RENORMALIZED SOLUTIONS TO A NONLINEAR PARABOLIC-ELLIPTIC SYSTEM*

MARÍA TERESA GONZÁLEZ MONTESINOS† AND FRANCISCO ORTEGÓN GALLEGO‡

Abstract. The aim of this paper is to show the existence of renormalized solutions to a parabolic-elliptic system with unbounded diffusion coefficients. This system may be regarded as a modified version of the well-known thermistor problem; in this case, the unknowns are the temperature in a conductor and the electrical potential.

Key words. renormalized solutions, nonlinear elliptic equations, nonlinear parabolic equations, weak solutions, Caratheodory functions, thermistor problem, Sobolev spaces

1. Introduction. This paper is concerned with the resolution of the nonlinear parabolic-elliptic system

$$
\left\{
\begin{array}{ll}
\dfrac{\partial u}{\partial t} - \nabla\cdot(a(u)\nabla u) = \sigma(u)|\nabla\varphi|^2 & \text{in } Q=\Omega\times(0,T),\\
-\nabla\cdot(\sigma(u)\nabla\varphi) = \nabla\cdot F(u) & \text{in } Q,\\
u=0 & \text{on } \partial\Omega\times(0,T),\\
\varphi=0 & \text{on } \partial\Omega\times(0,T),\\
u(\cdot,0)=u_0 & \text{in } \Omega,
\end{array}
\right. \tag{1}
$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain, $T>0$, $a(x,t,s)$, $\sigma(x,t,s)$, and $F(x,t,s)$, $F=(F_1,\dots,F_N)'$, are Caratheodory functions defined in $Q\times\mathbb{R}$. This problem has a similar structure to the so-called thermistor problem arising in electromagnetism ([4, 12]); in that particular context, Ω stands for the domain occupied by the thermistor, u is the temperature, u_0 the initial temperature, φ is a shifted electric potential, $F(x,t,s)=\sigma(s)\nabla\varphi_0(x,t)$, φ_0 is a given function, and σ is a continuous and bounded function. Indeed, the actual electric potential is $\psi=\varphi+\varphi_0$, and thus φ_0 is the electric potential Dirichlet boundary data on $\partial\Omega\times(0,T)$. In our analysis, and from a mathematical standpoint, we will consider more general functions $F(x,t,s)$.

A great deal of attention has been paid to the thermistor problem during the last two decades by several authors ([2, 4, 13, 26], etc.). In these works, many situations and different hypotheses have been considered, but both a and σ are assumed to be bounded in all these referred works.

The goal of this paper is to analyze problem (1) in the case of nonbounded diffusion coefficients a and σ. Moreover, no asymptotic behavior on a, σ, and F is assumed.

Under these general assumptions, one readily realizes that weak solutions (in the sense of distributions) are not well suited in this context. Note that even if u or φ

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†Departamento de Matemáticas, Facultad de Ciencias Económicas y Empresariales, Universidad de Cádiz, 11002 Cádiz, Spain (mariateresa.gonzalez@uca.es).

‡Departamento de Matemáticas, Facultad de Ciencias del Mar, Universidad de Cádiz, 11510 Puerto Real, Cádiz, Spain (francisco.ortegon@uca.es).

belong to some Banach space of the form $L^q(W^{1,q}(\Omega))$, the terms $a(u)\nabla u$, $\sigma(u)\nabla\varphi$, or $F(u)$ may not belong to any $L^r(Q)$ space, $r \geq 1$. For this reason, we consider the notion of renormalized solutions adapted to our setting. The concept of renormalized solution was fist introduced by DiPerna and Lions ([15, 16]) in the framework of the Fokker–Plank–Boltzmann equations; later on, it was applied to more general situations (for instance, in the resolution of nonlinear elliptic equations ([9, 22, 23]), or in the resolution of nonlinear parabolic equations ([6, 7, 8])).

The fact that a and σ are unbounded is not the only difficulty we may encounter in the resolution of problem (1). Indeed, the parabolic equation needs a special treatment due to the nonlinear right-hand side belonging to $L^1(Q)$.

In order to solve problem (1) under the assumptions stated below, we use truncation and approximate solutions. This work is organized as follows.

In section 2, we set up the notation used in the paper; this leads to the introduction of some functional spaces. We also recall certain compactness results and give an existence theorem for problem (1) in the case of bounded data.

Section 3 enumerates the hypotheses and introduces the concept of renormalized solution adapted to our context. Finally, we give the existence result.

Section 4 develops the proof of the existence result; it is split into three steps, namely: setting of approximate problems, derivation of estimates, and passing to the limit and conclusion.

2. Notation and functional spaces. Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be an open bounded domain, and $\partial\Omega$ its boundary. Then we define $\mathcal{D}(\Omega)$ as the space of all C^∞–functions in Ω with compact support.

For $p \in [1, +\infty]$, let $W^{1,p}(\Omega)$ be the first order Sobolev space given as

$$W^{1,p}(\Omega) = \left\{ v \in L^p(\Omega) \ / \ \nabla v \in L^p(\Omega)^N \right\},$$

where the gradient $\nabla v = \left(\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_N} \right)'$ is taken in the sense of distributions (here, the prime symbol stands for vector transposition). It is well-known that $W^{1,p}(\Omega)$ is a Banach space with norm

$$\begin{aligned} \|v\|_{W^{1,p}(\Omega)} &= \left(\|v\|^p_{L^p(\Omega)} + \|\nabla v\|^p_{L^p(\Omega)^N} \right)^{1/p}, \quad p \in [1, +\infty), \\ \|v\|_{W^{1,\infty}(\Omega)} &= \|v\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)^N}; \end{aligned}$$

moreover, if $p = 2$, then we write $H^1(\Omega) = W^{1,2}(\Omega)$, which is a Hilbert space.

