On an induction–conduction PDEs system in the harmonic regime

M.T. González Montesinos [a], F. Ortegón Gallego [b]

[a] Departamento de Matemática Aplicada I, Universidad de Sevilla, Avda. Reina Mercedes, s/n, 41012 Sevilla, Spain
[b] Departamento de Matemáticas, Universidad de Cádiz, Facultad de Ciencias, Campus del Río San Pedro, 11510 Puerto Real, Spain

ABSTRACT

We study the existence of weak solutions to a nonlinear coupled parabolic–elliptic system arising in the heating industrial process of a steel workpiece. The unknowns are the electric potential, the magnetic vector potential and the temperature. The different time scales related to the electric potential and the magnetic vector potential versus the temperature lead us to introduce the harmonic regime. This yields to a new system of nonlinear partial differential equations.

1. Introduction

This work is devoted to analyze the existence of weak solutions to a nonlinear coupled system of partial differential equations which constitutes the mathematical modeling of the heating industrial process by induction–conduction of a steel workpiece. The main goal of heat treating of steel is to attain a satisfactory hardness on certain critical parts of the workpiece while keeping the rest ductile. In this paper, the workpiece represents an automobile steering rack. The rack is a solid cylinder with a tooth line profile as shown in Fig. 1.

Among the different hardening surface procedures, we are interested in an induction–conduction industrial procedure. In this way, a copper inductor is put in contact with the rack as it is shown in Fig. 2. Then a high frequency alternating current is switched on and flows through the coil made up by the workpiece itself and the copper inductor. An alternating magnetic field is generated which in its turn induces eddy currents bringing about heat (Joule's effect) just where it is needed. Once the desired high level of temperature is reached at certain critical parts along the rack, the supplied electric current is switched off, and the workpiece is quenched in order to cool it down rapidly.

The supplied density current flow is modeled through a Neumann boundary condition on a fictitious cross-section Γ cutting across the copper inductor (see Fig. 2).

The heating–cooling industrial processes are governed by a coupled nonlinear system of partial differential equations and ordinary differential equations. The mathematical description of the setting corresponding to Fig. 2 can be found in [1] together with some numerical simulations. As it is shown in this figure, the inductor and the workpiece share the common boundary S. The heating model (1)–(8) reflects this fact mainly in the expression of the Joule term, which takes the form $\sigma(\theta)|\boldsymbol{\mathcal{A}}_{,t} + \nabla\phi|^2$. Note that in the case of direct current, $\boldsymbol{\mathcal{A}}$ does not depend on t, and the model reduces to the so-called thermistor problem [2–4]; on the other hand, if the inductor does not touch the workpiece, then the electric potential does not depend on x on the workpiece and the Joule term becomes $\sigma(\theta)|\boldsymbol{\mathcal{A}}_{,t}|^2$. This situation is considered, for instance, in [5,6] where the model is also rewritten in the harmonic regime (see Section 3). The novelty of this paper is the mathematical

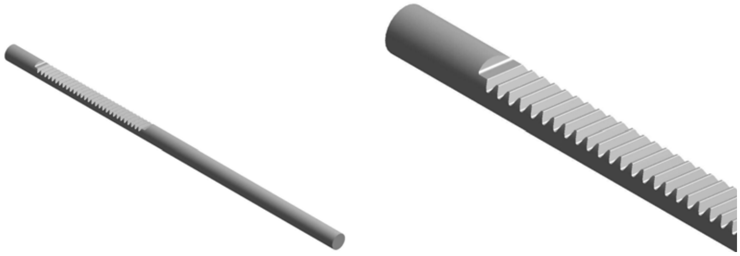

Fig. 1. Automobile steering rack.

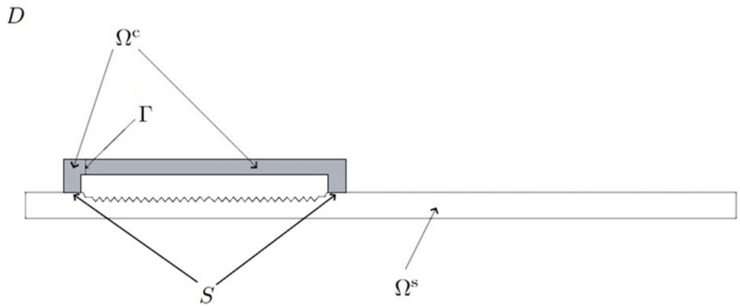

Fig. 2. Open sets D, Ω^s and Ω^c, contact surfaces S, and the auxiliary cross-section Γ.

analysis and resolution of the induction–conduction heating process corresponding with the situation described in Fig. 2 and the unknowns are expressed in the harmonic regime. The resulting model is given in (10)–(17). The analysis of this system is more difficult than the one of the thermistor problem since the Joule term cannot be written as the divergence of an L^2 function [2]. The same comment can be said about the induction case [6] as the Joule heating yields a more regular term.

This work is organized as follows. In Section 2 we establish the mathematical description for the heating process. Section 3 is devoted to introduce the harmonic regime. In Section 4 we set up the notation used in this paper; this leads to the introduction of some functional spaces. We also enumerate the hypotheses on data, recall certain compactness results, give the notion of weak solution adapted to our problem and state the main result. Finally, Section 5 develops the proof of the existence result; it is split into three steps, namely setting of the approximate problems, derivation of estimates, and passing to the limit and conclusion.

2. Setting of the problem

Our main task is to analyze the existence of weak solutions of a simplified model which does not take into account mechanical effects.

