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EXPONENTIAL STABILITY OF AN INCOMPRESSIBLE NON-NEWTONIAN FLUID WITH DELAY

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Abstract. The existence and uniqueness of stationary solutions to an incompressible non-Newtonian fluid are first established. The exponential stability of steady-state solutions is then analyzed by means of four different approaches. The first is the classical Lyapunov function method, while the second one is based on a Razumikhin type argument. Then, a method relying on the construction of Lyapunov functionals and another one using a Gronwall-like lemma are also exploited to study the stability, respectively. Some comments concerning several open research directions about this model are also included.

1. **Introduction.** The investigation of non-Newtonian fluids has been receiving much attention over the recent years, mainly due to their importance for the understanding of fluid materials motion in real life which cannot be characterized by Newtonian fluids (such as the classic Navier-Stokes equations). As examples of these fluids we can cite ketchup, toothpaste, saliva, and synovial fluid, Bingham plastics (like clay suspensions, drilling mud and mustard), latex paint, blood plasma, etc (see [2, 3, 16, 18]). On the other hand, delay effects have been proved to be useful in the modeling of physical and biological phenomena, as well as in other real world applications. For instance, when we want to use some types of external forcing terms to control a system in control engineering, it seems natural to assume that these forces should take into account not only the current state of the system, but also some part of its history, sometimes even the whole history.

The existence and uniqueness of solution, the existence of maximal compact attractor and global (or pullback) attractor for non-Newtonian equations have been studied in [1, 2, 3, 13, 17, 23, 24, 25], while Guo et al. analyzed in [12] the martingale stationary solutions for some stochastic non-Newtonian fluids without delay. However, to the best of our knowledge, there are no available works concerning the

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local stability analysis of incompressible non-Newtonian fluids containing hereditary characteristics (constant, distributed or variable delay, memory, etc).

Enlightened by the analysis carried out in [6] in the case of $2D$–Navier-Stokes equations, in the current paper we focus on the asymptotic behavior of the stationary solution of the following incompressible non-Newtonian fluid with delay in a $2D$ bounded domain $\Omega \subset \mathbb{R}^2$ whose existence and uniqueness of solutions have already been analyzed in [17] (see Theorem 3.1 in Section 3 for more details),

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nabla \cdot \mu(e(u)) + g(t, u_t) + f(x,t), \quad \text{in } (\tau, +\infty) \times \Omega, \tag{1}$$

$$\nabla \cdot u = 0, \quad \text{in } (\tau, +\infty) \times \Omega, \tag{2}$$

$$u(\tau + \theta, x) = \phi(\theta, x), \quad \theta \in [-h, 0], \; x \in \Omega. \tag{3}$$

The above system (1)-(3) is equipped with below boundary conditions

$$u = 0, \quad \tau_{ij}\nu_j\nu_l = 0, \; i, j, l = 1, 2, \; \text{on } \partial\Omega \times (\tau, +\infty), \tag{4}$$

where the velocity of the fluid $u = u(x,t) = (u^{(1)}, u^{(2)})$ is an unknown vector function, the external function $f(x,t) = (f^{(1)}, f^{(2)})$ is time-dependent, p is the pressure and $h > 0$ is the delay time. Besides, $g(t, u_t)$ represents the influence of an external force with some kind of delay, memory or hereditary characteristics. In (4), $u = 0$ is the no-slip condition, namely, the fluid has zero velocity relative to the boundary, and $\nu = (\nu_1, \nu_2)$ is the outer unit normal to $\partial\Omega$, while $\tau_{ij}\nu_j\nu_l = 0$ with $\tau_{ij}e = 2\mu_1 \frac{\partial e_{ij}}{\partial x_l}$, $i, j, l = 1, 2$, expresses that the first moments of the traction vanishes on $\partial\Omega$, which is an immediate consequence of the principle of practical work.

Problem (1)-(4) describes the motion of an isothermal incompressible viscous fluid with the extra stress tensor $\mu(e(u)) = (\mu_{ij}(e(u)))_{2\times 2}$, and which is a matrix of order 2×2 with

$$\begin{aligned} \mu_{ij}(e(u)) &= 2\mu_0(\epsilon + |e|^2)^{-\frac{\alpha}{2}} e_{ij} - 2\mu_1 \Delta e_{ij}, \quad i, j = 1, 2, \\ e_{ij} = e_{ij}(u) &= \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), \qquad |e|^2 = \sum_{i,j=1}^{2} |e_{ij}|^2, \end{aligned} \tag{5}$$

where μ_0, μ_1, ϵ and α $(0 < \alpha < 1)$ are temperature and pressure-depended positive constants.

Our primary objective is to show the applicability of various methods developed in [6, 9, 14, 15] to analyze the exponential stability of steady-state solutions of our problem. More specifically, the classical Lyapunov theory is used to show the exponential stability of stationary solutions in the cases that the delay terms are continuously differentiable. Fortunately, this assumption, which may be somehow restrictive, can be weaken by an appropriate application of the Razumikhin technique, where only the continuity on the operators of the model is needed but more general delay terms are allowed, in fact, continuity is the only requirement for delay terms. Furthermore, our third way to study the asymptotic behavior of problem (1)-(4) is to exploit the construction of Lyapunov functionals. It is worth pointing out that when we are able to construct suitable Lyapunov functionals, better stability results can be achieved. The fourth alternative is based on a Gronwall-like lemma, which only requires measurability of the delay functions but still provides us with exponential stability of the steady-state solutions under appropriate sufficient conditions.

Nevertheless, we first need to prove the existence and eventual uniqueness of stationary solutions, which is not a trivial task due to the difficulties in handling the nonlinear term $N(u)$. Indeed, the proof of the existence of stationary solutions is much more complicated and involved when we compare with other models, for example, Navier-Stokes. In other words, many more technicalities are required to deal with the nonlinear term $N(u)$ and to obtain the existence of stationary solution, which represents one of the main difficulties of this work. In this respect, it is worth mentioning that Guo and Lin studied in [11] the existence and uniqueness of stationary solutions of non-Newtonian viscous incompressible fluids without delay, but this reference does not contain a complete proof for the existence of such stationary solution, a gap which is solved in our current paper since the result in [11] can be obtained as a particular case of the analysis we are doing in this paper by just taking $h = 0$. We would also like to recall that the existence and uniqueness of solutions, and the existence of pullback attractors of our delay model have been investigated in our previous work [17].

The contents of this paper are the following. In Section 2, we recall some abstract phase spaces and operators that will be used in this work, and we present two typical examples of delay terms. Section 3 is devoted to proving existence and eventual uniqueness of stationary solutions for our problem (1)-(4). Four different methods are applied to study the exponential stability of the steady-state of Eq. (1) in Section 4. Finally, in Section 5, we include some remarks about possible generalizations and variants as well as some future open directions to continue investigating this challenging field of non-Newtonian fluids.

2. Preliminaries.

Although the content of this section with preliminary results and notations can be shortened and, instead, we could suggest the reader to read some already published literature (e.g. [1, 3, 13, 24]), we prefer to include them here for the reader convenience and to make the paper as much self-contained as possible.

Unless otherwise is stated, the letters c_i, $i = 1, 2, 3, \cdots$ denote positive constants, and for short we will write $\|\cdot\|$ instead of $\|\cdot\|_{L^2(\Omega)}$.

Let $\mathcal{V}$ denote the set $\{u \in C_0^\infty(\Omega) \times C_0^\infty(\Omega) : u = (u_1, u_2),\ \nabla \cdot u = 0\}$, then H is the closure of $\mathcal{V}$ in $L^2(\Omega)$ with norm $\|\cdot\|$, and H' is the dual space of H, W denotes the closure of $\mathcal{V}$ in $H^2(\Omega)$ with norm $\|\cdot\|_W$, and W' is the dual space of W, while $(\cdot,\cdot)$−the inner product in H, and $\langle \cdot, \cdot \rangle$ the dual pairing between W and W'.