Since we deal with homogenous Dirichlet boundary conditions, it is interesting to introduce the space $W_0^{1,p}(\Omega)$ defined as the closure of $\mathcal{D}(\Omega)$ with respect to $\|\cdot\|_{W^{1,p}(\Omega)}$, that is,

$$W_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1,p}(\Omega)}, \quad p \in [1, +\infty).$$

It is known that if $\partial\Omega$ is smooth enough (for instance, Lipschitz continuous), $W_0^{1,p}(\Omega)$ is characterized by the following property:

$$W_0^{1,p}(\Omega) = \{ v \in W^{1,p}(\Omega) \ / \ v_{|\partial\Omega} = 0 \}, \quad p \in [1, +\infty).$$

Also we put $H_0^1(\Omega) = W_0^{1,2}(\Omega)$. $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$ are, respectively, Banach and Hilbert spaces. By Poincaré's inequality, the seminorm $|v|_{W^{1,p}(\Omega)} = \|\nabla v\|_{L^p(\Omega)^N}$ is a norm in $W_0^{1,p}(\Omega)$ equivalent to $\|\cdot\|_{W^{1,p}(\Omega)}$ on $W_0^{1,p}(\Omega)$. The space $W^{-1,p'}(\Omega)$ stands for the dual space of $W_0^{1,p}(\Omega)$, $p \in [1,+\infty)$.

We now introduce some notation according to the parabolic equation of (1). For a Banach space X and $1 \le p \le +\infty$, let $L^p(X)$ denote the space $L^p([0,T];X)$, that is, the set of (equivalence class of) measurable functions $f:[0,T] \to X$ such that $t \in [0,T] \mapsto \|f(t)\|_X$ is in $L^p(0,T)$. If $f \in L^p(X)$, we define

$$\|f\|_{L^p(X)} = \left(\int_0^T \|f(t)\|_X^p\right)^{1/p}, \ 1 \le p < +\infty, \quad \|f\|_{L^\infty(X)} = \operatorname*{ess\,sup}_{t\in[0,T]} \|f(t)\|_X;$$

and thus $\big(L^p(X), \|\cdot\|_{L^p(X)}\big)$ is a Banach space. By Fubini's theorem we can identify the space $L^p(L^p(\Omega))$ with $L^p(Q)$, Q being the cylinder $\Omega \times (0,T)$.

Let X and Y be two Banach spaces, $X \hookrightarrow Y$ with continuous inclusion, and set

$$W = \left\{ v \in L^p(X) \;/\; \frac{\mathrm{d}v}{\mathrm{d}t} \in L^q(Y) \right\}, \ p, q \in [1,+\infty],$$

provided with the standard norm $\|w\|_W = \|w\|_{L^p(X)} + \left\|\frac{\mathrm{d}v}{\mathrm{d}t}\right\|_{L^q(Y)}$. Then $(W, \|\cdot\|_W)$ is a Banach space and the inclusion $W \hookrightarrow C^0([0,T];Y)$ holds and is continuous. However, it will be very interesting and useful to know if a particular compactness embedding involving these spaces holds. The answer is given by the following two lemmas ([24]).

Lemma 1. *Let X, B, and Y be three Banach spaces such that $X \hookrightarrow B \hookrightarrow Y$, every embedding being continuous and the inclusion $X \hookrightarrow B$ compact. Let $1 \le p < +\infty$ and $1 \le q \le +\infty$. Then, the inclusion $W \hookrightarrow L^p(B)$ holds and is compact.*

Lemma 2. *Let X, B and Y be as in Lemma 1, and $E \subset L^\infty(X)$ be a bounded set such that*

(i) $\frac{\mathrm{d}v}{\mathrm{d}t} \in L^1(Y)$ *for all $v \in E$, and*
(ii) *there exist $h \in L^1(0,T)$, $s > 1$ and a bounded set $\mathcal{Z} \subset L^s(0,T)$ such that*

$$\left\|\frac{\mathrm{d}v}{\mathrm{d}t}\right\|_Y \le h + z_v, \ \textit{for all } v \in E, \ z_v \in \mathcal{Z} \textit{ and a.e. in } (0,T).$$

Then, E is relatively compact in $C^0([0,T];B)$.

The approximate problems in section 4.1 are defined via truncation functions. For this purpose, we introduce, for each $j > 0$ in $\mathbb{R}$, the truncation function at height j to be

$$T_j(s) = \operatorname{sign}(s)\min(j,|s|), \quad \operatorname{sign}(s) = \begin{cases} 0 & \text{if } s = 0, \\ s/|s| & \text{if } s \ne 0. \end{cases} \tag{2}$$

We will also make use of the following lemma, due to Boccardo and Gallouët ([10]) and ([19]).

Lemma 3. *Let (v_n) be a sequence of measurable functions in Q such that*

1. *(v_n) is bounded in $L^\infty(L^1(\Omega))$.*
2. *For all $j > 0$, $n \ge 0$, $T_j(v_n) \in L^2(H_0^1(\Omega))$.*
3. *There exists a constant $C > 0$ such that*

$$\int_{\{m \le |v_n| < m+1\}} |\nabla v_n|^2 \le C \textit{ for all } m, n \ge 0.$$

Then (v_n) *is bounded in the space* $L^q(W^{1,q}(\Omega))$ *for all* $q < \frac{N+2}{N+1}$ *if* $N \geq 2$, *and for all* $q < 2$ *if* $N = 1$.

If $g : Q \times \mathbb{R}$ is a Caratheodory function and u is measurable in Q, we write $g(u)$ for the measurable function in Q defined as $(x,t) \in Q \mapsto g(x,t,u(x,t))$.