To this end, let $\Omega,\ D \subset \mathbb{R}^3$ be bounded, connected and Lipschitz-continuous open sets such that $\bar{\Omega} \subset D$, $\Omega = \Omega^c \cup \Omega^s \cup S$ is the set of conductors, Ω^c is the copper inductor, Ω^s is the steel workpiece containing a toothed part to be hardened, Ω^c and Ω^s being open sets, and $S = \bar{\Omega}^c \cap \bar{\Omega}^s$ is the surface contact between Ω^c and Ω^s, $\Omega^c \cap \Omega^s = \varnothing$ (see Fig. 2).

In our setting the high frequency current density supplied through the workpiece is about 80 kHz. Then we may neglect the electric displacement term in the set of Maxwell's equations. A high frequency current is supplied during a time interval $[0, T]$ passing through the set of conductors $\Omega = \Omega^s \cup \Omega^c \cup S$. Due to its shape, the set of conductors constitutes a coil which in its turn induces electromagnetic eddy currents inside the workpiece. The combined effect of both conduction and induction through the workpiece results in an energy dissipation (Joule's heating) leading to an increase in temperature in the critical parts of the workpiece to be hardened. This heating process takes about $T = 5.5$ s. Once the desired temperature is reached, the current is switched off and the cooling stage begins by spraying the workpiece with water. This process is called aquaquenching. Let $\phi\colon \Omega \times [0, T] \mapsto \mathbb{R}$ be the electric potential, $\boldsymbol{\mathcal{A}}\colon D \times [0, T] \mapsto \mathbb{R}^3$ the magnetic vector potential, and $\theta\colon \Omega \times [0, T] \mapsto \mathbb{R}$ the temperature. Neglecting mechanical effects, the heating process is described by the following system of elliptic–parabolic PDEs [1,7–13]:

$$\nabla \cdot [\sigma(\theta)\nabla\phi] = 0 \quad \text{in } \Omega_T = \Omega \times (0, T), \tag{1}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{2}$$

$$\left[\sigma(\theta)\frac{\partial \phi}{\partial \nu}\right]_{\Gamma} = j_S \quad \text{on } \Gamma \times (0, T), \tag{3}$$

$$\sigma_0(\theta)\mathcal{A}_{,t} + \nabla \times \left(\frac{1}{\mu}\nabla \times \mathcal{A}\right) - \delta\nabla(\nabla \cdot \mathcal{A}) + \sigma_0(\theta)\nabla\phi = 0 \quad \text{in } D_T = D \times (0, T), \tag{4}$$

$$\mathcal{A} = \mathbf{0} \quad \text{on } \partial D \times (0, T), \tag{5}$$

$$\rho c_\epsilon \theta_{,t} - \nabla \cdot (\kappa(\theta)\nabla\theta) = \sigma(\theta)|\mathcal{A}_{,t} + \nabla\phi|^2 + G \quad \text{in } \Omega \times (0, T), \tag{6}$$

$$\frac{\partial\theta}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{7}$$

$$\theta(\cdot, 0) = \theta_0 \quad \text{in } \Omega. \tag{8}$$

Here $\sigma(\theta)$ is the temperature-dependent electric conductivity. Notice that for a function $g: \Omega \times (0, T) \times \mathbb{R} \mapsto \mathbb{R}$, by $g(\theta)$ we mean the function $(x, t) \in \Omega \times (0, T) \mapsto g(x, t, \theta(x, t))$, and the same comment applies for functions defined in $D \times (0, T) \times \mathbb{R}$. The boundary datum j_S is the external source current density, $[\cdot]_\Gamma$ denotes the jump across the auxiliary surface Γ and ν is a unit normal vector on Γ; the function $\sigma_0: D \times (0, T) \times \mathbb{R} \mapsto \mathbb{R}$ is given as follows:

$$\sigma_0(x, t, s) = \begin{cases} \sigma(x, t, s) & \text{if } (x, t) \in \Omega_T,\ s \in \mathbb{R} \\ 0 & \text{elsewhere.} \end{cases}$$

The function $\mu: D \mapsto \mathbb{R}$ stands for the magnetic permeability and is given by

$$\mu(x) = \mu_1\chi_{\Omega^s} + \mu_2\chi_{\Omega^c} + \mu_3\chi_{D\setminus\Omega},$$

where μ_i, $1 \le i \le 3$, are constant values such that $0 < \mu_2 < \mu_3 \ll \mu_1$; $\delta > 0$ is a small parameter. ρ is the density, c_ϵ is the specific heat capacity at constant strain and κ is the temperature-dependent thermal conductivity; θ_0 is the initial temperature. The source term G takes into account the energy dissipation coming from mechanical deformations and the latent heat due to the transformations of the different phase transitions in steel. Notice that the electric current is modeled through the boundary condition on the auxiliary cross-section Γ given in (3).

A system like (1)–(8) has been studied in [11] where mechanical effects and phase transitions have also been taken into account.

Remark 1. The introduction of the term $-\delta\nabla(\nabla \cdot \mathcal{A})$ in (4) is artificial. This is a penalty term in order to regularize the divergence of the magnetic vector potential and assure the uniqueness of $\mathcal{A}$ for a given θ. In [11], instead of (4)–(5), another set of equations is proposed which takes into account the Coulomb-gauge condition, namely

$$\sigma_0(\theta)\mathcal{A}_{,t} + \nabla \times \left(\frac{1}{\mu}\nabla \times \mathcal{A}\right) + \sigma_0(\theta)\nabla\phi = 0 \quad \text{in } D_T = D \times (0, T), \tag{4a}$$

$$\nabla \cdot \mathcal{A} = \mathbf{0} \quad \text{in } D \times (0, T), \tag{4b}$$

$$\mathcal{A} \cdot \boldsymbol{n} = \mathbf{0} \quad \text{on } \partial D \times (0, T). \tag{5a}$$

Along with a straightforward framework adaptation, we may derive a similar existence result for the variational formulation of (1)–(8) where Eqs. (4)–(5) have been replaced by (4a), (4b) and (5a). Nevertheless, (4a) cannot be retrieved from its own variational formulation, because the Coulomb-gauge condition (4b) makes appear an undesired gradient term. Notice that $-\delta\nabla(\nabla \cdot \mathcal{A})$ is already a gradient term, but here the parameter δ can be chosen small enough.