And $\mathrm{dist}_M(X, Y)$ is the Hausdorff semi-distance between $X, Y \subset M$, where M is a normed space, defined by

$$\mathrm{dist}_M(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|x - y\|_M.$$

Set

$$a(u, v) = \sum_{i,j,k=1}^{2} \left(\frac{\partial e_{ij}(u)}{\partial x_k}, \frac{\partial e_{ij}(v)}{\partial x_k} \right) = \sum_{i,j,k=1}^{2} \int_\Omega \frac{\partial e_{ij}(u)}{\partial x_k} \cdot \frac{\partial e_{ij}(v)}{\partial x_k} dx, \quad u, v \in W, \tag{6}$$

which defines a positive definite symmetric bilinear form on W (see [24]). It follows from Lax-Milgram theorem that we can define an isometric operator $A \in \mathcal{L}(W, W')$ by

$$\langle Au, v \rangle = a(u, v), \quad u, v \in W.$$

Notice that $D(A) = \{u \in W : Au \in H\}$, it turns out that $D(A)$ is a Hilbert space and A is also an isometry from $D(A)$ to H. In fact, $A = P\Delta^2$, where P is the Leray projector from $L^2(\Omega)$ to H. We also define a continuous trilinear form on $H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$b(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx, \quad u, v, w \in H_0^1(\Omega).$$

Since $W \subset H_0^1(\Omega)$, $b(\cdot, \cdot, \cdot)$ is continuous on $W \times W \times W$ and it is easy to check (see [22]) that

$$b(u, v, w) = -b(u, w, v), \quad b(u, v, v) = 0, \quad \forall u, v, w \in W. \tag{7}$$

Now we can define a continuous functional $B(u) := B(u, u)$ from $W \times W$ to W', for any $u \in W$,

$$\langle B(u), w\rangle = b(u, u, w), \qquad \forall w \in W. \tag{8}$$

To finish, we set

$$\mu(u) = 2\mu_0(\epsilon + |e(u)|^2)^{-\alpha/2},$$

for $u \in W$, and define a nonlinear form on $W \times W \times W$ by

$$n(u, v, w) = \sum_{i,j=1}^{2} \int_\Omega \mu(u) e_{ij}(v) e_{ij}(w) dx, \quad u, v, w \in W.$$

Similarly, we define a nonlinear functional $N(u)$

$$\langle N(u), v\rangle = \sum_{i,j=1}^{2} \int_\Omega \mu(u) e_{ij}(u) e_{ij}(v) dx, \quad \forall v \in W. \tag{9}$$

Then the functional $N(u)$ is continuous from W to W'. When $u \in D(A)$, we can extend $N(u)$ to H by setting

$$\langle N(u), v\rangle = -\int_\Omega \{\nabla \cdot [\mu(u)e(u)] \cdot v\} \, dx, \quad \forall v \in H. \tag{10}$$

From a physical point of view, problem (1)-(4) can be formulated as

$$\begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p &= \nabla \cdot \left(2\mu_0(\epsilon + |e|^2)^{-\frac{\alpha}{2}} e - 2\mu_1 \Delta e\right) \\ &\qquad + g(t, u_t) + f(x, t), \text{ in } (\tau, +\infty) \times \Omega, \end{aligned} \tag{11}$$

$$\nabla \cdot u = 0, \quad \text{in } (\tau, +\infty) \times \Omega, \tag{12}$$

$$u = 0, \ \tau_{ij}\nu_j\nu_l = 0, \quad \text{on } \partial\Omega \times (\tau, \infty), \tag{13}$$

$$u(\tau + \theta, x) = \phi(\theta, x), \quad \theta \in [-h, 0], \ x \in \Omega. \tag{14}$$

As usual, with an abstract formulation we can ignore the pressure and rewrite our problem (11)-(14) as (see [3, 23])

$$\frac{\partial u}{\partial t} + 2\mu_1 Au + B(u) + N(u) = g(t, u_t) + f(x, t), \text{ in } (\tau, +\infty) \times \Omega, \tag{15}$$

$$u(\tau + \theta, x) = \phi(\theta, x), \quad \theta \in [-h, 0], \ x \in \Omega. \tag{16}$$

Here, we define the mapping $u_t : [-h, 0] \times \Omega \to \mathbb{R}^2$ by

$$u_t(\theta, x) = u(t + \theta, x), \quad \theta \in [-h, 0], \ x \in \Omega.$$

It is worth mentioning that this abstract formulation includes several types of delay terms in a unified way. Readers are referred to [4, 7, 8] for more details.

Let X be a Banach space. Denote by C_X the Banach space $C([-h,0];X)$ endowed with the norm $\|\phi\|_{C_X} = \sup_{\theta\in[-h,0]} \|\phi(\theta)\|_X$.

We assume that $g : [\tau, T] \times C_H \mapsto (L^2(\Omega))^2$ satisfies

(g1) For any $\xi \in C_H$, the mapping $[\tau, T] \ni t \mapsto g(t,\xi) \in (L^2(\Omega))^2$ is measurable,
(g2) $g(\cdot, 0) = 0$,
(g3) $\exists\ L_g > 0$ such that for any $t \in [\tau, T]$ and all $\xi, \eta \in C_H$,

$$\|g(t,\xi) - g(t,\eta)\| \le L_g \|\xi - \eta\|_{C_H},$$

Remark 1. As it is pointed out in [7, 10, 19], $(g2)$ is not really a restriction, and condition $(g2)$ and $(g3)$ imply that

$$\|g(t,\xi)\| \le L_g \|\xi\|_{C_H},$$

so that $\|g(\cdot,\xi)\| \in L^\infty(\tau, T)$.

Two typical examples of delay terms satisfying (g1) − (g3) are presented below. The proofs can be found, e.g., in the references [6, 10].

Example 1. Forcing term with variable delay
Let $G : [\tau, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ be a measurable function satisfying $G(t, 0) = 0$ for all $t \in [\tau, T]$, and assume that there exists $L_G > 0$ such that

$$|G(t,u) - G(t,v)|_{\mathbb{R}^2} \le L_G |u - v|_{\mathbb{R}^2}, \quad \forall u, v \in \mathbb{R}^2.$$

Consider a function $\rho(t)$, which plays the role of the delay. Suppose that $\rho(\cdot)$ is measurable and define $g(t,\xi)(x) = G(t, \xi(-\rho(t))(x))$ for each $\xi \in C_H$, $x \in \Omega$ and $t \in [\tau, T]$. Notice that, in this case, the delay term g in our problem becomes

$$g(t,\xi) = G(t, \xi(-\rho(t))),$$

and conditions (g1)-(g3) are fulfilled.

Example 2. Forcing term with distributed delay
Let $G : [\tau, T] \times [-h, 0] \times \mathbb{R}^2 \to \mathbb{R}^2$ be a measurable function satisfying $G(t, s, 0) = 0$ for all $(t, s) \in [\tau, T] \times [-h, 0]$, and there exists a function $\alpha(\cdot) \in L^2(-h, 0)$ such that

$$|G(t,s,u) - G(t,s,v)|_{\mathbb{R}^2} \le \alpha(s)|u - v|_{\mathbb{R}^2}, \quad \forall u, v \in \mathbb{R}^2,\ \forall (t,s) \in [\tau, T] \times [-h, 0].$$

Define $g(t,\xi)(x) = \int_{-h}^{0} G(t, s, \xi(s)(x))ds$ for each $\xi \in C_H$, $t \in [\tau, T]$, and $x \in \Omega$. Then the delay term g in our problem becomes

$$g(t,\xi) = \int_{-h}^{0} G(t, s, \xi(s))ds,$$

and conditions (g1)-(g3) are satisfied.

3. **Existence and uniqueness of stationary solutions.** In this paragraph, we first recall a result on the existence and uniqueness of solutions for our model, complemented with a statement about the regularity of solutions. Next we prove a result ensuring the existence and uniqueness of stationary solutions to our problem by exploiting the techniques of Galerkin's approximations, Lax-Milgram's theorem as well as Schauder's fixed pointed theorem. The presence of the nonlinear term $N(\cdot)$ requires of a more involved and technical analysis compared with the Newtonian case, which implies the nontrivial character of this proof.

In the sequel, we will use the following inequalities.

$$\|Au\|^2 \ge \lambda_1 \|u\|_W^2, \qquad \|u\|_W^2 \ge \|u\|^2, \tag{17}$$

where constant $\lambda_1 > 0$ denotes the first eigenvalue of the operator A.

Let us recall a result ensuring existence and uniqueness of solution to our problem which was stated and proved in [17].