In what follows, $C > 0$ stands for generic constant values which only depend on initial data.

The introduction of the approximate solutions relies on the following result.

Theorem 4. *Assume that the Caratheodory functions* a, σ *and* F *are such that* $a, \sigma \in L^\infty(Q \times \mathbb{R})$, $F \in L^\infty(Q \times \mathbb{R})^N$ *and there exist two constant values* $a_0 > 0$ *and* σ_0 *satisfying*

$$a(x,t,s) \geq a_0,\ \sigma(x,t,s) \geq \sigma_0,\ \textit{for all } s \in \mathbb{R},\ \text{a.e. } (x,t) \in Q.$$

Finally, let $u_0 \in L^2(\Omega)$. *Then, for every* $j > 0$, *there exists* $u \in L^2(H_0^1(\Omega))$ *and* $\varphi \in L^\infty(H_0^1(\Omega))$ *such that*

$$\frac{\mathrm{d}u}{\mathrm{d}t} \in L^2(H^{-1}(\Omega)), \quad u(\cdot,0) = u_0 \ \textit{in } \Omega,$$

and

$$(3)\begin{cases} \displaystyle\int_0^T \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, v \right\rangle + \int_Q a(u)\nabla u \nabla v = \int_Q T_j\left(\sigma(u)|\nabla\varphi|^2\right) v,\ \textit{for all } v \in L^2(H_0^1(\Omega)), \\ \displaystyle\int_\Omega \sigma(u)\nabla\varphi\nabla\psi = \int_\Omega F(u)\nabla\psi,\ \textit{for all } \psi \in H_0^1(\Omega),\ \text{a.e. } t \in (0,T). \end{cases}$$

For the proof of this result one may follow the same arguments as in the proof of the existence theorem for the thermistor problem ([4]).

3. The main result. We make the following assumptions:

(H.1) $a, \sigma : Q \times \mathbb{R} \to \mathbb{R}$ and $F : Q \times \mathbb{R} \to \mathbb{R}^N$ are Caratheodory functions and there exists a nondecreasing function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\max\left(a(x,t,s), \sigma(x,t,s), |F(x,t,s)|\right) \leq \gamma(|s|),\ \text{for all } s \in \mathbb{R},\ \text{a.e. in } Q.$$

(H.2) There exist two constant values $a_0 > 0$ and $\sigma_0 > 0$ such that

$$a(x,t,s) \geq a_0,\ \sigma(x,t,s) \geq \sigma_0,\ \text{for all } s \in \mathbb{R},\ \text{a.e. in } Q.$$

(H.3) There exists a function $\Gamma \in L^1(Q)$ such that

$$|F(x,t,s)|^2 \leq \Gamma(x,t)\sigma(x,t,s),\ \text{for all } s \in \mathbb{R},\ \text{a.e. in } Q.$$

(H.4) $\displaystyle\max_{k \leq |s| \leq 2k} \operatorname*{ess\,sup}_Q \frac{1}{k}\frac{\sigma(x,t,s)}{a(x,t,s)} = \omega(k)$ as $k \to +\infty$, where $\omega(k)$ stands for a null sequence, that is, $\lim_{k\to\infty}\omega(k) = 0$.

(H.5) $u_0 \in L^1(\Omega)$.

Hypothesis (H.1) is one of the main difficulties in the resolution of problem (1). As it has been stated in section 1, we cannot expect to search for weak solutions. However, assumptions (H.3) and (H.4) give a relation of the asymptotic behavior of $a(s)$, $\sigma(s)$ and $F(s)$ for large values of s.

We introduce now the definition of renormalized solutions to problem (1).

Definition 5. *A couple of functions* (u,φ) *is called a renormalized solution to problem* (1) *if the following conditions are fulfilled:*

(R.1) $u \in L^1(\Omega)$, $\varphi \in L^2(H_0^1(\Omega))$, *and* $\int_Q \sigma(u)|\nabla\varphi|^2 < +\infty$;

(R.2) $T_M(u) \in L^2(H_0^1(\Omega))$ *for all* $M > 0$;

(R.3) $\displaystyle\lim_{n\to\infty} \int_{\{n\le|u|<n+1\}} a(u)\nabla u\nabla u = 0$;

(R.4) *For all* $S \in C^\infty(\mathbb{R})$ *with* supp S' *compact,*

$$\frac{\partial S(u)}{\partial t} - \nabla\cdot[a(u)\nabla u S'(u)] + S''(u)a(u)\nabla u\nabla u = \sigma(u)|\nabla\varphi|^2 S'(u) \ \textit{in}\ \mathcal{D}'(Q),$$
$$S(u(\cdot,0)) = S(u_0) \ \textit{in}\ \Omega;$$

(R.5) *For all* $\psi \in L^2(H_0^1(\Omega))$ *such that* $\int_Q \sigma(u)|\nabla\psi|^2 < +\infty$, *we have*

$$\int_Q \sigma(u)\nabla\varphi\nabla\psi = -\int_Q F(u)\nabla\psi.$$

Remark. Properties (R.1)–(R.4) on u are the usual conditions verified by renormalized solutions of parabolic equations ([7]). On the other hand, (R.5) says in particular that the set of test functions in the equation for φ depends upon the solution u.

We can now state the main result of this work.

THEOREM 6. *Under hypotheses* (H.1)–(H.5), *system* (1) *admits a renormalized solution* (u,φ) *in the sense of Definition* 5.

4. Proof of Theorem 6. The proof is divided into three steps: first, we introduce a sequence of approximate problems; then, we derive certain estimates for the approximate solutions; and finally, we pass to the limit and conclude.