3. The harmonic regime

High frequency sinusoidal currents generate electromagnetic fields in time. Thus, both the electric potential, ϕ, and the magnetic vector potential, $\mathcal{A}$, may be written as $\mathcal{F}(x, t) = \mathrm{Re}\,[\exp(i\omega t)F(x, t)]$ [14–17], where F is a complex-valued function or vector field, and $\omega = 2\pi f$ is the angular frequency, f being the electric current frequency. In general, F depends on t at a time scale much greater than $1/\omega$. Then, we may introduce the complex-valued fields φ, $\boldsymbol{A}$ and $\boldsymbol{j}$ as

$$\begin{cases} \phi = \mathrm{Re}[\exp(i\omega t)\varphi(x, t)], \\ \mathcal{A} = \mathrm{Re}[\exp(i\omega t)\boldsymbol{A}(x, t)], \\ j_S = \mathrm{Re}[\exp(i\omega t)\boldsymbol{j}(x, t)]. \end{cases} \tag{9}$$

From (1)–(8) and (9), neglecting the term $\boldsymbol{A}_{,t}$, the harmonic regime is obtained. Moreover, in the energy equation, the term $|\mathcal{A}_{,t} + \nabla\phi|^2$ is replaced by its mean value taken over a time period $[t, t + \omega]$, namely,

$$\frac{1}{\omega}\int_t^{t+\omega} |\mathcal{A}_{,t} + \nabla\phi|^2 \approx \frac{1}{2}|i\omega\boldsymbol{A} + \nabla\varphi|^2.$$

In this way, the term $\sigma(\theta)|i\omega\boldsymbol{A}+\nabla\varphi|^2/2$ corresponds to the effective Joule's heating, and the system (1)–(8) in the harmonic regime becomes

$$\nabla\cdot(\sigma(\theta)\nabla\varphi)=0 \quad \text{in } \Omega_T, \tag{10}$$

$$\frac{\partial\varphi}{\partial n}=0 \quad \text{on } \partial\Omega\times(0,T), \tag{11}$$

$$\left[\sigma(\theta)\frac{\partial\varphi}{\partial\nu}\right]_\Gamma=\boldsymbol{j} \quad \text{on } \Gamma\times(0,T), \tag{12}$$

$$i\omega\sigma_0(\theta)\boldsymbol{A}+\nabla\times\left(\frac{1}{\mu}\nabla\times\boldsymbol{A}\right)-\delta\nabla(\nabla\cdot\boldsymbol{A})+\sigma_0(\theta)\nabla\varphi=0 \quad \text{in } D_T, \tag{13}$$

$$\boldsymbol{A}=\boldsymbol{0} \quad \text{on } \partial D\times(0,T), \tag{14}$$

$$\rho c_\epsilon\theta_{,t}-\nabla\cdot(\kappa(\theta)\nabla\theta)=\frac{\sigma(\theta)}{2}|i\omega\boldsymbol{A}+\nabla\varphi|^2+G \quad \text{in } \Omega_T, \tag{15}$$

$$\frac{\partial\theta}{\partial n}=0 \quad \text{on } \partial\Omega\times(0,T), \tag{16}$$

$$\theta(\cdot,0)=\theta_0 \quad \text{in } \Omega. \tag{17}$$

4. Assumptions and main result

We consider the system (10)–(17), where the unknowns, $\boldsymbol{A}$, the magnetic vector potential, and φ, the electric potential, are complex-valued. Consequently, we must deal with complex-valued functional spaces. Also, without loss of generality, we assume in this analysis that $\rho c_\epsilon=1$.

Let $H^1(\Omega)=\{v\in L^2(\Omega)/\nabla v\in(L^2(\Omega))^3\}$ be the complex-valued usual Sobolev space, where the derivatives of v are taken in the sense of distributions. Since Ω is a bounded, connected and Lipschitz-continuous open set, the quotient space $H^1(\Omega)/\mathbb{C}$ provided with the norm

$$\|\hat{u}\|_{H^1(\Omega)/\mathbb{C}}=\inf_{u\in\hat{u}}\|u\|_{H^1(\Omega)} \tag{18}$$

is a Hilbert space. Furthermore, the seminorm $\hat{u}\in H^1(\Omega)/\mathbb{C}\mapsto|u|_{H^1(\Omega)}$ is equivalent to norm (18), that is, there are two constant positive values c_1 and c_2 such that, for all $\hat{u}\in H^1(\Omega)/\mathbb{C}$ and $u\in\hat{u}$, one has

$$c_1\|\hat{u}\|_{H^1(\Omega)/\mathbb{C}}\le\left(\int_\Omega|\nabla u|^2\right)^{1/2}\le c_2\|\hat{u}\|_{H^1(\Omega)/\mathbb{C}}.$$

We also use the complex-valued Sobolev space $H_0^1(D)=\{v\in H^1(D)/v=0 \text{ on } \partial D\}$. Then we put $\boldsymbol{H}_0^1(D)=(H_0^1(D))^3$. For this Hilbert space we have the following result (see [18]).

