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SOME REMARKS ON AN ENVIRONMENTAL DEFENSIVE EXPENDITURES MODEL

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Abstract. In this paper, we consider the environmental defensive expenditures model with delay proposed by Russu in [15] and obtain different results about stability of equilibria in the case of absence of delay. Moreover we provide a more detailed analysis of the stability for equilibria and Hopf bifurcation in the case with delay. Finally, we discuss possible modifications of the model in order to make it more accurate and realistic.

1. **Introduction.** We consider a model introduced in [15] that deals with management of tourism in protected areas (PAs). The model is based on the interactions among visitors V, quality of ecosystem goods E, and capital K, intended as accommodation and entertainment facilities in PAs, which is described by the system:

$$\begin{cases} \dot{V} &=& m_1 E + m_2 K - aV^2, \\ \dot{E} &=& r\left(\bar{P} - E\right) - (b - c\eta)V_d, \\ \dot{K} &=& (1-\eta)\, V_d - \delta K, \end{cases} \tag{1}$$

where $V_d = V(t-\tau)$, $\tau \geq 0$ is the time delay and $\bar{P}$ is the maximum tolerable pollution stock P; $m_1, m_2, a, b, c, \eta, \delta, \bar{P}$ are strictly positive constants, $0 \leq \eta < 1$ and $0 < r < 1$. A detailed discussion of the model can be found in [15] but it is worth mentioning that a part, η, of total revenues is invested to defend the environmental resources while the other part,$(1-\eta)$, is used to increase the capital stock K. Also, we point out that the quality of ecosystem goods is measured as $E = \bar{P} - P$. The idea is to study the stability of the model, that is the sustainability of the tourism in PAs,

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depending on the parameters involved (see also [1], [2], [6]-[10], for more general discussions on this the topic). For this reason the existence and stability of positive equilibria play a crucial role in this context. Our analysis of the equilibria is different from that in [15], in particular we find a necessary and sufficient condition for the existence of a positive equilibrium $(V_\infty, E_\infty, K_\infty)$ inside $\mathbb{R}^+ \times (0, \bar{P}) \times \mathbb{R}^+$ that is (see next section for details):

$$b - c\eta > 0, \qquad \bar{P}a\delta r - m_2(1-\eta)(b - c\eta) > 0,$$

which is in contrast with (a) of Proposition 1 in [15]. We also consider interesting to revise (in section 3 below) the bifurcation analysis for $\tau > 0$ providing more detailed results with respect to those in [15]. Moreover we prove the existence of stability switches for the fixed point in the model with delay. We did not prove that the hypotheses are compatible with the positiveness of E_∞, but still we consider the result interesting enough. In fact, in an attempt of using the model in real cases, the existence of stability switches makes the strategy of the managers of PAs more complicate in order to stabilize the system (see section 3 below). In section 5 of this work we discuss some critical issues about the model and propose some changes in order to improve it.

2. **Fixed point and stability analysis with $\tau = 0$.** The steady states of system (1) in absence of delay are obtained as non-negative solutions of the algebraic system

$$m_1 E + m_2 K - aV^2 = 0, \qquad r\left(\bar{P} - E\right) - (b - c\eta)V = 0, \qquad (1-\eta)\,V - \delta K = 0. \tag{2}$$

As $0 \le P \le \overline{P}$ and $E = \overline{P} - P$, one has that $\overline{P} - E \ge 0$. Hence, when $b - c\eta \le 0$, it is immediately seen that system (2) has no solution (we look for interior steady states, so we exclude the solution $E = \overline{P}$ obtained for $b - c\eta = 0$). Let $b - c\eta > 0$. Then, the existence of fixed points for (1) is related to existence of solutions for the following second order equation in V

$$a\delta r V^2 - \left[(1-\eta)\,r m_2 - (b - c\eta)\,\delta m_1\right] V - m_1 \delta r \overline{P} = 0. \tag{3}$$

A direct calculation shows that Eq. (3) possesses only one positive solution, independently of the sign of the coefficient of V.