4.1. Setting of the approximate problems. For every $j > 0$, we consider the truncation functions defined by

$$a_j(x,t,s) = a(x,t,T_j(s)), \quad \sigma_j(x,t,s) = \sigma(x,t,T_j(s)), \quad F_j(x,t,s) = F(x,t,T_j(s)),$$

where T_j is defined in (2). Thanks to $a_j, \sigma_j \in L^\infty(Q\times\mathbb{R})$ and $F_j \in L^\infty(Q\times\mathbb{R})^N$.

The approximate problems are stated as follows: to find $u_j \in L^2(H_0^1(\Omega))$ and $\varphi_j \in L^\infty(H_0^1(\Omega))$ such that $\frac{\mathrm{d}u_j}{\mathrm{d}t} \in L^2(H^{-1}(\Omega))$, $u_j(\cdot,0) = T_j(u_0)$ in Ω and

$$(4)\quad \begin{cases} \displaystyle\int_0^T \left\langle \frac{\mathrm{d}u_j}{\mathrm{d}t}, v\right\rangle + \int_Q a_j(u_j)\nabla u_j\nabla v = \int_Q T_j\left(\sigma_j(u_j)|\nabla\varphi_j|^2\right) v, \ \text{for all } v \in L^2(H_0^1(\Omega)), \\ \displaystyle\int_\Omega \sigma_j(u_j)\nabla\varphi_j\nabla\psi = -\int_\Omega F_j(u_j)\nabla\psi, \ \text{for all } \psi\in H_0^1(\Omega), \text{ a.e. } t\in(0,T). \end{cases}$$

By virtue of Theorem 4, we know that for each $j > 0$, there exists (u_j,φ_j) verifying all these conditions.

4.2. Estimates for (u_j) and (φ_j). Choosing $\psi = \varphi_j$ in the equation for φ_j and integrating over Q yields,

$$\int_Q \sigma_j(u_j)|\nabla\varphi_j|^2 = -\int_Q F_j(u_j)\nabla\varphi_j \le \left(\int_Q \sigma_j(u_j)^{-1}|F_j(u_j)|^2\right)^{1/2}\left(\int_Q \sigma_j(u_j)|\nabla\varphi_j|^2\right)^{1/2},$$

hence, using (H.3),

$$(5)\qquad \int_Q \sigma_j(u_j)|\nabla\varphi_j|^2 \le \int_Q \sigma_j(u_j)^{-1}|F_j(u_j)|^2 \le \int_Q \Gamma = C.$$

In this way, the sequence $\left(\sigma_j(u_j)|\nabla\varphi_j|^2\right)$ is bounded in $L^1(Q)$. We may rewrite the parabolic equation of (4) as

$$\text{(6)}\qquad \begin{cases} \displaystyle\int_0^T \left\langle \frac{\mathrm{d}u_j}{\mathrm{d}t}, v\right\rangle + \int_Q a_j(u_j)\nabla u_j \nabla v = \int_Q f_j v, \ \text{for all } v \in L^2(H_0^1(\Omega)), \\ \qquad\qquad\qquad u_j(\cdot,0) = T_j(u_0), \end{cases}$$

where $f_j = T_j\left(\sigma_j(u_j)|\nabla\varphi_j|^2\right)$. Since the sequences (f_j) and $(T_j(u_0))$ are bounded in $L^1(Q)$ and $L^1(\Omega)$, respectively, we may deduce some well-known estimates for the sequence of solutions to (6) (u_j) in suitable Banach spaces ([7, 10]), namely

$$\text{(7)}\qquad (u_j) \text{ is bounded in } L^\infty(L^1(\Omega));$$

for all $M > 0$ and $j \geq 1$, there exists a constant $C > 0$, not depending upon M and j, such that

$$\text{(8)}\qquad \int_Q |\nabla T_M(u_j)|^2 \leq CM,$$

$$\text{(9)}\qquad \int_{\{M\leq|u_j|<M+1\}} |\nabla u_j|^2 \leq C,$$

and also

$$\text{(10)}\qquad \int_{\{M\leq|u_j|<M+1\}} a_j(u_j)\,|\nabla u_j|^2 \leq \int_{\{|u_j|>M\}} |f_j| + \int_{\{|u_0|>M\}} |u_0|.$$

Owing to (7), (9), and Lemma 3, we have

$$\text{(11)}\quad (u_j) \text{ is bounded in } L^q(W_0^{1,q}(\Omega)), \ \text{for all } q < \frac{N+2}{N+1} \text{ if } N \geq 2,\ q < 2 \text{ if } N = 1.$$

As far as the parabolic term $\frac{\mathrm{d}u_j}{\mathrm{d}t}$ is concerned, we proceed as follows. Let $S \in C^\infty(\mathbb{R})$ with supp $S' \subset [-M, M]$. Taking $v = S'(u_j)\phi$, $\phi \in \mathcal{D}(\Omega)$, in (6), it yields

$$\text{(12)}\quad \frac{\mathrm{d}S(u_j)}{\mathrm{d}t} - \nabla\cdot[a_j(u_j)\nabla u_j S'(u_j)] + S''(u_j)a_j(u_j)\nabla u_j\nabla u_j = f_j S'(u_j) \text{ in } \mathcal{D}'(\Omega).$$

Thanks to (8) and (H.1) we obtain

$$\left(\frac{\mathrm{d}S(u_j)}{\mathrm{d}t}\right) \text{ is bounded in } L^2(H^{-1}(\Omega)) + L^1(Q).$$