Theorem 1. *For all $\boldsymbol{v}\in\boldsymbol{H}_0^1(D)$, one has*

$$|\boldsymbol{v}|^2_{\boldsymbol{H}^1(D)}=\|\nabla\times\boldsymbol{v}\|^2_{\boldsymbol{L}^2(D)}+\|\nabla\cdot\boldsymbol{v}\|^2_{\boldsymbol{L}^2(D)}.$$

For $1\le p\le\infty$, we consider the Banach space $W^{1,p}(\Omega)$ provided with their standard norm, and $(W^{1,p}(\Omega))'$ its dual (topological and algebraic) space.

If X is a Banach space, we put $L^p(X)=L^p(0,T;X)$ and $W^{1,p}(X)=W^{1,p}(0,T;X)$, that is $W^{1,p}(X)=\{v\in L^p(X)/v_{,t}\in L^p(X)\}$, the derivative $v_{,t}$ taken in the sense of distributions in $(0,T)$. Both $L^p(X)$ and $W^{1,p}(X)$ are Banach spaces. Remember that $W^{1,p}(X)\hookrightarrow C([0,T];X)$ with continuous embedding. The following result is very well known (see [19]).

Lemma 1. *Let X, B and Y three Banach spaces such that $X\hookrightarrow B\hookrightarrow Y$, every embedding being continuous and the inclusion $X\hookrightarrow B$ compact. For $1\le p,\ q<+\infty$, let $\mathcal{W}$ be the Banach space defined as $\mathcal{W}=\left\{v\in L^p(X)/v_{,t}\in L^q(Y)\right\}$. Then, the inclusion $\mathcal{W}\hookrightarrow L^p(B)$ holds and is compact.*

Finally, let V be a close subspace of $H^1(\Omega)$ so that $H_0^1(\Omega)\subset V\subset H^1(\Omega)$, and write $\mathcal{V}=L^2(V)$ and $\mathcal{V}'=L^2(V')$. Throughout this paper C stands for a generic positive constant value independent of the subscript k associated with the approximate problems stated below. Now we assume the following hypotheses on data of system (10)–(17):

(H.1) $\sigma,\ \kappa:\Omega\times(0,T)\times\mathbb{R}\mapsto\mathbb{R}$ are Carathéodory functions and there exist some constant values $\sigma_1,\ \sigma_2\,\kappa_1,\ \kappa_2$ such that $0<\sigma_1\le\sigma(x,t,s)\le\sigma_2$, $0<\kappa_1\le\kappa(x,t,s)\le\kappa_2$, almost everywhere $(x,t)\in\Omega_T$ and for all $s\in\mathbb{R}$.
(H.2) $\boldsymbol{j}\in L^2(H^{-1/2}(\Gamma))$ and $\langle\boldsymbol{j}(t),1\rangle_\Gamma=0$, almost everywhere $t\in(0,T)$, where $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pair between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$.
(H.3) $\mu\in L^\infty(D)$ and there is a constant value μ_* such that $0<\mu_*\le\mu$ in D.
(H.4) $G\in L^1(\Omega_T)$.
(H.5) $\theta_0\in L^1(\Omega)$.

Definition 1. A triplet $(\varphi, \boldsymbol{A}, \theta)$ is said to be a weak solution of (10)–(17) if

$$\varphi \in L^2(H^1(\Omega)/\mathbb{C}), \tag{19}$$

$$\boldsymbol{A} \in L^2(\boldsymbol{H}_0^1(D)), \tag{20}$$

$$\theta \in L^p(W^{1,p}(\Omega)) \cap C\left([0,T]; (W^{1,p'}(\Omega))'\right) \quad \text{for all } p \in \left[1, \frac{5}{4}\right), \ \frac{1}{p} + \frac{1}{p'} = 1, \tag{21}$$

$$\theta(\cdot, 0) = \theta_0 \quad \text{in } \Omega, \tag{22}$$

$$\int_{\Omega_T} \sigma(\theta)\nabla\varphi \cdot \nabla\bar{\psi} + \int_0^T \langle \boldsymbol{j}, \bar{\psi}\rangle_\Gamma = 0, \quad \text{for all } \psi \in L^2(H^1(\Omega)/\mathbb{C}), \tag{23}$$

$$i\omega \int_{\Omega_T} \sigma(\theta)\boldsymbol{A} \cdot \bar{\boldsymbol{v}} + \int_{D_T} \frac{1}{\mu}\nabla \times \boldsymbol{A} \cdot \nabla \times \bar{\boldsymbol{v}} + \delta \int_{D_T} \nabla \cdot \boldsymbol{A}\, \nabla \cdot \bar{\boldsymbol{v}} + \int_{\Omega_T} \sigma(\theta)\nabla\varphi \cdot \bar{\boldsymbol{v}} = 0, \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0^1(D), \tag{24}$$

$$-\int_{\Omega_T} \theta\zeta_{,t} + \int_{\Omega_T} \kappa(\theta)\nabla\theta\nabla\zeta = \int_{\Omega_T} F\zeta + \int_\Omega \theta_0(x)\zeta(x, 0),$$
$$\text{for all } \zeta \in C^1(\bar{\Omega} \times [0,T]) \text{ such that } \zeta(\cdot, T) = 0 \text{ in } \Omega, \tag{25}$$

where $F = \sigma(\theta)|i\omega\boldsymbol{A} + \nabla\varphi|^2/2 + G$.

Remark 2. From the regularity given in (19) and (20), the right-hand side of Eq. (6) is in $L^1(\Omega_T)$. According to the Boccardo–Gallouët estimates [5,20], for $\Omega \subset \mathbb{R}^N$ one obtains the regularity $\theta \in L^p(W^{1,p}(\Omega))$ for all $p \in [1, (N+2)/(N+1))$. Since we are working in $N = 3$, this yields the upper value 5/4 in (21).