Lemma 1. *Let $b - c\eta > 0$. System (1) has a unique fixed point $(V_\infty, E_\infty, K_\infty)$, where*

$$V_\infty = \frac{(1-\eta)\,r m_2 - (b - c\eta)\,\delta m_1 + \sqrt{\left[(1-\eta)\,r m_2 - (b - c\eta)\,\delta m_1\right]^2 + 4a\delta^2 r^2 m_1 \overline{P}}}{2a\delta r},$$

$$E_\infty = \overline{P} - \frac{(b - c\eta)}{r} V_\infty \qquad \textit{and} \qquad K_\infty = \frac{(1-\eta)}{\delta} V_\infty.$$

If we want $E_\infty > 0$ and $b - c\eta > 0$ we obtain the following necessary condition:

$$\bar{P}a\delta r - m_2(1-\eta)(b - c\eta) > 0.$$

The study of local stability of equilibrium solution is based on the localization on the complex plane of the eigenvalues of the Jacobian matrix of (1) evaluated at $(V_\infty, E_\infty, K_\infty)$. The characteristic equation is

$$\det \begin{pmatrix} -2aV_\infty - \lambda & m_1 & m_2 \\ -b + c\eta & -r - \lambda & 0 \\ 1 - \eta & 0 & -\delta - \lambda \end{pmatrix} = 0,$$

namely

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0, \tag{4}$$

where

$$A = 2aV_\infty + \delta + r > 0, \qquad B = 2a\,(\delta + r)\,V_\infty + \delta r - (1-\eta)\,m_2 + (b - c\eta)\,m_1,$$

and

$$\begin{aligned} C &= 2a\delta rV_\infty - (1-\eta)\, rm_2 + (b-c\eta)\,\delta m_1 \\ &= \sqrt{\left[(1-\eta)\, rm_2 - (b-c\eta)\,\delta m_1\right]^2 + 4a\delta^2 r^2 m_1 \overline{P}} \; > 0. \end{aligned} \tag{5}$$

Proposition 2. *Let $b-c\eta>0$. The fixed point $(V_\infty, E_\infty, K_\infty)$ of system* (1) *is locally asymptotically stable in absence of delay if and only if the following condition holds*

$$\begin{aligned} &\left[2a\,(\delta+r)\,V_\infty - (1-\eta)\,m_2 + (b-c\eta)\,m_1\right] 2aV_\infty + (\delta+r)\,\delta r + (\delta+r)^2\, 2aV_\infty \\ &\qquad\qquad - (1-\eta)\,\delta m_2 + (b-c\eta)\, rm_1 > 0. \end{aligned} \tag{S}$$

Proof. By Routh–Hurwitz's condition, all roots of Eq. (4) have negative real parts if and only if $A>0$, $B>0$, $C>0$ and $AB>C$. Since $A>0$ and $C>0$, it is clear that $AB>C$ implies $B>0$. Expliciting the terms A, B and C in $AB>C$ gives

$$\begin{aligned} &4a^2\,(\delta+r)\,V_\infty^2 + 2a\,(\delta+r)^2\,V_\infty + (\delta+r)\,\delta r + \left[-(1-\eta)\,m_2 + (b-c\eta)\,m_1\right] 2aV_\infty \\ &\qquad\qquad - (1-\eta)\,\delta m_2 + (b-c\eta)\, rm_1 > 0. \end{aligned}$$

The statement follows by rewriting the previous inequality. □

Remark 3. From (5) we have $(b-c\eta)\,m_1 > \left[(1-\eta)\, rm_2\right]/\delta - 2arV_\infty$. Therefore,

$$\begin{aligned} &4a^2\,(\delta+r)\,V_\infty^2 + \left[-(1-\eta)\,m_2 + (b-c\eta)\,m_1\right] 2aV_\infty \\ &\qquad = \left[2a\,(\delta+r)\,V_\infty - (1-\eta)\,m_2 + (b-c\eta)\,m_1\right] 2aV_\infty \\ &\qquad > \left[2a\delta V_\infty + \left(\frac{r-\delta}{\delta}\right)(1-\eta)\,m_2\right] 2aV_\infty \end{aligned}$$

and

$$2a\,(\delta+r)^2\,V_\infty - (1-\eta)\,\delta m_2 + (b-c\eta)\, rm_1 > 2a\delta^2 V_\infty + 4a\delta rV_\infty + \left(\frac{r^2-\delta^2}{\delta}\right)(1-\eta)\,m_2.$$

Consequently, if $r \ge \delta$, then condition (S) holds true and the fixed point is always stable.