Theorem 3.1. *([17]) Assume that $(g1)-(g3)$ hold. Let $f \in L^2_{loc}(\mathbb{R}, W')$ and $\phi \in C_H$. Then, for any $\tau \in \mathbb{R}$,*

(a) there exists a unique weak solution u to problem (15) satisfying

$$u \in C([\tau-h,T];H) \cap L^\infty(\tau,T;H) \cap L^2(\tau,T;W), \quad \forall T > \tau.$$

(b) If $\phi(0) \in C_W$, and $f \in L^2_{loc}(\mathbb{R}, H)$, then there exists a unique strong solution u to problem (15) satisfying

$$u \in C([\tau-h,T];W) \cap L^\infty(\tau,T;W) \cap L^2(\tau,T;D(A)), \quad \forall T > \tau.$$

Although our interest in this paper is to analyze the stability properties of solutions in the case of variable delays, we can consider the existence of steady-state solutions in a much more general case which is described below. Indeed, to start our analysis, we suppose that there exists a function $G : \mathbb{R}^2 \to \mathbb{R}^2$ such that for any constant function $\xi(\cdot) : [-h, 0] \to W$, i.e. $\xi(\theta) = \xi^* \in W$ for all $\theta \in [-h, 0]$, it holds

$$g(t,\xi^*)(x) = G(\xi^*(x)), \text{ for all } t \in \mathbb{R}, x \in \Omega \tag{18}$$

where G satisfies

$$G(0) = 0 \tag{19}$$

and that there exits $L_G > 0$ for which

$$|G(u) - G(v)|_{\mathbb{R}^2} \le L_G|u-v|_{\mathbb{R}^2}, \quad \forall\, u, v \in \mathbb{R}^2. \tag{20}$$

Now, we can study existence and uniqueness of steady-state solutions to the equation

$$\frac{du}{dt} + 2\mu_1 Au + B(u) + N(u) = g(t, u_t) + f, \tag{21}$$

with $f \in W'$ independent of t. Recall that such a stationary (or steady-state) solution to (21) is a $u^* \in W$ such that

$$2\mu_1 Au^* + B(u^*) + N(u^*) = g(t, u^*) + f$$

for all $t \ge 0$, which can be written, according to our assumption, as

$$2\mu_1 Au^* + B(u^*) + N(u^*) = f + G(u^*). \tag{22}$$

Theorem 3.2. *Suppose that G satisfies (19)-(20) and $2\lambda_1\mu_1 > L_G$. Then,*

(a) for all $f \in W'$, there exists a stationary solution to (21);

(b) if $f \in (L^2(\Omega))^2$, the stationary solutions belong to $D(A)$;

(c) there exists a constant $C_0(\Omega) > 0$, such that if $(2\lambda_1\mu_1 - L_G)^2 > C_0(\Omega)\|f\|_*$, then the stationary solution to (21) is unique.

Proof. (a) Denote $W_m = \text{span}\{w_1, w_2, \cdots, w_m\}$, where $\{w_n\}_{n=1}^\infty \subset W \cap D(A)$ form a basis of W and are orthonormal in H. Now we claim that for fixed $z^m \in W_m$, there exists u^m satisfying, for every $v^m \in W_m$,

$$2\mu_1(Au^m, v^m) + b(z^m, u^m, v^m) + n(z^m, u^m, v^m) = (G(z^m), v^m) + \langle f, v^m\rangle. \tag{23}$$

Indeed, notice that for each $z^m \in W_m$, the functional $(u, v) \mapsto 2\mu_1(Au, v) + b(z^m, u, v) + n(z^m, u, v)$ is bilinear, continuous and coercive in $W_m \times W_m$. On the other hand, the functional $v \mapsto (G(z^m), v) + \langle f, v\rangle$ is obviously linear and continuous. Thanks to Lax-Milgram's theorem, for each $z^m \in W_m$, there exists a unique $u^m \in W_m$ such that (23) holds true.

Define the mapping $T_m : W_m \mapsto W_m$ given by

$$T(z^m) = u^m.$$

We will see that for each m we can apply Schauder's fixed point theorem to the map T_m (restricted to a suitable subset $K_m \subset W_m$) and ensure that we obtain $u^m \in W_m$ such that

$$\begin{aligned} 2\mu_1(Au^m, v^m) + b(u^m, u^m, v^m) + n(u^m, u^m, v^m) \\ = (G(u^m), v^m) + \langle f, v^m\rangle, \ \forall v^m \in W_m. \end{aligned} \tag{24}$$

Indeed, setting $v^m = u^m$ in (23) yields

$$2\mu_1(Au^m, u^m) + n(z^m, u^m, u^m) = (G(z^m), u^m) + \langle f, u^m\rangle. \tag{25}$$

By (17),

$$2\mu_1(Au^m, u^m) \geq 2\lambda_1\mu_1\|u^m\|_W^2,$$

and

$$\begin{aligned} (G(z^m), u^m) + \langle f, u^m\rangle &\leq L_G\|z^m\|\|u^m\| + \|f\|_*\|u^m\|_W \\ &\leq L_G\|z^m\|\|u^m\|_W + \|f\|_*\|u^m\|_W. \end{aligned}$$

Since $n(z^m, u^m, u^m) \geq 0$, the previous inequalities imply

$$2\lambda_1\mu_1\|u^m\|_W \leq L_G\|z^m\| + \|f\|_*.$$

Because $2\lambda_1\mu_1 > L_G$, one may take $k > 0$ such that $k(2\lambda_1\mu_1 - L_G) \geq \|f\|_*$ and, consequently, $2\lambda_1\mu_1\|u^m\|_W \leq L_G\|z^m\| + k(2\lambda_1\mu_1 - L_G)$.

Define $K_m = \{z \in W_m : \|z\|_W \leq k\}$, which is a convex set of W, and also compact since the inclusion $W \subset H_0^1(\Omega)$ is compact as well. Obviously, $T_m : K_m \to K_m$ is well defined due to the choice of the constant k. Now we will use Schauder's fixed point theorem to establish the existence of stationary solutions. To do this, we still need to verify the continuity of T_m. Actually, take $z_1^m, z_2^m \in W_m$, and denote $u_i^m = T(z_i^m),\ i = 1, 2$, the respective solutions of (23). For any $v^m \in W_m$ we deduce

$$\begin{aligned} 2\mu_1(A(u_1^m - u_2^m), v^m) + b(z_1^m, u_1^m, v^m) - b(z_2^m, u_2^m, v^m) \\ +n(z_1^m, u_1^m, v^m) - n(z_2^m, u_2^m, v^m) = (G(z_1^m) - G(z_2^m), v^m). \end{aligned} \tag{26}$$

In particular, we set $v^m = u_1^m - u_2^m$ in (26), and then using (17) once more,

$$\begin{aligned} 2\lambda_1\mu_1\|u_1^m - u_2^m\|_W^2 \leq\ & b(z_2^m, u_2^m, v^m) - b(z_1^m, u_1^m, v^m) \\ & +n(z_2^m, u_2^m, v^m) - n(z_1^m, u_1^m, v^m) \\ & +(G(z_1^m) - G(z_2^m), v^m). \end{aligned} \tag{27}$$

As for the trilinear term,

$$\begin{aligned} & b(z_2^m, u_2^m, v^m) - b(z_1^m, u_1^m, v^m) \\ & \quad = b(z_2^m - z_1^m, u_1^m, u_1^m - u_2^m) \\ & \quad \leq \|z_2^m - z_1^m\|_{(L^4(\Omega))^2}\|\nabla u_1^m\|_{(L^2(\Omega))^2}\|u_2^m - u_1^m\|_{(L^4(\Omega))^2} \\ & \quad \leq c_0\|z_2^m - z_1^m\|_W\|u_1^m\|_W\|u_2^m - u_1^m\|_W \\ & \quad \leq c_1\|z_2^m - z_1^m\|_W\|u_2^m - u_1^m\|_W. \end{aligned} \tag{28}$$

Then we estimate the nonlinear term,

$$\begin{aligned} & n(z_2^m, u_2^m, v^m) - n(z_1^m, u_1^m, v^m) \\ & \quad = \sum_{i,j=1}^{2} \int_\Omega [\mu(z_2^m)e_{ij}(u_2^m) - \mu(z_1^m)e_{ij}(u_1^m)]e_{ij}(v^m)dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j=1}^{2} \int_{\Omega} [\mu(z_2^m) - \mu(z_1^m)] e_{ij}(u_2^m) e_{ij}(v^m) dx \\
&- \sum_{i,j=1}^{2} \int_{\Omega} \mu(z_1^m) |e_{ij}(u_2^m - u_1^m)|^2 dx \\
&\le \sum_{i,j=1}^{2} \int_{\Omega} [\mu(z_2^m) - \mu(z_1^m)] e_{ij}(u_2^m) e_{ij}(v^m) dx. \qquad (29)
\end{aligned}
$$