Remark 3. Since $N = 3$, the Sobolev embedding implies that $L^1(\Omega) \subset (W^{1,q}(\Omega))'$ for all $q > 3$. Furthermore, since $p < 5/4$ we have $p' > 5$. In particular, $L^1(\Omega) \subset (W^{1,p'}(\Omega))'$ for all $p \in [1, 5/4)$. Therefore, in view of (H.5) and the regularity $\theta \in C([0,T]; (W^{1,p'}(\Omega))')$ stated in (21), the initial condition for the temperature (22) makes sense at least in the space $(W^{1,p'}(\Omega))'$. Under a more restrictive assumption on the thermal conductivity κ (see (H.6) below), it can be shown that $\theta \in C([0,T]; L^1(\Omega))$. Thus, the initial condition (22) also makes sense in $L^1(\Omega)$.

Before giving the existence result to the system (10)–(17), we also consider the following hypothesis on the thermal conductivity κ.

(H.6) There exist $\varepsilon_0 > 0$ and $L_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0]$, one has

$$|\kappa(x,t,s_1) - \kappa(x,t,s_2)| \le L_0|s_1 - s_2|,$$

almost everywhere $(x,t) \in \Omega_T$ and for all $s_1, s_2 \in \mathbb{R}$ such that $|s_1 - s_2| < \varepsilon$.

Theorem 2. *Under the hypotheses* (H.1)–(H.5), *there exists a weak solution to the system* (10)–(17) *in the sense of Definition* 1. *Moreover, if the thermal conductivity* κ *verifies* (H.6), *then* $\theta \in C([0,T]; L^1(\Omega))$ *and satisfies the variational formulation*

$$-\int_{\Omega_T} \theta\zeta_{,t} + \int_\Omega \theta(x,T)\zeta(x,T) - \int_\Omega \theta_0(x)\zeta(x,0) + \int_{\Omega_T} \kappa(\theta)\nabla\theta\nabla\zeta = \int_{\Omega_T} F\zeta, \quad \textit{for all } \zeta \in C^1(\bar{\Omega} \times [0,T]). \tag{26}$$

5. Proof of the main result

This section is devoted to the Proof of Theorem 2. We first introduce a sequence of approximate problems, then a priori estimates are derived and, finally, we pass to the limit.

5.1. Approximate problems

For each $k \in \mathbb{N}$, let $T_k(s) = \operatorname{sign} s \min(k, |s|)$ be the truncation function at height k and define the sequence of functions $(F_k) \subset L^\infty(\Omega_T)$ as

$$F_k = \frac{1}{2}\sigma(\theta_k)T_k\left(|i\omega\boldsymbol{A}_k + \nabla\varphi_k|^2\right) + T_k(G),$$

where $\boldsymbol{A}_k$ and φ_k are defined below. We also consider $(\boldsymbol{j}_k) \subset C([0,T], H^{-1/2}(\Gamma))$ satisfying

$$\boldsymbol{j}_k \to \boldsymbol{j} \quad \text{strongly in } L^2(H^{-1/2}(\Gamma)). \tag{27}$$

Then we set the sequence of approximate problems (10)–(17) as follows:

$$\nabla \cdot (\sigma(\theta_k)\nabla\varphi_k) = 0 \quad \text{in } \Omega_T, \tag{28}$$

$$\frac{\partial \varphi_k}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{29}$$

$$\left[\sigma(\theta_k)\frac{\partial\varphi_k}{\partial\nu}\right]_\Gamma = \boldsymbol{j}_k \quad \text{on } \Gamma \times (0,T), \tag{30}$$

$$i\omega\sigma(\theta_k)\boldsymbol{A}_k + \nabla \times \left(\frac{1}{\mu}\nabla \times \boldsymbol{A}_k\right) - \delta\nabla(\nabla \cdot \boldsymbol{A}_k) + \sigma(\theta_k)\nabla\varphi_k = 0 \quad \text{in } D_T, \tag{31}$$

$$\boldsymbol{A}_k = 0 \quad \text{on } \partial D \times (0,T), \tag{32}$$

$$\theta_{k,t} - \nabla \cdot (\kappa(\theta_k)\nabla\theta_k) = F_k \quad \text{in } \Omega_T, \tag{33}$$

$$\frac{\partial\theta_k}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{34}$$

$$\theta_k(\cdot,0) = T_k(\theta_0) \quad \text{in } \Omega. \tag{35}$$

The next result can be shown by a straightforward application of Schauder's fixed point theorem.

Theorem 3. *For each $k \in \mathbb{N}$, the approximate problem (28)–(35) has a weak solution $(\varphi_k, \boldsymbol{A}_k, \theta_k)$ in the following sense:*

$$\varphi_k \in L^2(H^1(\Omega)/\mathbb{C}),\ \boldsymbol{A}_k \in L^2(\boldsymbol{H}_0^1(\Omega)),\ \theta_k \in L^2(H^1(\Omega)) \cap C\left([0,T]; L^2(\Omega)\right), \tag{36}$$

$$\int_\Omega \sigma(\theta_k)\nabla\varphi_k \cdot \nabla\bar{\psi} + \int_\Gamma \boldsymbol{j}_k\bar{\psi} = 0, \quad \psi \in H^1(\Omega)/\mathbb{C},\ \text{a.e. } t \in (0,T), \tag{37}$$