Since $L^2(H^{-1}(\Omega)) + L^1(Q) \hookrightarrow L^1(W^{-1,r}(\Omega))$, $r < \frac{N}{N-1}$, with continuous inclusion, we have

$$\text{(13)}\qquad \left(\frac{\mathrm{d}S(u_j)}{\mathrm{d}t}\right) \text{ is bounded in } L^1(W^{-1,r}(\Omega)) \text{ for all } r < \frac{N}{N-1}.$$

Furthermore, using (11), we readily have

$$(S(u_j)) \text{ is bounded in } L^q(W_0^{1,q}(\Omega)), \text{ for all } q < \frac{N+2}{N+1}, \text{ if } N \geq 2,\ q < 2 \text{ if } N = 1.$$

Now we apply the compactness result stated in Lemma 1. To do so, we take

$$X = W_0^{1,q}(\Omega), \quad B = L^q(\Omega), \quad Y = W^{-1,r}(\Omega);$$

therefore,

$$(S(u_j)) \text{ is relatively compact in } L^q(Q) \text{ for all } q < \frac{N+2}{N+1}. \tag{14}$$

Property (14) is not enough to deduce the almost everywhere convergence of (u_j) modulo a subsequence. We must also use the estimates derived above. To this end, let $M > 0$ and consider a function $S \in C^\infty(\mathbb{R})$ satisfying

(i) supp S' is compact,
(ii) S is nondecreasing, and
(iii) $S(s) = s$ if $|s| \leq M$.

Therefore, we have the identity $T_M(s) = T_M(S(s))$ for all $s \in \mathbb{R}$, and, in particular,

$$T_M(u_j) = T_M(S(u_j)). \tag{15}$$

According to (8), for every $M > 0$ there exist a subsequence, which will be denoted in the same way, and a function $z_M \in L^2(H_0^1(\Omega))$ such that

$$T_M(u_j) \to z_M \text{ weakly in } L^2(H_0^1(\Omega)). \tag{16}$$

On the other hand, from (14), there exist a subsequence, still denoted in the same way, and a function $\varsigma_S \in L^q(Q)$ such that

$$S(u_j) \to \varsigma_S \text{ strongly in } L^q(Q) \text{ and a.e. in } Q. \tag{17}$$

Notice that (15) and (17) imply that $T_M(u_j)$ converges almost everywhere to $T_M(\varsigma_S)$; this fact, together with (16), implies that $z_M = T_M(\varsigma_S)$.

Furthermore, from (11), there exist $u \in L^q(W_0^{1,q}(\Omega))$ and a subsequence of (u_j) such that

$$u_j \to u \text{ weakly in } L^q(W_0^{1,q}(\Omega)), \text{ for all } q < \frac{N+2}{N+1} \text{ if } N \geq 2,\ q < 2 \text{ if } N = 1.$$

All these convergences lead to (modulo a subsequence) the almost everywhere convergence of (u_j). Indeed, this property can be readily derived from the next result ([19]).

Lemma 7. *Let $q \geq 1$, $A \subset \mathbb{R}^N$ a nonnegligible measurable set, $(w_j) \subset L^q(A)$, $w \in L^q(A)$ be such that*

$$w_j \to w \textit{ weakly in } L^q(A).$$

Assume that for every $M > 0$ there exists $v_M \in L^1(A)$ such that

$$T_M(v_j) \to v_M \text{ a.e. } \textit{in } A,$$

then $T_M(w) = v_M$, for all $M > 0$ (and in particular $w_j \to w$ almost everywhere in A).

Summing up, we have shown the existence of subsequences, still denoted in the same way, (u_j), (φ_j), and functions $u \in L^q(W_0^{1,q}(\Omega))$ and $\varphi \in L^2(H_0^1(\Omega))$ such that

$$u_j \to u \text{ weakly in } L^q(W_0^{1,q}(\Omega)), \text{ for all } q < \frac{N+2}{N+1} \text{ if } N \geq 2,\ q < 2 \text{ if } N = 1, \tag{18}$$

$$
\begin{aligned}
&(19) \qquad T_M(u_j) \to T_M(u) \quad \text{weakly in } L^2(H_0^1(\Omega)),\\
&(20) \qquad u_j \to u \quad \text{a.e. in } Q,\\
&(21) \qquad S(u_j) \to S(u) \quad \text{strongly in } L^r(Q) \text{ for all } r < +\infty,\\
&(22) \qquad \frac{\mathrm{d}S(u_j)}{\mathrm{d}t} \to \frac{\mathrm{d}S(u)}{\mathrm{d}t} \quad \text{in } \mathcal{D}'(Q),\\
&(23) \qquad \varphi_j \to \varphi \quad \text{weakly in } L^2(H_0^1(\Omega)),
\end{aligned}
$$

where (21) and (22) are valid for all $S \in C^\infty(\Omega)$ with supp S' compact, and (23) is obtained from (5) and (H.2).

Now we turn our attention to (φ_j) and φ. First of all, we show that

$$
\sigma_j(u_j)^{1/2}\nabla\varphi_j \to \sigma(u)^{1/2}\nabla\varphi \text{ weakly in } L^2(Q)^N. \tag{24}
$$

Indeed, from (5), there exist a subsequence and $\Phi \in L^2(Q)^N$ such that

$$
\sigma_j(u_j)^{1/2}\nabla\varphi_j \to \Phi \text{ weakly in } L^2(Q)^N. \tag{25}
$$

Using (20) and (H.2), it yields

$$
\sigma_j(u_j)^{-1/2} \to \sigma(u)^{-1/2} \text{ weakly}-* \text{ in } L^\infty(Q) \text{ and a.e. in } Q. \tag{26}
$$

Putting

$$
\nabla\varphi_j = \sigma_j(u_j)^{-1/2}\sigma_j(u_j)^{1/2}\nabla\varphi_j, \tag{27}
$$

and passing to the limit, gathering (25)–(27), we obtain $\Phi = \sigma(u)^{1/2}\nabla\varphi$, and this shows the statement (24). Notice that, in particular, $\sigma(u)|\nabla\varphi|^2 \in L^1(Q)$.