In [15] the claim (without proof) is that $AB>C$ and as a consequence the fixed point is always stable. However, we only were able to prove the case $r\ge\delta$ leaving the case $r<\delta$ unsolved.

3. **Stability analysis and Hopf bifurcation with $\tau>0$.** The appearance of constant time delay does not affect equilibria. Hence, under the above conditions $(V_\infty, E_\infty, K_\infty)$ is still the unique positive equilibrium of system (1). In this case the associated characteristic equation of the linearization of system (1) at the equilibrium is given by

$$\det\begin{pmatrix} -2aV_\infty-\lambda & m_1 & m_2 \\ (c\eta-b)\,e^{-\lambda\tau} & -r-\lambda & 0 \\ (1-\eta)\,e^{-\lambda\tau} & 0 & -\delta-\lambda \end{pmatrix} = 0,$$

i.e.

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_1\lambda + b_0)\,e^{-\lambda\tau} = 0, \tag{6}$$

where

$$a_2 = 2aV_\infty + r + \delta > 0, \qquad a_1 = 2a\,(\delta+r)\,V_\infty + \delta r > 0, \qquad a_0 = 2a\delta rV_\infty > 0$$

and

$$b_1 = -\left(1-\eta\right)m_2 + \left(b - c\eta\right)m_1, \qquad b_0 = -\left(1-\eta\right)rm_2 + \left(b - c\eta\right)\delta m_1.$$

The stability of the equilibrium point $(V_\infty, E_\infty, K_\infty)$ will change when the system under consideration has zero or a pair of purely imaginary eigenvalues. The former occurs when $\lambda = 0$ in (6), i.e. if $a_0 + b_0 = 2a\delta rV_\infty - (1-\eta)\,rm_2 + (b - c\eta)\,\delta m_1 = 0$, but we know this value to be positive from (5). Thus, $\lambda = 0$ is not a root of the characteristic equation (6). The latter deals with assuming that (6) has a purely imaginary root $\lambda = i\omega$, with $\omega > 0$. Then, it follows from (6) that

$$-\omega^3 i - a_2\omega^2 + a_1\omega i + a_0 + (b_1\omega i + b_0)(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{7}$$

By separating real and imaginary parts in (7), we obtain

$$\omega^3 - a_1\omega = b_1\omega\cos\omega\tau - b_0\sin\omega\tau, \qquad a_2\omega^2 - a_0 = b_1\omega\sin\omega\tau + b_0\cos\omega\tau \tag{8}$$

Adding up the squares of Eqs. (8), we obtain

$$\omega^6 + \left(a_2^2 - 2a_1\right)\omega^4 + \left(a_1^2 - 2a_0a_2 - b_1^2\right)\omega^2 + a_0^2 - b_0^2 = 0. \tag{9}$$

Setting $z = \omega^2$, Eq. (9) becomes

$$h(z) = z^3 + pz^2 + qz + s = 0, \tag{10}$$

where

$$\begin{aligned} p &= a_2^2 - 2a_1 = 4a^2V_\infty^2 + (r+\delta)^2 + \delta r > 0, \\ q &= a_1^2 - 2a_0a_2 - b_1^2 = 4a^2\left(\delta^2 + r^2\right)V_\infty^2 + \delta^2r^2 - \left[-\left(1-\eta\right)m_2 + \left(b - c\eta\right)m_1\right]^2, \qquad (11) \\ s &= a_0^2 - b_0^2 = 4a^2\delta^2r^2V_\infty^2 - \left[-\left(1-\eta\right)rm_2 + \left(b - c\eta\right)\delta m_1\right]^2. \qquad (12) \end{aligned}$$

Lemma 4.

1) *Let $q \geq 0$ and $s \geq 0$. Then Eq. (10) has no positive roots.*
2) *Let $q \geq 0$ and $s < 0$ or $q < 0$ and $s \leq 0$. Then Eq. (10) has one positive root, say z_0.*
3) *Let $q < 0$ and $s > 0$. Set*

$$z_* = \frac{-p + \sqrt{p^2 - 3q}}{3}. \tag{13}$$

Then Eq. (10) has no positive roots if $h(z_*) > 0$, Eq. (10) has one positive root z_* if $h(z_*) = 0$, and it has two positive roots, say z_- and z_+, with $z_- < z_+$, if $h(z_*) < 0$. Furthermore, $h'(z_*) = 0$, $h'(z_-) < 0$ and $h'(z_+) > 0$.