$$\begin{aligned} &i\omega\int_\Omega \sigma(\theta_k)\boldsymbol{A}_k \cdot \bar{\boldsymbol{v}} + \int_D \frac{1}{\mu}\nabla \times \boldsymbol{A}_k \cdot \nabla \times \bar{\boldsymbol{v}} + \delta\int_D \nabla \cdot \boldsymbol{A}_k \nabla \cdot \bar{\boldsymbol{v}} \\ &\quad + \int_\Omega \sigma(\theta_k)\nabla\varphi_k \cdot \bar{\boldsymbol{v}} = 0, \quad \boldsymbol{v} \in \boldsymbol{H}_0^1(D),\ \text{a.e. } t \in (0,T), \end{aligned} \tag{38}$$

$$\langle\theta_{k,t}, v\rangle_{V',V} + \int_\Omega \kappa(\theta_k)\nabla\theta_k\nabla v = \int_\Omega F_k v \quad \text{a.e. } t \in (0,T),\ v \in V = H^1(\Omega), \tag{39}$$

$$\theta_k(\cdot,0) = T_k(\theta_0). \tag{40}$$

Moreover, the following estimate holds:

$$\begin{aligned} \frac{1}{2}\int_\Omega |\theta_k(t)|^2 + \kappa_1\int_0^t\int_\Omega |\nabla\theta_k|^2 &\le \frac{1}{2}\int_\Omega |\theta_k(t)|^2 + \int_0^t\int_\Omega \kappa(\theta_k)|\nabla\theta_k|^2 \\ &= \frac{1}{2}\int_\Omega |T_k(\theta_0)|^2 + \int_0^t\int_\Omega F_k\theta_k, \quad \text{for all } t \in [0,T]. \end{aligned} \tag{41}$$

Remark 4. In view of the regularity of $\boldsymbol{j}_k$, it can be shown that $\varphi_k \in L^\infty(H^1(\Omega)/\mathbb{C})$ and $\boldsymbol{A}_k \in L^\infty(\boldsymbol{H}_0^1(D))$.

5.2. A priori estimates

Applying Lax–Milgram's theorem we deduce that

$$(\varphi_k) \quad \text{is bounded in } L^2(H^1(\Omega)/\mathbb{C}), \tag{42}$$

$$(\boldsymbol{A}_k) \quad \text{is bounded in } L^2(\boldsymbol{H}_0^1(D)). \tag{43}$$

Then, from the definition of F_k, (42) and (43), we have that

$$(F_k) \quad \text{is bounded in } L^1(\Omega_T). \tag{44}$$

The estimate (44) together with (H.5) lead us to the following:

$$(\theta_k) \quad \text{is bounded in } L^p(W^{1,p}(\Omega)),\ \text{for all } 1 \le p < 5/4. \tag{45}$$

Remark 5. A Proof of (45) can be found in [20] in the case of homogeneous Dirichlet boundary conditions. An extension to the case of homogeneous Neumann boundary conditions was first studied in [5]. Using the same techniques derived in these works, it can be shown that (45) also holds in our framework.

From (H.1) and (45) we obtain that $(\kappa(\theta_k)\nabla\theta_k)$ is bounded in $L^p(\boldsymbol{L}^p(\Omega))$, wherefrom $(\nabla\cdot(\kappa(\theta_k)\nabla\theta_k))$ is bounded in $L^1((W^{1,p'}(\Omega))')$. Also, in view of Remark 3, one has the continuous embedding

$$L^1(\Omega)+(W^{1,p'}(\Omega))' \hookrightarrow (W^{1,p'}(\Omega))', \quad \text{with } 1\le p<5/4,$$

and, consequently,

$$(\theta_{k,t}) \quad \text{is bounded in } L^1((W^{1,p'}(\Omega))'),\ 1\le p<5/4. \tag{46}$$

5.3. *Passing to the limit*

Consider $1\le q<3p/(3-p)$, $X=W^{1,p}(\Omega)$, $B=L^{\bar{p}}(\Omega)$ and $Y=(W^{1,p'}(\Omega))'$. Since the embeddings $X\hookrightarrow B$ and $B\hookrightarrow Y$ are compact and continuous, respectively, Lemma 1 implies that the Banach space

$$\mathcal{W}=\left\{v\in L^1(W^{1,p}(\Omega))/v_{,t}\in L^1((W^{1,p'}(\Omega))')\right\}$$

is compactly embedded in $L^p(L^q(\Omega))$. On the other hand, if $1\le p<5/4$ then $1\le q<15/7$ and, thanks to (45) and (46),

$$(\theta_k) \quad \text{is relatively compact in } L^p(L^q(\Omega)),\ 1\le p<\frac{5}{4},\ 1\le q<\frac{15}{7}. \tag{47}$$

Thus there exists a function $\theta\in L^p(L^q(\Omega))$ such that, up to a subsequence,

$$\theta_k\to\theta \quad \text{strongly in } L^p(L^q(\Omega)) \text{ and a.e. in } \Omega_T. \tag{48}$$

Furthermore, since $(\sigma(\theta_k))$ is bounded in $L^\infty(\Omega_T)$, from the definition of σ, (H.1) and (48) we deduce that

$$\sigma(\theta_k)\rightharpoonup\sigma(\theta) \quad \text{weakly-}* \text{ in } L^\infty(\Omega_T) \text{ and a.e. in } \Omega_T. \tag{49}$$

If we also assume (H.6) then

$$\theta\in C([0,T];L^1(\Omega)). \tag{50}$$

The Proof of (50) is well-known. In fact, it can be shown that $(\theta_k)\subset C([0,T];L^1(\Omega))$ is a Cauchy sequence in this space and, therefore, $\theta_k\to\theta$ strongly in $C([0,T];L^1(\Omega))$ (see [21]).