Using the mean value theorem to $\mu(z_2^m) - \mu(z_1^m)$, there exists a constant s with $|e(z_1^m)| < s < |e(z_2^m)|$, such that

$$
\begin{aligned}
\mu(z_2^m) - \mu(z_1^m) &= 2\mu_0(\epsilon + |e(z_2^m)|^2)^{-\frac{\alpha}{2}} - 2\mu_0(\epsilon + |e(z_1^m)|^2)^{-\frac{\alpha}{2}} \\
&= 2\mu_0(-\frac{\alpha}{2})(\epsilon + s^2)^{-\frac{\alpha+2}{2}} (|e(z_2^m)|^2 - |e(z_1^m)|^2) \qquad (30) \\
&= -\alpha\mu_0(\epsilon + s^2)^{-\frac{\alpha+2}{2}} (|e(z_2^m)| + |e(z_1^m)|)(|e(z_2^m)| - |e(z_1^m)|).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&n(z_2^m, u_2^m, v^m) - n(z_1^m, u_1^m, v^m) \\
&\le 2\alpha\mu_0 \sum_{i,j=1}^{2} \int_{\Omega} (\epsilon + |e(z_1^m)|^2)^{-\frac{\alpha+2}{2}} |e(z_2^m)||e(z_2^m - z_1^m)||e_{ij}(u_2^m)||e_{ij}(v^m)| dx \\
&\le 2\alpha\mu_0 \epsilon^{-\frac{\alpha+2}{2}} \|e(z_2^m)\|_{(L^4(\Omega))^2} \|e(z_2^m - z_1^m)\|_{(L^4(\Omega))^2} \|e_{ij}(u_2^m)\|_{(L^4(\Omega))^2} \times \qquad (31) \\
&\qquad \times \|e_{ij}(u_1^m - u_2^m)\|_{(L^4(\Omega))^2} \\
&\le 2\alpha\mu_0 \epsilon^{-\frac{\alpha+2}{2}} c_2 \|z_2^m\|_W \|z_1^m - z_2^m\|_W \|u_2^m\|_W \|u_1^m - u_2^m\|_W \\
&\le 2\alpha\mu_0 \epsilon^{-\frac{\alpha+2}{2}} c_3 \|z_1^m - z_2^m\|_W \|u_1^m - u_2^m\|_W.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(G(z_1^m) - G(z_2^m), u_1^m - u_2^m) &\le L_G \|z_1^m - z_2^m\| \|u_1^m - u_2^m\| \\
&\le L_G \|z_1^m - z_2^m\|_W \|u_1^m - u_2^m\|_W.
\end{aligned}
$$

Thanks to all above inequalities we obtain

$$
2\lambda_1\mu_1 \|u_1^m - u_2^m\|_W^2 \le (c_1 + 2\alpha\mu_0\epsilon^{-\frac{\alpha+2}{2}} c_3 + L_G) \|z_1^m - z_2^m\|_W \|u_1^m - u_2^m\|_W. \qquad (32)
$$

The continuity of the mapping $T : z \mapsto u$ in K_m follows from (32). Therefore, by Schauder's fixed point theorem, there exists $z^m \in K_m$ such that $T(z^m) = z^m$, which means that (24) holds true for every m. Next, we pass to the limit on the solutions and conclude the existence of a stationary solution u to (21). Choosing $v^m = u^m$ in (24), we deduce

$$
2\mu_1(Au^m, u^m) + n(u^m, u^m, u^m) = (G(u^m), u^m) + \langle f, u^m \rangle.
$$

Due to some standard computations,

$$
(2\lambda_1\mu_1 - L_G) \|u^m\|_W \le \|f\|_*,
$$

which provides a uniform bound of u^m in W (namely, $\|u^m\|_W \le (2\lambda_1\mu_1 - L_G)^{-1} \|f\|_*$). We can extract a weakly convergent subsequence (relabeled the same) $u^m \rightharpoonup u$ in W, by the compact injections $((H^2(\Omega))^2 \subset (H_0^1(\Omega))^2 \subset (L^2(\Omega))^2)$, we have $\|u^m - u\|_{(H_0^1(\Omega))^2} \to 0$ and $\|u^m - u\|_{(L^2(\Omega))^2} \to 0$.

To proceed, we fix any $w_j \in W_m$. Since we have a subsequence of equations (24) for every m greater than j, it is clear that we can pass to the limit on every term to obtain

$$2\mu_1(Au, w_j) + b(u, u, w_j) + n(u, u, w_j) = (G(u), w_j) + \langle f, w_j\rangle. \qquad (33)$$

The first term is obtained by the weak convergence $u^m \rightharpoonup u$ in W. Indeed,

$$2\mu_1(Au^m, w_j) = 2\mu_1(\frac{\partial e_{ij}(u^m)}{\partial x_k}, \frac{\partial e_{ij}(w_j)}{\partial x_k}) \rightharpoonup 2\mu_1(\frac{\partial e_{ij}(u)}{\partial x_k}, \frac{\partial e_{ij}(w_j)}{\partial x_k}) = 2\mu_1(Au, w_j),$$

as $m \to \infty$.

The trilinear term

$$\begin{aligned}
&b(u^m, u^m, w_j) - b(u, u, w_j) = -b(u^m - u, w_j, u^m) - b(u, w_j, u^m - u)\\
&\le c_4\|u^m - u\|_{(L^4(\Omega))^2}\|w_j\|_{(H_0^1(\Omega))^2}\|u^m\|_{(L^4(\Omega))^2}\\
&\quad + c_5\|u\|_{(L^4(\Omega))^2}\|w_j\|_{(H_0^1(\Omega))^2}\|u^m - u\|_{(L^4(\Omega))^2}\\
&\le c_6\|u^m - u\|^{1/2}_{(L^2(\Omega))^2}\|u^m - u\|^{1/2}_{(H_0^1(\Omega))^2}\|w_j\|_{(H_0^1(\Omega))^2}\|u^m\|_{(L^2(\Omega))^2}\|u^m\|^{1/2}_{(H_0^1(\Omega))^2}\\
&\quad + c_7\|u\|_{(L^2(\Omega))^2}\|u\|^{1/2}_{(H_0^1(\Omega))^2}\|w_j\|_{(H_0^1(\Omega))^2}\|u^m - u\|^{1/2}_{(L^2(\Omega))^2}\|u^m - u\|^{1/2}_{(H_0^1(\Omega))^2} \to 0.
\end{aligned}$$

The nonlinear term

$$\begin{aligned}
n(u^m, u^m, w_j) - n(u, u, w_j) &= \langle N(u^m) - N(u), w_j\rangle\\
&\le |\langle N(u^m) - N(u), w_j\rangle|\\
&\le c_8\|u^m - u\|_{(H_0^1(\Omega))^2}\|w_j\|_{(H_0^1(\Omega))^2} \to 0.
\end{aligned}$$

And the delay term

$$(G(u^m) - G(u), w_j) \le L_G\|u^m - u\|_{(L^2(\Omega))^2}\|w_j\|_{(L^2(\Omega))^2} \to 0.$$

Thus, (33) holds true for every $w_j \in W_m$. Since the set of linear combinations of $w_1, w_2, \cdots, w_m, \cdots$ is dense in W, we deduce that (21) is satisfied at least by $u^* = u$.