One of the most delicate parts in the passing to the limit consists in showing the convergence

$$
\sigma_j(u_j)^{1/2}\nabla\varphi_j \to \sigma(u)^{1/2}\nabla\varphi \text{ strongly in } L^2(Q)^N. \tag{28}
$$

From (24), it is enough to show that

$$
\int_Q \sigma_j(u_j)|\nabla\varphi_j|^2 \to \int_Q \sigma(u)|\nabla\varphi|^2. \tag{29}
$$

To do this, we first introduce the function $S_k \in W^{1,\infty}(\mathbb{R})$, $k > 0$, defined as

$$
S_k(s) = \begin{cases} 1 & \text{if } |s| \le k,\\ (2k - |s|)/k & \text{if } k < |s| \le 2k,\\ 0 & \text{if } |s| > 2k. \end{cases} \tag{30}
$$

Note that supp $S_k = [-2k, 2k]$ and $S_k'(s) = \frac{1}{k}\left(\chi_{(-2k,-k)} - \chi_{(k,2k)}\right)$. Then, we take in (4) the test function $\psi = S_k(u_j)T_M(\varphi) \in L^\infty(H_0^1(\Omega))$. The integration over $(0,T)$ leads to

$$
\begin{aligned}
&\int_Q \sigma_j(u_j)\nabla\varphi_j\nabla T_M(\varphi)S_k(u_j) + \int_Q \sigma_j(u_j)\nabla\varphi_j\nabla u_j S_k'(u_j)T_M(\varphi)\\
&\quad = -\int_Q F_j(u_j)\nabla T_M(\varphi)S_k(u_j) - \int_Q F_j(u_j)\nabla u_j S_k'(u_j)T_M(\varphi);
\end{aligned}
$$

we call these terms (I)–(IV) and study them separately.
(I). Since $\sigma_j(u_j)S_k(u_j) = \sigma_j(T_{2k}(u_j))S_k(u_j) \in L^\infty(Q)$ and is bounded in this space, using (20) it yields

$$\sigma_j(u_j)S_k(u_j) \to \sigma(u)S_k(u) \text{ weakly}-* \text{ in } L^\infty(Q) \text{ and a.e. in } Q.$$

From (23), making $j \to \infty$, we readily obtain

$$\int_Q \sigma_j(u_j)\nabla\varphi_j \nabla T_M(\varphi)S_k(u_j) \to \int_Q \sigma(u)\nabla\varphi T_M(\varphi)S_k(u).$$

Owing to Lebesgue's theorem, we finally deduce

$$\lim_{M\to\infty}\lim_{k\to\infty}\lim_{j\to\infty}\int_Q \sigma_j(u_j)\nabla\varphi_j\nabla T_M(\varphi)S_k(u_j) = \int_Q \sigma(u)|\nabla\varphi|^2.$$

(II). We first derive another estimate for (u_j). Let $H_k \in W^{1,\infty}(\mathbb{R})$ be the function

$$H_k(s) = \begin{cases} 0 & \text{if } |s| \le k, \\ (|s|-k)/k & \text{if } k < |s| \le 2k, \\ |s|/s & \text{if } |s| > 2k, \end{cases}$$

then put $\tilde{H}_k(s) = \int_0^s H_k(\tau)\mathrm{d}\tau$ and $E_j^k = \{k < |u_j| < 2k\}$. Choosing $v = H_k(u_j)$ in (4) yields

$$\int_\Omega \tilde{H}_k(u_j(T)) + \frac{1}{k}\int_{E_j^k} a_j(u_j)|\nabla u_j|^2 = \int_Q f_j H_k(u_j) + \int_\Omega \tilde{H}_k(T_j(u_0));$$

therefore, for all $j \ge 1$ and $k > 0$, there exists a constant $C > 0$, not depending upon j and k, such that

$$\frac{1}{k}\int_Q a_j(u_j)|\nabla u_j|^2\chi_{E_j^k} \le C,$$

that is,

$$\left(\frac{1}{\sqrt{k}}a_j(u_j)^{1/2}\nabla u_j\chi_{E_j^k}\right) \text{ is bounded (in } j \text{ and } k\text{) in } L^2(Q)^N. \tag{31}$$

Going back to (II)

$$(II) = \int_Q \sigma_j(u_j)^{1/2}\nabla\varphi_j\sigma_j(u_j)^{1/2}a_j(u_j)^{-1/2}a_j(u_j)^{1/2}\nabla u_j S_k'(u_j)T_M(\varphi),$$

thus

$$\begin{aligned} |(II)| &\le M\int_Q \left|\sigma_j(u_j)^{1/2}\nabla\varphi_j\,\frac{1}{\sqrt{k}}\sigma_j(u_j)^{1/2}a_j(u_j)^{-1/2}\,\frac{1}{\sqrt{k}}a_j(u_j)^{1/2}\nabla u_j\chi_{E_j^k}\right| \\ &\le M\left\|\sigma_j(u_j)^{1/2}\nabla\varphi_j\right\|_{L^2(Q)}\cdot\left\|\frac{1}{\sqrt{k}}a_j(u_j)^{1/2}\nabla u_j\chi_{E_j^k}\right\|_{L^2(Q)} \\ &\qquad\cdot\left\|\frac{1}{\sqrt{k}}\sigma_j(u_j)^{1/2}a_j(u_j)^{-1/2}\chi_{E_j^k}\right\|_{L^\infty(Q)}. \end{aligned}$$