As far as φ_k is concerned, we firstly have that (42) implies the existence of a function $\varphi\in L^2(H^1(\Omega)/\mathbb{C})$ such that, up to a subsequence,

$$\varphi_k\rightharpoonup\varphi \quad \text{weakly in } L^2(H^1(\Omega)/\mathbb{C}). \tag{51}$$

From (49) and (51) it is straightforward that

$$\sigma(\theta_k)\nabla\varphi_k\rightharpoonup\sigma(\theta)\nabla\varphi \quad \text{weakly in } L^2(\boldsymbol{L}^2(\Omega)). \tag{52}$$

Making $k\to\infty$ in (37) and bearing in mind (27) and (52), it yields

$$\int_{\Omega_T}\sigma(\theta)\nabla\varphi\cdot\nabla\bar{\psi}=-\int_0^T\int_\Gamma \boldsymbol{j}\,\bar{\psi},\quad \psi\in L^2(H^1(\Omega)/\mathbb{C}). \tag{53}$$

Choosing $\psi=\varphi_k$ in (37) and thanks to (53),

$$\int_{\Omega_T}\sigma(\theta_k)|\nabla\varphi_k|^2=-\int_0^T\int_\Gamma \boldsymbol{j}_k\,\bar{\varphi}_k\overset{k\to\infty}{\longrightarrow}-\int_0^T\int_\Gamma \boldsymbol{j}\,\bar{\varphi}=\int_{\Omega_T}\sigma(\theta)|\nabla\varphi|^2,$$

that is, $\|\sigma(\theta_k)^{1/2}\nabla\varphi_k\|_{L^2(\boldsymbol{L}^2(\Omega))}\to\|\sigma(\theta)^{1/2}\nabla\varphi\|_{L^2(\boldsymbol{L}^2(\Omega))}$. This convergence together with (52) lead us directly to

$$\sigma(\theta_k)^{1/2}\nabla\varphi_k\to\sigma(\theta)^{1/2}\nabla\varphi \quad \text{strongly in } L^2(\boldsymbol{L}^2(\Omega)) \tag{54}$$

and, clearly, by (49),

$$\sigma(\theta_k)\nabla\varphi_k\to\sigma(\theta)\nabla\varphi \quad \text{strongly in } L^2(\boldsymbol{L}^2(\Omega)). \tag{55}$$

Moreover, (H.1) and (55) implies directly that

$$\varphi_k\to\varphi \quad \text{strongly in } L^2(H^1(\Omega)/\mathbb{C}). \tag{56}$$

For the magnetic vector potential one has, from (43), that there exists a function $\boldsymbol{A}\in L^2(\boldsymbol{H}_0^1(D))$ such that, up to subsequence,

$$\boldsymbol{A}_k\rightharpoonup\boldsymbol{A} \quad \text{weakly in } L^2(\boldsymbol{H}_0^1(D)). \tag{57}$$

On the other hand, (49) and (57) implies that

$$\sigma(\theta_k)^{1/2}\boldsymbol{A}_k \rightharpoonup \sigma(\theta)^{1/2}\boldsymbol{A} \quad \text{weakly in } L^2(\boldsymbol{L}^2(\Omega)). \tag{58}$$

Making $k \to \infty$ in (38) and taking into account (49), (55), (57) and (58), we deduce that, for all $\boldsymbol{v} \in L^2(\boldsymbol{H}_0^1(D))$,

$$i\omega \int_{\Omega_T} \sigma(\theta)\boldsymbol{A}\cdot\bar{\boldsymbol{v}} + \int_{D_T} \frac{1}{\mu}\nabla\times\boldsymbol{A}\cdot\nabla\times\bar{\boldsymbol{v}} + \delta\int_{D_T} \nabla\cdot\boldsymbol{A}\nabla\cdot\bar{\boldsymbol{v}} + \int_{\Omega_T} \sigma(\theta)\nabla\varphi\cdot\bar{\boldsymbol{v}} = 0. \tag{59}$$

After some calculations we have that

$$\int_{\Omega_T} \sigma(\theta_k)|\boldsymbol{A}_k|^2 \to \int_{\Omega_T} \sigma(\theta)|\boldsymbol{A}|^2,$$

$$\int_{D_T} \frac{1}{\mu}|\nabla\times\boldsymbol{A}_k|^2 + \delta\int_{D_T} |\nabla\cdot\boldsymbol{A}_k|^2 \to \int_{D_T} \frac{1}{\mu}|\nabla\times\boldsymbol{A}|^2 + \delta\int_{D_T} |\nabla\cdot\boldsymbol{A}|^2,$$

wherefrom, in view of (57) and (58) we obtain, respectively,

$$\boldsymbol{A}_k \to \boldsymbol{A} \quad \text{strongly in } L^2(\boldsymbol{H}_0^1(D)), \tag{60}$$

$$\sigma(\theta_k)^{1/2}\boldsymbol{A}_k \to \sigma(\theta)^{1/2}\boldsymbol{A} \quad \text{strongly in } L^2(\boldsymbol{L}^2(\Omega)). \tag{61}$$

Finally, from (54) and (61) we infer

$$\sigma(\theta_k)|i\omega\boldsymbol{A}_k + \nabla\varphi_k|^2 \to \sigma(\theta)|i\omega\boldsymbol{A} + \nabla\varphi|^2 \quad \text{fuerte en } L^1(\Omega_T), \tag{62}$$

and, therefore,

$$F_k \to F \quad \text{strongly in } L^1(\Omega_T). \tag{63}$$

Remark 6. In practical situations the electric conductivity is not uniformly elliptic. Specifically, σ_0 is given by

$$\sigma_0(x,s) = \begin{cases} \sigma_s(s) & \text{if } x \in \Omega^s,\ s \in \mathbb{R}, \\ \sigma_c(s) & \text{if } x \in \Omega^c,\ s \in \mathbb{R}, \\ 0 & \text{if } x \in D\setminus\bar{\Omega},\ s \in \mathbb{R}. \end{cases}$$