(b) Regularity. From (a) we know that

$$2\mu_1 Au + B(u) + N(u) = G(u) + f, \qquad (34)$$

which must be understood in the sense of $\mathcal{D}'$. Now taking the inner product of (34) with u gives

$$2\mu_1(Au, u) + (N(u), u) = (G(u), u) + (f, u).$$

By standard calculations,

$$\|u\|_W \le (2\lambda_1\mu_1 - L_G)^{-1}\|f\|. \qquad (35)$$

From (34), we have

$$2\mu_1\|Au\| \le \|B(u, u)\| + \|N(u)\| + \|G(u)\| + \|f\|.$$

Notice that

$$\|B(u, u)\| \le c_9\|u\|\|u\|_{(H_0^1(\Omega))^2} \le c_{10}\|u\|_W^2,$$

and

$$\begin{aligned}
\|N(u)\| &= 2\mu_0(\int_\Omega(\epsilon + |\nabla u|^2)^{-\alpha}|\Delta u|^2 dx)^{1/2}\\
&\le 2\mu_0\epsilon^{-\alpha/2}\|\Delta u\|\\
&\le 2\mu_0\epsilon^{-\alpha/2}c_{11}\|u\|_W.
\end{aligned}$$

Hence,

$$\begin{aligned}2\mu_1\|Au\| &\leq c_{10}\|u\|_W^2 + 2\mu_0\epsilon^{-\alpha/2}c_{11}\|u\|_W + L_G\|u\|_W + \|f\| \\ &\leq \Big(c_{10}(2\lambda_1\mu_1 - L_G)^{-1}\|f\| + (2\mu_0\epsilon^{-\alpha/2}c_{11} + 2\lambda_1\mu_1)\Big)(2\lambda_1\mu_1 - L_G)^{-1}\|f\|,\end{aligned}$$

which implies $u \in D(A)$.

(c) Uniqueness. Let u_1, u_2 be two stationary solutions of (21), and $v = u_1 - u_2$, then

$$\begin{aligned}2\mu_1(A(u_1 - u_2), u_1 - u_2) + b(u_1, u_1, v) - b(u_2, u_2, v) + n(u_1, u_1, v) - n(u_2, u_2, v) \\ = (G(u_1) - G(u_2), v).\end{aligned}$$

Note that $n(u_1, u_1, v) - n(u_2, u_2, v) \geq 0$, and

$$\begin{aligned}|b(u_1, u_1, v) - b(u_2, u_2, v)| &= |b(v, u_2, v)| \\ &\leq C_0(\Omega)\|v\|_W^2\|u_2\|_W \\ &\leq C_0(\Omega)(2\lambda_1\mu_1 - L_G)^{-1}\|f\|_*\|u_1 - u_2\|_W^2,\end{aligned}$$

$$(G(u_1) - G(u_2), v) \leq L_G\|u_1 - u_2\|_W^2,$$

whence

$$2\lambda_1\mu_1\|u_1 - u_2\|_W^2 \leq \left(L_G + C_0(\Omega)(2\lambda_1\mu_1 - L_G)^{-1}\|f\|_*\right)\|u_1 - u_2\|_W^2,$$

and therefore

$$\left[(2\lambda_1\mu_1 - L_G) - C_0(\Omega)(2\lambda_1\mu_1 - L_G)^{-1}\|f\|_*\right]\|u_1 - u_2\|_W^2 \leq 0.$$

Since $(2\lambda_1\mu_1 - L_G)^2 - C_0(\Omega)\|f\|_* > 0$,

$$\|u_1 - u_2\|_W^2 = 0.$$

This completes the proof. □

4. **Local asymptotic behavior: Exponential stability of steady-state solutions.** In this section we will describe four approaches to analyze the long time behavior of solutions. They are: the classical Lyapunov function method, the Lyapunov-Razumikhin type argument, the construction of Lyapunov functionals approach, and a Gronwall-like Lemma technique.

It is worth pointing out that the first method requires a differentiability assumption on the delay term, which can be relaxed by a Razumikhin method approach but at the price of requiring more continuity with respect to time t for the operators in the problem, in addition to the fact that we have to work with strong solutions instead of weak ones. However, a better stability result can be obtained by constructing appropriate Lyapunov functionals as long as one can find the appropriate ones, which is not a straightforward task. Finally, a Gronwall like technique is exploited for the stability analysis by only assuming measurability on the delay term. This scheme has already been used in the analysis of stability properties for the stationary solutions of 2D Navier-Stokes equations with delay (see [6] for more details).

4.1. **Exponential stability: Lyapunov function.** First we show that, under appropriate conditions, our model possesses a unique stationary solution, u_∞, and every weak solution of (1) converges to u_∞ exponentially as $t \to +\infty$.

Theorem 4.1. *Suppose that $g(t, u_t) = G(u(t - \rho(t)))$ with $\rho \in C^1(\mathbb{R}^+; [0, h])$ such that $\rho'(t) \leq \rho_* < 1$ for all $t \geq 0$. Assume that there exists $l_1 = l_1(\Omega) > 0$, such that if $f \in (L^2(\Omega))^2$ and $2\lambda_1\mu_1 > L_G$, in addition,*

$$4\lambda_1\mu_1 > \frac{(2-\rho_*)L_G}{1-\rho_*} + \frac{l_1}{2\lambda_1\mu_1 - L_G}\|f\|. \tag{36}$$

Then, there is a unique stationary solution u_∞ of (21) and every solution of (1) converges to u_∞ exponentially as $t \to +\infty$. More precisely, there exist two positive constant C and λ, such that for all $u_0 \in H$ and $\phi \in L^2(-h, 0; W)$, the solution u of (1) satisfies

$$\|u(t) - u_\infty\|^2 \leq Ce^{-\lambda t}(\|u_0 - u_\infty\|^2 + \|\phi - u_\infty\|^2_{L^2(-h,0;W)}), \quad \forall t \geq 0. \tag{37}$$

Proof. Let $u(t)$ be solution of (1), and $u_\infty \in D(A)$ be a stationary solution to (1). Denote $w(t) = u(t) - u_\infty$, then

$$\frac{dw(t)}{dt} + 2\mu_1 Aw + B(u(t)) - B(u_\infty) + N(u(t)) - N(u_\infty) = G(u(t - \rho(t))) - G(u_\infty).$$

By standard computations

$$\begin{aligned}\frac{d}{dt}e^{\lambda t}\|w(t)\|^2 \quad &\leq \quad (\lambda - 4\lambda_1\mu_1 + L_G)e^{\lambda t}\|w(t)\|^2_W + 2e^{\lambda t}|b(w, w, u_\infty)| \\ &\quad\ + L_G e^{\lambda t}\|w(t - \rho(t))\|^2. \end{aligned}\tag{38}$$

Notice that

$$|b(w, w, u_\infty)| \leq l_0\|w\|_{(L^4(\Omega))^2}\|\nabla w\|_{(L^4(\Omega))^2}\|u_\infty\| \leq l_1\|w\|^2_W\|u_\infty\|_W.$$

On the other hand,

$$2\mu_1(Au_\infty, u_\infty) + (N(u_\infty), u_\infty) = (G(u_\infty), u_\infty) + (f, u_\infty),$$

which implies, arguing as in (34)-(35)

$$\|u_\infty\|_W \leq (2\lambda_1\mu_1 - L_G)^{-1}\|f\|,$$

and

$$\begin{aligned}\frac{d}{dt}e^{\lambda t}\|w(t)\|^2 \quad &\leq \quad \left(\lambda - 4\lambda_1\mu_1 + L_G + l_1(2\lambda_1\mu_1 - L_G)^{-1}\|f\|\right)e^{\lambda t}\|w(t)\|^2_W \\ &\quad\ + L_G e^{\lambda t}\|w(t - \rho(t))\|^2.\end{aligned}$$

Denote by $r(t) = t - \rho(t)$. Then the function $r(\cdot)$ is strictly increasing in $[0, +\infty)$, and there exists $\mu > 0$ such that $r^{-1}(t) \leq t + \mu$ for all $t \geq -\rho(0)$. Thus, by performing the change of variable $\eta = s - \rho(s) = r(s)$ in the integral containing the delay, we obtain

$$\begin{aligned} e^{\lambda t}\|w(t)\|^2 \le\ & \|w(0)\|^2 \\ & + \left(\lambda - 4\lambda_1\mu_1 + L_G + l_1(2\lambda_1\mu_1 - L_G)^{-1}\|f\|\right)\int_0^t e^{\lambda s}\|w(s)\|_W^2 ds \\ & + \int_{-\rho(0)}^{t-\rho(t)} e^{\lambda r^{-1}(\eta)}\|w(\eta)\|^2 \frac{1}{r'(r^{-1}(\eta))} d\eta \\ \le\ & \|w(0)\|^2 \\ & + \left(\lambda - 4\lambda_1\mu_1 + L_G + l_1(2\lambda_1\mu_1 - L_G)^{-1}\|f\|\right)\int_0^t e^{\lambda s}\|w(s)\|_W^2 ds \\ & + \frac{e^{\lambda\mu}}{1-\rho_*}\int_0^t e^{\lambda\eta}\|w(\eta)\|^2 d\eta + \frac{e^{\lambda\mu}}{1-\rho_*}\int_{-h}^0 e^{\lambda\eta}\|w(\eta)\|^2 d\eta \\ \le\ & \|w(0)\|^2 \\ & + \left(\lambda - 4\lambda_1\mu_1 + L_G + l_1(2\lambda_1\mu_1 - L_G)^{-1}\|f\| + \frac{e^{\lambda\mu}}{1-\rho_*}\right) \times \\ & \times \int_0^t e^{\lambda s}\|w(s)\|_W^2 ds + \frac{e^{\lambda\mu}}{1-\rho_*}\int_{-h}^0 e^{\lambda\eta}\|w(\eta)\|^2 d\eta. \end{aligned} \tag{39}$$