Hence, from (H.4), (5), and (31), we deduce

$$|(II)| \le C\omega(k),$$

which implies

$$\lim_{k\to\infty} \limsup_{j\to\infty} \int_Q \sigma_j(u_j)\nabla\varphi_j \nabla u_j S_k'(u_j) T_M(\varphi) = 0.$$

(III). Lebesgue's theorem easily shows that

$$\lim_{j\to\infty} \int_Q F_j(u_j)\nabla T_M(\varphi) S_k(u_j) = \int_Q F(u)\nabla T_M(\varphi) S_k(u).$$

We now express this last integral as

$$\int_Q F(u)\sigma(u)^{-1/2}\sigma(u)^{1/2}\nabla T_M(\varphi) S_k(u).$$

Owing to (H.3) and (24) we can apply again Lebesgue's theorem, first in k, then in M, to deduce finally that

$$\lim_{M\to\infty} \lim_{k\to\infty} \lim_{j\to\infty} \int_Q F_j(u_j)\nabla T_M(\varphi) S_k(u_j) = \int_Q F(u)\nabla\varphi. \tag{32}$$

(IV). Following the same techniques as in (II) and (III), it is straightforward that

$$\lim_{k\to\infty} \limsup_{j\to\infty} \int_Q F_j(u_j)\nabla u_j S_k'(u_j) T_M(\varphi) = 0.$$

Gathering (27)–(32),

$$\int_Q \sigma(u)|\nabla\varphi|^2 = -\int_Q F(u)\nabla\varphi. \tag{33}$$

On the other hand, taking $\psi = \varphi_j$ in (4) and integrating over $(0,T)$, we obtain

$$\int_Q \sigma_j(u_j)|\nabla\varphi_j|^2 = -\int_Q F_j(u_j)\nabla\varphi_j;$$

since $F_j(u_j)\nabla\varphi_j = F_j(u_j)\sigma_j(u_j)^{-1/2}\sigma_j(u_j)^{1/2}\nabla\varphi_j$, and bearing in mind (H.3), (20), and (24), we conclude that

$$\int_Q F_j(u_j)\nabla\varphi_j \to \int_Q F(u)\nabla\varphi; \tag{34}$$

putting together (33)–(34) gives directly (29), that is, $\sigma_j(u_j)^{1/2}\nabla\varphi_j \to \sigma(u)^{1/2}\nabla\varphi$ strongly in $L^2(Q)^N$. This also implies that

$$f_j = T_j\left(\sigma_j(u_j)|\nabla\varphi_j|^2\right) \to \sigma(u)|\nabla\varphi|^2 \text{ strongly in } L^1(Q). \tag{35}$$

The last relevant convergence to be shown before passing to the limit in the approximate problems (4) is,

$$T_M(u_j) \to T_M(u) \text{ strongly in } L^2(H_0^1(\Omega)), \text{ for every } M > 0. \tag{36}$$

In fact, this is a consequence of (6), (19), and (35), but it is not an immediate result; for details of the proof of this property the reader is referred to [8].

4.3. Passing to the limit and conclusion. Let u and φ be the limit functions given in (18) and (23). Here we show that both functions verify (R.1)–(R.5) of Definition 5.

In fact, (R.1) and (R.2) have been already obtained.

By virtue of (19), (20), and (35), making $j \to \infty$ in (10) yields

$$\int_{\{M \le |u| < M+1\}} a(u)|\nabla u|^2 \le \int_{\{|u|>M\}} \sigma(u)|\nabla\varphi|^2 + \int_{|u_0|>M} |u_0|;$$

due to hypothesis (H.5) and making $M \to \infty$ in this last expression, we can easily derive (R.3).

In order to obtain (R.4), we just take $v = S(u_j)\phi$ in (4) with $S \in C^\infty(\mathbb{R})$, supp S' compact and $\phi \in \mathcal{D}(\Omega)$. Thanks to the convergence properties derived in the preceding section, we can make $j \to \infty$ and this yields the variational formulation (R.4). Note that the strong convergence of the truncations function $T_M(u_j) \to T_M(u)$ in $L^2(H_0^1(\Omega))$ is essential in this stage. It remains to state the initial condition $S(u(\cdot,0)) = S(u_0)$; to do so, we apply Lemma 2 with the following choices:

$$X = L^\infty(\Omega), \quad B = Y = W^{-1,r}(\Omega), \text{ any } r < \frac{N}{N-1},$$

and put $E = \{S(u_j)\}_{j\ge1}$, supp $S' = [-M, M]$. Obviously, E is bounded in $L^\infty(X)$ and, according to (13), $\frac{\mathrm{d}v}{\mathrm{d}t} \in L^1(Y)$ for all $v \in E$. Also, by virtue of (12), we can write

$$\frac{\mathrm{d}S(u_j)}{\mathrm{d}t} = f_j S'(u_j) - S''(u_j)a_j(T_M(u_j))|\nabla T_M(u_j)|^2 + \nabla\cdot[a_j(T_M(u_j))\nabla T_M(u_j)S'(u_j)].$$

Now, from (20) and (35), $f_j S'(u_j)$ converges strongly in $L^1(Q)$ and from (20) and (36), $S''(u_j)a_j(T_M(u_j))|\nabla T_M(u_j)|^2$ converges strongly in $L^1(Q)$. Owing to Lebesgue's inverse theorem, there exists $\bar{h} \in L^1(Q)$ such that

$$|\Phi_j| \le \bar{h} \text{ for all } j \ge 1 \text{ and a.e. in } Q,$$

where $\Phi_j = f_j S'(u_j) - S''(u_j)a_j(u_j)|\nabla u_j|^2$. Consequently,

$$\|\Phi_j\|_{W^{-1,r}(\Omega)} \le C\|\Phi_j\|_{L^1(\Omega)} \le C\|\bar{h}\|_{L^1(\Omega)}, \text{ for all } r < \frac{N}{N-1},\ j \ge 1, \text{a.e. } t \in (0,T).$$