In this case $\sigma = \sigma_0|_{\Omega\times\mathbb{R}}$, $\sigma_1,\ \sigma_2 \in W^{1,\infty}(\Omega)$, and there exist some constant values $C_2, cK_1, K_2 > 0$ such that, for all $s \in \mathbb{R}$,

$$0 < \sigma_s(s) \le C_2, \quad 0 < K_1 \le \sigma_c(s) \le K_2. \tag{64}$$

Since $\sigma_s(s)$ is not bounded below far away from zero, we cannot expect in general the regularity $\phi(\cdot,t) \in H^1(\Omega)$. Moreover, both ϕ and $\mathcal{A}$ are affected by this assumption on σ_s. A common expression for a material like steel used in numerical simulations is $\sigma_s(s) = 1/(a + bs + cs^2 + ds^3)$ for some constants values $a, b, c, d \in \mathbb{R}$.

The main difficulty lies in the assumption (64). In fact the existence of weak solutions to (1)–(8) under (64) is an open problem.

Acknowledgments

This research was partially supported by Ministerio de Economía y Competitividad of the Spanish government under grant MTM2010-16401 with the participation of FEDER, and Consejería de Educación y Ciencia de la Junta de Andalucía, research group FQM-315.

The authors wish to thank the referees for their comments and suggestions, which have improved the presentation of this paper.

References

[1] J.M. Díaz Moreno, C. García Vázquez, M.T. González Montesinos, F. Ortegón Gallego, Analysis and numerical simulation of a induction–conduction model arising in steel heat treating, J. Comput. Appl. Math. 236 (2012) 3007–3015.
[2] S.N. Antontsev, M. Chipot, The thermistor problem: existence, smoothness, uniqueness, blowup, SIAM J. Math. Anal. 25 (4) (1994) 1128–1156.
[3] G. Cimatti, Existence of weak solutions for the nonstationary problem of the Joule heating of a conductor, Ann. Mat. Pura Appl. 162 (1) (1992) 33–42.
[4] M.T. González Montesinos, F. Ortegón Gallego, The evolution thermistor problem with degenerate thermal conductivity, Commun. Pure Appl. Anal. 1 (3) (2002) 313–325.
[5] S. Clain, Analyse mathématique et numérique d'un modèle de chauffage par induction, Ph.D. Thesis, N. 1240, Laussan, EPFL, 1994.
[6] F. Tröltzsch, I. Yousept, PDE-constrainted optimization of time-dependent 3D electromagnetic induction heating by alternating voltages, ESAIM Math. Model. Numer. Anal. 46 (2012) 709–729.
[7] K. Chełmiński, D. Hömberg, D. Kern, On a thermomechanical model of phase transitions in steel, Adv. Math. Sci. Appl. 18 (2008) 119–140.
[8] J.M. Díaz Moreno, C. García Vázquez, M.T. González Montesinos, F. Ortegón Gallego, Un modelo para la descripción de las transiciones de fases en una barra de acero, in: Actas XX Congreso de Ecuaciones Diferenciales y Aplicaciones/X Congreso de Matemática Aplicada, Sevilla, 2007, pp. 24–28.

[9] J.M. Díaz Moreno, C. García Vázquez, M.T. González Montesinos, F. Ortegón Gallego, Numerical simulation of a induction–conduction model arising in steel hardening, in: World Congress on Engineering 2009, in: Lecture Notes in Engineering and Computer Science, vol. II, 2009, pp. 1251–1255.
[10] J. Fuhrmann, D. Hömberg, M. Uhle, Numerical simulation of induction hardening of steel, COMPEL 18 (3) (1999) 482–493.
[11] D. Hömberg, A mathematical model for induction hardening including mechanical effects, Nonlinear Anal. RWA 5 (2004) 55–90.
[12] D. Hömberg, J. Sokolowski, Optimal shape design of inductor coils for surface hardening, SIAM J. Control Optim. 42 (3) (2003) 1087–1117.
[13] D. Hömberg, W. Weiss, PID control of laser surface hardening of steel, IEEE Trans. Control Syst. Technol. 14 (5) (2006) 896–904.
[14] A. Bermúdez, J. Bullón, F. Pena, P. Salgado, A numerical method for transient simulation of metallurgical compound electrodes, Finite Elem. Anal. Des. 39 (2003) 283–299.
[15] A. Bermúdez, D. Gómez, M.C. Muñiz, P. Salgado, Transient numerical simulation of a thermoelectrical problem in cylindrical induction heating furnaces, Adv. Comput. Math. 26 (2007) 39–62.
[16] F.J. Pena Brage, Contribución al modelado matemático de algunos problemas en la metalurgia del silicio, Ph.D. Thesis, Universidade de Santiago de Compostela, 2003.
[17] H.M. Yin, Regularity of weak solution to Maxwell's equations and applications to microwave heating, J. Differential Equations 200 (2004) 137–161.
[18] C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potencials in three-dimensional non-smooth domais, Math. Methods Appl. Sci. 21 (1998) 823–864.
[19] J. Simon, Compact sets in the space $L^p(0, T; B)$, Ann. Mat. Pura Appl. 146 (4) (1987) 65–96.
[20] L. Boccardo, T. Gallouët, Non-linear elliptic and parabolic equations involving measure data, J. Funct. Anal. 87 (1989) 149–169.
[21] P.L. Lions, Mathematical Topics in Fluid Mechanics, Vol. 1, Oxford Science Publications, 1998.