Since (36) is satisfied, then there exists $\lambda > 0$, small enough, such that

$$\lambda - 4\lambda_1\mu_1 + L_G + l_1(2\lambda_1\mu_1 - L_G)^{-1}\|f\| + \frac{e^{\lambda\mu}}{1-\rho_*} \le 0,$$

which combines with (39), we conclude that for this λ,

$$e^{\lambda t}\|w(t)\|^2 \le \|w(0)\|^2 + \frac{e^{\lambda\mu}}{1-\rho_*}\int_{-h}^0 e^{\lambda\eta}\|w(\eta)\|^2 d\eta,$$

which implies (37).

Finally, if $\hat{u}_\infty$ is another stationary solution of (21), then applying (37) and letting $t \to +\infty$, one conclude that $\|\hat{u}_\infty - u_\infty\|^2 \le 0$. Because $u_0 = \hat{u}_\infty$ and $\phi = \hat{u}_\infty$. Hence, the stationary solution is unique. The the proof is then completed. □

4.2. **Exponential stability: A Lyapunov-Razumikhin approach.** In the previous part we established the exponential convergence of weak solutions of problem (1) to the unique stationary solution when the variable delay term is continuously differentiable. We will now relax this condition by a Razumikhin method. Only the continuity with respect to time t of operators in this model and the solutions is required, but we need to work with strong solution rather than the weak ones.

Theorem 4.2. *Suppose that g satisfies $(g1)-(g3)$, and for each $\xi \in C([-h,0];W)$, the mapping $t \in [0,+\infty) \mapsto g(t,\xi) \in (L^2(\Omega))^2$ is continuous. Assume that $2\lambda_1\mu_1 > L_G$ and $f \in (L^2(\Omega))^2$, and there exists a unique stationary solution u_∞ of (21) such that for some $\lambda > 0$ it holds*

$$\begin{aligned} & -2\mu_1(A(\phi(0)-u_\infty),\phi(0)-u_\infty) - (B(\phi(0)) - B(u_\infty),\phi(0)-u_\infty) \\ & \quad - (N(\phi(0)) - N(u_\infty),\phi(0)-u_\infty) + (g(t,\phi) - g(t,u_\infty),\phi(0)-u_\infty) \\ & \quad < -\lambda\|\phi(0)-u_\infty\|^2, \end{aligned} \tag{40}$$

whenever $\phi \in C([-h,0];H)$ *with* $\phi(0) \in W$ *satisfies*

$$\|\phi - u_\infty\|^2_{C([-h,0];H)} \leq e^{\lambda h}\|\phi(0) - u_\infty\|^2.$$

Then, the strong solution $u(t;\phi)$ *of* (1) *converges exponentially to the unique stationary solution* u_∞ *as follows*

$$\|u(t;\phi) - u_\infty\|^2 \leq e^{-\lambda t}\|\phi - u_\infty\|^2_{C([-h,0];H)}. \tag{41}$$

Proof. We argue by contradiction. Indeed, if (41) is false, then there exists an initial datum $\phi \in C([-h,0];H)$ with $\phi(0) \in W$ such that

$$\|u(t;\phi) - u_\infty\|^2 > e^{-\lambda t}\|\phi - u_\infty\|^2_{C([-h,0];H)}, \text{ for some values of } t.$$

Write

$$\sigma = \inf\left\{t : \|u(t;\phi) - u_\infty\|^2 > e^{-\lambda t}\|\phi - u_\infty\|^2_{C([-h,0];H)}\right\}.$$

Thus, for $0 \leq t \leq \sigma$,

$$e^{\lambda t}\|u(t;\phi) - u_\infty\|^2 \leq e^{\lambda\sigma}\|u(\sigma;\phi) - u_\infty\|^2 = \|\phi - u_\infty\|^2_{C([-h,0];H)}.$$

Besides, for any $t \in [\sigma, \sigma+\varepsilon]$, there exists $t_k \searrow \sigma$ such that

$$e^{\lambda t_k}\|u(t_k;\phi) - u_\infty\|^2 > e^{\lambda\sigma}\|u(\sigma;\phi) - u_\infty\|^2. \tag{42}$$

However,

$$e^{\lambda(\sigma+\theta)}\|u(\sigma+\theta;\phi) - u_\infty\|^2 \leq e^{\lambda\sigma}\|u(\sigma;\phi) - u_\infty\|^2, \ \theta \in [-h,0],$$

from which we infer that

$$\|u_\sigma - u_\infty\|^2_{C([-h,0];H)} \leq e^{\lambda h}\|u(\sigma;\phi) - u_\infty\|^2 = e^{\lambda h}\|u_\sigma(0) - u_\infty\|^2,$$

which means

$$\begin{aligned}
&-2\mu_1(A(u_\sigma(0) - u_\infty), u_\sigma(0) - u_\infty) - (B(u_\sigma(0)) - B(u_\infty), u_\sigma(0) - u_\infty)\\
&-(N(u_\sigma(0)) - N(u_\infty), u_\sigma(0) - u_\infty) + (g(t,u_\sigma) - g(t,u_\infty), u_\sigma(0) - u_\infty)\\
&\quad < -\lambda\|u_\sigma(0) - u_\infty\|^2.
\end{aligned}$$

Notice that $u(\cdot;\phi) \in C([-h,+\infty);W)$, which together with the continuity of the operators of our model gives that there exists $\epsilon_* > 0$ such that for all $\varepsilon \in (0,\epsilon_*]$ and $t \in [\sigma, \sigma+\varepsilon]$,

$$\begin{aligned}
&-2\mu_1(A(u(t;\phi) - u_\infty), u(t;\phi) - u_\infty) - (B(u(t;\phi)) - B(u_\infty), u(t;\phi) - u_\infty)\\
&-(N(u(t;\phi)) - N(u_\infty), u(t;\phi) - u_\infty) + (g(t,u_t(\cdot;\phi)) - g(t,u_\infty), u(t;\phi) - u_\infty)\\
&\quad < -\lambda\|u(t;\phi) - u_\infty\|^2.
\end{aligned}$$

Thus, denoting by $w(t) = u(t;\phi) - u_\infty$,

$$\frac{dw(t)}{dt} + 2\mu_1 Aw + B(u) - B(u_\infty) + N(u) - N(u_\infty) = g(t,u_t) - g(t,u_\infty).$$

Take inner product of above equation with w, for all $t \in [\sigma, \sigma+\varepsilon]$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + 2\mu_1(Aw,w) + (B(u) - B(u_\infty), w) + (N(u) - N(u_\infty), w)\\
&\qquad = (g(t,u_t) - g(t,u_\infty), w),
\end{aligned}$$

and integrate over $[\sigma, \sigma+\varepsilon]$,

$$e^{\lambda(\sigma+\varepsilon)}\|w(\sigma+\varepsilon;\phi)\|^2 - e^{\lambda\sigma}\|u(\sigma;\phi) - u_\infty\|^2$$

$$
\begin{aligned}
&= \lambda \int_\sigma^{\sigma+\varepsilon} e^{\lambda t}\|w(t;\phi)\|^2 dt - 4\mu_1 \int_\sigma^{\sigma+\varepsilon} e^{\lambda t}(Aw, w)dt \\
&\quad - 2\int_\sigma^{\sigma+\varepsilon} e^{\lambda t}(B(u) - B(u_\infty), w)dt \\
&\quad - 2\int_\sigma^{\sigma+\varepsilon} e^{\lambda t}(N(u) - N(u_\infty), w)dt + 2\int_\sigma^{\sigma+\varepsilon} e^{\lambda t}(g(t,u_t) - g(t,u_\infty), w)dt \\
&\le 0,
\end{aligned}
$$

which contradicts (42). □

The next corollary offers a sufficient condition which implies (40) but easier to verify.