On the other hand, the last term $\nabla\cdot[a_j(T_M(u_j))\nabla T_M(u_j)S'(u_j)]$ is bounded in $L^2(H^{-1}(\Omega))$, and therefore it is also bounded in $L^2(W^{-1,r}(\Omega))$, for all $r < \frac{N}{N-1}$. Hence, we may take $h = C\|\bar{h}\|_{L^1(\Omega)} \in L^1(0,T)$ and $s = 2$ to deduce that

$$\left\|\frac{\mathrm{d}S(u_j)}{\mathrm{d}t}\right\|_Y \le h + \|\nabla\cdot[a_j(T_M(u_j))\nabla T_M(u_j)S'(u_j)]\|_Y, \text{ for all } j \ge 1, \text{ a.e. } t \in (0,T).$$

By Lemma 2, this means that $(S(u_j))$ is relatively compact in $C^0\left([0,T]; W^{-1,r}(\Omega)\right)$ for any $r < \frac{N}{N-1}$ and thus, there exists a subsequence, still denoted in the same way, such that $(S(u_j))$ converges in $C^0\left([0,T]; W^{-1,r}(\Omega)\right)$. From (21), this limit must be $S(u)$. In particular,

$$S(u_j(\cdot,0)) \to S(u(0)) \text{ in } W^{-1,r}(\Omega),$$

and since $S(u_j(0)) = S(T_j(u_0)) \to S(u_0)$ in $L^1(\Omega)$-strongly, we deduce the initial condition

$$S(u(\cdot,0)) = S(u_0) \text{ in } W^{-1,r}(\Omega),\ r < \frac{N}{N-1}.$$

Finally, in order to derive (R.5), we just take $\psi = S_k(u_j)T_M(\phi)$ in (3), where S_k is defined in (30) and $\phi \in L^2(H_0^1(\Omega))$ is such that $\int_Q \sigma(u)|\nabla\phi|^2 < +\infty$. In this situation, we can proceed as in (I)–(IV) above: taking the iterate limits, first in j, then in k, then in M, and the last expression becomes (R.5).

This ends the proof of Theorem 6. □

5. Concluding remarks. The diffusion coefficients a and σ are scalar functions in the setting given by hypotheses (H.1)–(H.4). We may consider a more general setting in which a and σ are diffusion matrices of order $N \times N$. The hypotheses on this data read as follows:

(H.1) $a, \sigma : Q \times \mathbb{R} \to \mathbb{R}^{N\times N}$ and $F : Q \times \mathbb{R} \to \mathbb{R}^N$ are Caratheodory functions and there exists a nondecreasing function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\max\left(\|a(x,t,s)\|, \|\sigma(x,t,s)\|, |F(x,t,s)|\right) \le \gamma(|s|), \text{ for all } s \in \mathbb{R}, \text{ a.e. in } Q,$$

where $\|\cdot\|$ stands for the spectral norm.

(H.2) There are two constant values $a_0 > 0$ and $\sigma_0 > 0$ so that

$$a(x,t,s)\xi\xi \ge a_0|\xi|^2,\ \sigma(x,t,s)\xi\xi \ge \sigma_0|\xi|^2, \text{ for all } s \in \mathbb{R},\ \xi \in \mathbb{R}^N, \text{ a.e. in } Q.$$

(H.3) $\Gamma \in L^1(Q)$ is a function satisfying

$$|\sigma(x,t,s)^{-\mathrm{S}/2}F(x,t,s)|^2 \le \Gamma(x,t), \text{ for all } s \in \mathbb{R}, \text{ a.e. in } Q.$$

(H.4) $\displaystyle \max_{k\le|s|\le 2k} \operatorname*{ess\,sup}_Q \frac{1}{\sqrt{k}}\|\sigma(x,t,s)^{\mathrm{S}/2}a(x,t,s)^{-\mathrm{S}/2}\| = \omega(k)$ as $k \to +\infty$.

(H.5) $u_0 \in L^1(\Omega)$.

The notation in (H.3) and (H.4) is now explained: for a matrix $B \in \mathbb{R}^{N\times N}$, we denote by B^{S} the symmetric part of B, that is, $B^{\mathrm{S}} = (B + B')/2$. From (H.2), $\sigma(x,t,s)^{\mathrm{S}}$ and $a(x,t,s)^{\mathrm{S}}$ are positive definite; then $\sigma(x,t,s)^{\mathrm{S}/2}$ stands for the unique positive definite square root of $\sigma(x,t,s)$, whereas $a(x,t,s)^{-\mathrm{S}/2}$ represents the inverse matrix of the unique positive definite square root of $a(x,t,s)^{\mathrm{S}}$.

In this situation, the existence result given in Theorem 6 still holds true.

The analysis described in this paper shows that the concept of renormalized solutions may be applied to systems of parabolic-elliptic equations with unbounded diffusion coefficients. The existence result relies on certain assumptions on data, apart from the standard ones, describing the relation of the asymptotic behavior between them.

The uniqueness of renormalized solution to problem (1) is a very complex task to be deduced; this is due to the fact that all known uniqueness results for the thermistor problem are derived from L^∞ estimates verified by u and φ; this regularity may be obtained under certain restrictive assumptions, including for instance $F \in L^\infty$, $a, \sigma \in L^\infty$. In that setting, there is no need to search for renormalized solutions: one reencounters the setting of weak solutions.

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