$, every inclusion being continuous, and $\mathbf{W}_r$ the space defined as $\mathbf{W}_r = \left\{v \in L^\infty(W^{-1,r'}(\Omega)) \,/\, \frac{\mathrm{d}v}{\mathrm{d}t} \in L^2(W^{-1,r'}(\Omega))\right\}$. This means that we may expect that the initial condition $u(0) = u_0$ makes sense at least in $W^{-1,r'}(\Omega)$, for all $r' < \frac{N}{N-1}$ if $N \geq 2$, $r' < \infty$ if $N = 1$.

4.3.5. *More regularity on u.* Now, we show the last assertion of Theorem 1. To do so, let $S \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $S' \in L^\infty(\mathbb{R})$ and $\operatorname{supp} S \subset \mathbb{R} \setminus (-\delta_0, \delta_0)$ for some $\delta_0 > 0$. Consider the functions $S(g_\delta(u)) \in \mathcal{V}$ for $\delta \in (0, \delta_0)$. Then, $S(g_\delta(u)) \to S(u)$ when $\delta \downarrow 0$ almost everywhere in Ω_T. On the other hand, using (35),

$$|\nabla S(g_\delta(u))| \leq \|S'\|_\infty |\nabla g_\delta(u)| \chi_{\{u > \delta_0\}} = \|S'\|_\infty |\nabla u| \chi_{\{u > \delta_0\}},$$

and thus, $S(g_\delta(u))$ is bounded in $\mathcal{V}$ for $\delta \in (0, \delta_0)$.

This ends the proof of theorem 1.

4.4. **Identification of** $\nabla A(u)$**.** Since in general $u \notin \mathcal{V}$, we cannot assure that $\nabla A(u) = a(u)\nabla u$ in Ω_T. This is due to the fact that A^{-1} is not globally Lipschitz-continuous. However, if (u, φ, Φ) is a capacity solution obtained in Theorem 1, it can be shown that $a(u)\nabla u \in L^2(\Omega_T)^N$ and $\nabla A(u) = a(u)\nabla u$ in Ω_T.

5. **Concluding remarks.** The assumptions on the diffusion coefficients include as a particular case the Wiedemann–Franz law and also metallic conduction. In previous works by the authors ([6]-[11]) only one of these two hypotheses was assumed. From the mathematical point of view, these two simultaneous assumptions have led to a very complex situation. The existence of a capacity solution (u, φ, Φ) to system (1) has been established in the preceding sections. The functions u and φ verify the thermistor problem in divergence form, that is,

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \nabla \cdot (a(u)\nabla u) &= \nabla \cdot (\sigma(u)\varphi\nabla\varphi) && \text{in } \Omega_T, \\ \nabla \cdot (\sigma(u)\nabla\varphi) &= 0 && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ \varphi &= \varphi_0 && \text{on } \Gamma_T, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}\right\} \tag{47}$$

where $u \in L^\infty(L^1(\Omega))$, $\nabla\varphi$ may be defined in an almost everywhere sense (see Remark 3 and [13]), $\frac{\mathrm{d}u}{\mathrm{d}t} \in \mathcal{V}'$, $\varphi \in L^\infty(\Omega_T)$, $\sigma(u)\nabla\varphi \in L^\infty(0,T;L^2(\Omega))$, $\nabla u \chi_{\{u>\delta\}} \in L^2(\Omega_T)$ for all $\delta > 0$, and $a(u)\nabla u \in L^2(\Omega_T)$.

We may consider if the identity $\nabla \cdot (\sigma(u)\varphi\nabla\varphi) = \sigma(u)|\nabla\varphi|^2$ holds true in a certain sense. It is very well known that this is true in $N = 1$, or when u or φ are smooth enough, for instance, $u \in L^\infty(\Omega_T)$, or $\varphi \in L^2(0,T;H^1(\Omega))$. In the general case $N \geq 2$, this question is far from being trivial.

The uniqueness of capacity solutions to system (1) has not been analyzed in this paper. Notice that here the assumptions on data are weaker than the ones considered in other works where uniqueness is established ([1, 2, 3, 14, 16, 21]). In those works there is no need to search for capacity solutions: the regularity of the solutions leads to the usual setting of weak solutions.

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