Corollary 1. *Suppose that g satisfies $(g1)-(g3)$, and for all $\xi \in C([-h,0];W)$ the mapping $t \in [0,+\infty) \mapsto g(t,\xi) \in (L^2(\Omega))^2$ is continuous. Assume $2\lambda_1\mu_1 > L_G$, $f \in (L^2(\Omega))^2$ and there exists a constant $l_1 > 0$ such that if*

$$
2\lambda_1\mu_1 > L_G + l_1(2\lambda_1\mu_1 - L_G)^{-1}\|f\|, \tag{43}
$$

then there is a unique stationary solution u_∞ of (21) , and for all $\phi \in C([-h,0];H)$ with $\phi(0) \in W$, it holds

$$
\|u(t;\phi) - u_\infty\|^2 \le e^{-\lambda t}\|\phi - u_\infty\|^2_{C([-h,0];H)}, \text{ for all } t \ge 0.
$$

Proof. Let $\phi \in C([-h,0];H)$ with $\phi(0) \in W$ satisfying

$$
\|\phi - u_\infty\|^2_{C([-h,0];W)} \le e^{\lambda h}\|\phi(0) - u_\infty\|^2,
$$

where $\lambda > 0$ will be specified later. Then

$$
\begin{aligned}
&- 2\mu_1(A(\phi(0) - u_\infty), \phi(0) - u_\infty) - \langle B(\phi(0)) - B(u_\infty), \phi(0) - u_\infty\rangle \\
&- \langle N(\phi(0)) - N(u_\infty), \phi(0) - u_\infty\rangle + (g(t,\phi) - g(t,u_\infty), \phi(0) - u_\infty) \\
&\le -2\lambda_1\mu_1\|\phi(0) - u_\infty\|^2_W - b(\phi(0) - u_\infty, u_\infty, \phi(0) - u_\infty) \\
&\qquad + L_G\|\phi - u_\infty\|\|\phi(0) - u_\infty\| \\
&\le -2\lambda_1\mu_1\|\phi(0) - u_\infty\|^2_W + L_G e^{\lambda h}\|\phi(0) - u_\infty\|^2_W \\
&\qquad + l_1(2\lambda_1\mu_1 - L_G)^{-1}\|f\|\|\phi(0) - u_\infty\|^2_W \\
&= \left(-2\lambda_1\mu_1 + L_G e^{\lambda h} + l_1(2\lambda_1\mu_1 - L_G)^{-1}\|f\|\right)\|\phi(0) - u_\infty\|^2_W.
\end{aligned} \tag{44}
$$

As long as (43) is satisfied, there exists $\lambda > 0$ such that

$$
\lambda - 2\lambda_1\mu_1 + L_G e^{\lambda h} + l_1(2\lambda_1\mu_1 - L_G)^{-1}\|f\| < 0,
$$

and with this λ, it follows directly from (44) that

$$
\begin{aligned}
&- 2\mu_1(A(\phi(0) - u_\infty), \phi(0) - u_\infty) - \langle B(\phi(0)) - B(u_\infty), \phi(0) - u_\infty\rangle \\
&- \langle N(\phi(0)) - N(u_\infty), \phi(0) - u_\infty\rangle + (g(t,\phi) - g(t,u_\infty), \phi(0) - u_\infty) \\
&< -\lambda\|\phi(0) - u_\infty\|^2_W \le -\lambda\|\phi(0) - u_\infty\|^2.
\end{aligned}
$$

□

4.3. **Exponential stability: Construction of Lyapunov functionals.** Our interest in this subsection is to study the exponential stability of solutions to problem (1) by constructing suitable Lyapunov functionals, a method which was proposed by V. Kolmanovskii and L. Shaikhet in [14, 15] and has been extensively used in delay differential equations, as well as in difference equations with discrete or continuous time (see [20, 21] for more details and references).

Let $\tilde{A} : W \to W'$; $f_1(t,\cdot) : C([-h,0];H) \to W'$; $f_2(t,\cdot) : C([-h,0];W) \to W'$ be three families of nonlinear operators satisfying $\tilde{A}(t,0) = 0$, $f_1(t,0) = 0$, $f_2(t,0) = 0$, $t > 0$.

Consider the equation

$$\begin{cases} \dfrac{du}{dt} = \tilde{A}(t,u(t)) + f_1(t,u_t) + f_2(t,u_t), & t > 0, \\ u(s) = \psi(s), & s \in [-h,0], \end{cases} \tag{45}$$

where $u(\cdot;\psi)$ is the solution to (45) corresponding to initial value ψ. A crucial theorem is recalled first, which is a key to our stability investigation.

Theorem 4.3. *(See [5]) Assume that there exists a functional* $V(\cdot,\cdot) : \mathbb{R}_+ \times C_H \longmapsto [0,+\infty)$ *such that the following conditions hold for some positive numbers* δ_1, δ_2 *and* λ*:*

$$V(t,u_t) \geq \delta_1 e^{\lambda t}\|u(t)\|^2, \quad t > 0,$$
$$V(0,u_0) \leq \delta_2 \|\psi\|^2_{C_H},$$
$$\frac{d}{dt}V(t,u_t) \leq 0, \quad t \geq 0,$$

for any $\psi \in C_H$ *such that* $u(\cdot;\psi) \in C([-h,+\infty);H)$*. Then the trivial solution of (45) is exponentially stable.*

Notice that this theorem implies that the stability analysis of Eq. (45) can be reduced to the construction of appropriate Lyapunov functionals.

To this end, consider the evolution equation

$$\frac{du}{dt} = \tilde{A}(t,u(t)) + G(u(t-\rho(t))), \tag{46}$$

where $\tilde{A}(t,\cdot)$, $G : W \to W'$ are proper partial differential operators (see conditions below), which is a particular case of Eq. (45). Now we are going to study exponential stability to problem (46).

Theorem 4.4. *(See [6]) Suppose that the operators in (46) satisfy*

$$\langle \tilde{A}(t,u), u\rangle \leq -\gamma\|u\|^2_W, \gamma > 0$$
$$G : W \to W', \quad \|G(u)\|_* \leq \beta\|u\|_W, \quad u \in W,$$
$$\rho(t) \in [0,h], \ \rho'(t) \leq \rho_* < 1.$$

Then the trivial solution of Eq.(46) is exponentially stable provided

$$\gamma > \frac{\beta}{\sqrt{1-\rho_*}}.$$

We only give a sketchy proof here. The Lyapunov functional V for our model Eq. (21) with $f(t) \equiv 0$ is constructed in the form

$$V = e^{\lambda t}\|u(t)\|^2 + \frac{\varepsilon L_G}{1-\rho_*}\int_{t-\rho(t)}^{t} e^{\lambda(s+h)}\|u(s)\|^2_W ds,$$

and we obtain

$$\begin{aligned}\frac{d}{dt}V(t,u_t) &= \frac{d}{dt}\left(e^{\lambda t}\|u(t)\|^2 + \frac{\varepsilon L_G}{1-\rho_*}\int_{t-\rho(t)}^{t} e^{\lambda(s+h)}\|u(s)\|_W^2 ds\right) \\ &= \lambda e^{\lambda t}\|u(t)\|^2 + 2e^{\lambda t}(-2\mu_1 Au(t) - B(u(t)) - N(u(t)), u(t)) \\ &\quad + 2e^{\lambda t}(G(u(t-\rho(t))), u(t)) \\ &\quad + \frac{\varepsilon L_G}{1-\rho_*}e^{\lambda(t+h)}\|u(t)\|_W^2 - \frac{\varepsilon L_G}{1-\rho_*}(1-\rho')e^{\lambda(t-\rho(t)+h)}\|u(t-\rho(t))\|_W^2 \\ &\le (\lambda - 4\lambda_1\mu_1 + \frac{L_G}{\varepsilon})e^{\lambda t}\|u(t)\|_W^2 + \varepsilon L_G e^{\lambda t}\|u(t-\rho(t))\|_W^2 \\ &\quad + \frac{\varepsilon L_G e^{\lambda h}}{1-\rho_*}e^{\lambda t}\|u(t)\|_W^2 - \varepsilon L_G e^{\lambda t}\|u(t-\rho(t))\|_W^2 \\ &= -(4\lambda_1\mu_1 - \frac{L_G}{\varepsilon} - \lambda - \frac{\varepsilon L_G e^{\lambda h}}{1-\rho_*})e^{\lambda t}\|u(t)\|_W^2\end{aligned}$$

Choosing $\varepsilon = \sqrt{1-\rho_*}$, we have

$$\begin{aligned}\frac{d}{dt}V(t,u_t) &\le -(4\lambda_1\mu_1 - \frac{L_G}{\sqrt{1-\rho_*}} - \lambda - \frac{L_G e^{\lambda h}}{\sqrt{1-\rho_*}})e^{\lambda t}\|u(t)\|_W^2 \\ &= -(4\lambda_1\mu_1 - \frac{2L_G}{\sqrt{1-\rho_*}} - \lambda - \frac{L_G(e^{\lambda h}-1)}{\sqrt{1-\rho_*}})e^{\lambda t}\|u(t)\|_W^2.\end{aligned} \tag{47}$$

Writing

$$h(\lambda) = \lambda + \frac{L_G(e^{\lambda h}-1)}{\sqrt{1-\rho_*}}, \quad h(0)=0,$$

since the function $h(\lambda)$ is continuous respect to λ, there exists $\lambda > 0$ small enough such that

$$2(2\lambda_1\mu_1 - \frac{L_G}{\sqrt{1-\rho_*}}) \ge h(\lambda).$$

Then it follows directly from (47) that $\frac{d}{dt}V(t,u_t) \le 0$, and the Lyapunov functional $V(t,u_t) = e^{\lambda t}\|u(t)\|^2 + \frac{\varepsilon L_G}{1-\rho_*}\int_{t-\rho(t)}^{t} e^{\lambda(s+h)}\|u(s)\|_W^2 ds$ satisfies the conditions in Theorem 4.3, which implies that the trivial solution of Eq. (21) is exponentially stable.

Remark 2. (a) Here $G : W \to W'$ is a Lipschitz continuous operator with Lipschitz constant $L_G > 0$ and $G(0) = 0$. If $G : H \to H$ with Lipschitz constant L_g with $L_g \ge L_G$, then $G : W \to W'$ is Lipschitz, and from $2\lambda_1\mu_1 > \frac{L_g}{\sqrt{1-\rho_*}}$, we obtain that $2\lambda_1\mu_1 > \frac{L_G}{\sqrt{1-\rho_*}}$.
(b) Although applying this method, we also need the differentiability of variable delay function, the stability result that we obtained is better than the first case, in which $2\lambda_1\mu_1 > \frac{(2-\rho_*)L_G}{2(1-\rho_*)}$ is required, but here we only need $2\lambda_1\mu_1 > \frac{L_G}{\sqrt{1-\rho_*}}$, which means we have more choices for μ_1.

4.4. **Exponential stability: A Gronwall argument.** Now we investigate the stability of stationary solutions to Eq. (21) via a Gronwall-like lemma. For convenience, we will consider Eq. (21) with $f(t) \equiv 0$ and $g(t,\phi) = G(\phi(-\rho(t)))$, for $\phi \in C_H$, where $G : H \to H$ is Lipschitz continuous with Lipschitz constant $L_g > 0$ and $G(0) = 0$. For the delay term ρ we only assume that it is measurable and bounded, i.e., $\rho : [0,+\infty) \to [0,h]$. Compared with the ones required in the three previous approaches, this is the weakest assumption. But we still can prove exponential stability of the steady-state solution.

Lemma 4.5. *([9]) Let $y(\cdot) : [-h, +\infty) \to [0, +\infty)$ be a function. Assume that there exist positive numbers $\gamma, \alpha_1, \alpha_2$ such that the following inequality holds:*

$$y(t) \le \begin{cases} \alpha_1 e^{-\gamma t} + \alpha_2 \int_0^t e^{-\gamma(t-s)} \sup\limits_{\theta\in[-h,0]} y(s+\theta) ds, & t \ge 0, \\ \alpha_1 e^{-\gamma t}, & t \in [-h, 0]. \end{cases}$$

Then,

$$y(t) \le \alpha_1 e^{-\mu t}, \quad \text{for all } t \ge -h,$$

where $\mu \in (0, \gamma)$ is given by the unique root of the equation $\frac{\alpha_2}{\gamma-\mu} e^{\mu h} = 1$ in this interval.

Theorem 4.6. *Suppose that $f(t) \equiv 0$ and $g(t, u_t) = G(u(t-\rho(t)))$, where $G : H \to H$ is Lipschitz constant $L_g > 0$ and satisfies $G(0) = 0$. Assume that $\rho : [0, +\infty) \mapsto [0, h]$ is a measurable function. Then the zero solution of (1) is exponentially stable provided*

$$4\lambda_1\mu_1 > L_g.$$

Proof. Let us choose a positive constant λ such that

$$\lambda - 4\lambda_1\mu_1 + L_g > 0.$$

Notice that the weak solution $u(\cdot)$ to model (1) corresponding to the initial datum ϕ satisfies

$$\begin{aligned}
e^{\lambda t}\|u(t)\|^2 &= \|\phi(0)\|^2 + \lambda \int_0^t e^{\lambda s}\|u(s)\|^2 ds - 4\mu_1 \int_0^t e^{\lambda s}(Au(s), u(s)) ds \\
&\quad - 2\int_0^t e^{\lambda s}\langle N(u(s)), u(s)\rangle ds + 2\int_0^t e^{\lambda s}(G(u(s-\rho(s))), u(s)) ds \\
&\le \|\phi(0)\|^2 + \lambda \int_0^t e^{\lambda s}\|u(s)\|^2 ds - 4\lambda_1\mu_1 \int_0^t e^{\lambda s}\|u(s)\|^2 ds \\
&\quad + 2L_g \int_0^t e^{\lambda s}\|u(s-\rho(s))\|\|u(s)\| ds \\
&\le \|\phi(0)\|^2 + (\lambda - 4\lambda_1\mu_1 + L_g) \int_0^t e^{\lambda s}\|u(s)\|^2 ds \\
&\quad + L_g \int_0^t e^{\lambda s}\|u(s-\rho(s))\|^2 ds \\
&\le \|\phi\|^2_{C([-h,0];H)} + (\lambda - 4\lambda_1\mu_1 + 2L_g) \int_0^t e^{\lambda s} \sup_{\theta\in[-h,0]} \|u(s+\theta)\|^2 ds.
\end{aligned}$$

Hence, from the Lemma 4.5, we know that the unique zero solution to Eq. (1) is exponentially stable.

□

5. **Conclusions and comments.** In this work we have exhibited several methods to analyze the exponential stability of incompressible non-Newtonian fluids when some hereditary properties are taken into account in the forcing term of the model, and our analysis has been carried out when the delays are bounded.

In the case of constant delays, the autonomous theory of global attractor may provide an appropriate framework to study the problem. But for more general delay terms, such as variable or distributed delays, the problem becomes non-autonomous and it is necessary to consider a non-autonomous framework for the global asymptotic behavior of the model. Several options, for instance, the theories of skew-product and uniform attractor are available, but we would like to emphasize that the theory of pullback attractors may allow more general non-autonomous terms in the models. In this respect, the existence of pullback attractor of an incompressible non-Newtonian fluids with bounded delay has been established in [17].

Although many other aspects on this model have already been investigated (see [1, 2, 3, 11, 12, 13, 17, 23, 24, 25] and the references therein), there are still many interesting problems related to incompressible non-Newtonian fluids that need to be studied in future. For instance, what are the effects that some environmental noise may produce in the phenomenon, which will then become a stochastic non-Newtonian fluid. Amongst the many topics that we could analyze within the field of stochastic non-Newtonian fluids with delay (bounded or unbounded), we could wonder about the existence and uniqueness of solutions, in particular the stationary one, their stability properties, and the existence and structure of random attractors as well. We plan to work on these problems in some forthcoming papers.

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