Proof. 1) is straightforward. For 2) and 3), we apply Descartes' rule of signs that states that the possible number of positive roots (10) is equal to the number of sign changes in the sequence formed by the polynomial's coefficients, or less than the sign changes by a multiple of 2. Finally, notice that $h(0) = s$, $h(+\infty) = +\infty$, $h'(z) = 3z^2 + 2pz + q$, $h''(z) = 6z + 2p > 0$; $h(z)$ is a convex function, and it has a unique minimum $z_* = (-p + \sqrt{p^2 - 3q})/3$ if $q < 0$ in the positive semiaxes. □

By using (8), we obtain

$$\sin\omega\tau = \frac{(b_1a_2 - b_0)\,\omega^3 + (a_1b_0 - a_0b_1)\omega}{b_1^2\omega^2 + b_0^2}, \qquad \cos\omega\tau = \frac{b_1\omega^4 + (a_2b_0 - a_1b_1)\omega^2 - a_0b_0}{b_1^2\omega^2 + b_0^2}.$$

If Eq. (9) has the unique positive root $\omega_0 = \sqrt{z_0}$, then we can determine the sequence ($j = 0, 1, 2, ...$)

$$\tau_j^0 = \begin{cases} \dfrac{1}{\omega_0}\cos^{-1}\left\{\dfrac{b_1\omega_0^4 + (a_2b_0 - a_1b_1)\omega_0^2 - a_0b_0}{b_1^2\omega_0^2 + b_0^2}\right\} + \dfrac{2j\pi}{\omega_0}, & \text{if } M_0 \geq 0, \\ \dfrac{2(j+\pi)}{\omega_0} - \dfrac{1}{\omega_0}\cos^{-1}\left\{\dfrac{b_1\omega_0^4 + (a_2b_0 - a_1b_1)\omega_0^2 - a_0b_0}{b_1^2\omega_0^2 + b_0^2}\right\}, & \text{if } M_0 < 0, \end{cases} \tag{14}$$

with $M_0 = (b_1a_2 - b_0)\,\omega_0^2 + a_1b_0 - a_0b_1$, at which the characteristic equation (6) has a pair of purely imaginary roots $\lambda = \pm i\omega_0$. Similarly, if Eq. (9) has the two positive roots $\omega_\pm = \sqrt{z_\pm}$, with $\omega_- < \omega_+$, then Eq. (6) has purely imaginary roots $\lambda = \pm i\omega_\pm$ when τ takes the critical values ($j = 0, 1, 2, ...$)

$$\tau_j^\pm = \begin{cases} \dfrac{1}{\omega_\pm}\cos^{-1}\left\{\dfrac{b_1\omega_\pm^4 + (a_2b_0 - a_1b_1)\omega_\pm^2 - a_0b_0}{b_1^2\omega_\pm^2 + b_0^2}\right\} + \dfrac{2j\pi}{\omega_\pm}, & \text{if } M_\pm \geq 0, \\ \dfrac{2(j+\pi)}{\omega_\pm} - \dfrac{1}{\omega_\pm}\cos^{-1}\left\{\dfrac{b_1\omega_\pm^4 + (a_2b_0 - a_1b_1)\omega_\pm^2 - a_0b_0}{b_1^2\omega_\pm^2 + b_0^2}\right\}, & \text{if } M_\pm < 0, \end{cases} \tag{15}$$

with $M_\pm = (b_1a_2 - b_0)\,\omega_\pm^2 + a_1b_0 - a_0b_1$.

Proposition 5. *Let $\lambda(\tau)$ be the root of* (6) *satisfying $\Re(\tau_j^0) = 0$ (resp. $Re(\tau_j^\pm) = 0$) and $\Im(\tau_j^0) = \omega_0$ (resp. $\Im(\tau_j^\pm) = \omega_\pm$). Then $\lambda = \pm i\omega_0$ (resp. $\lambda = \pm i\omega_\pm$) are simple roots of* (6) *when $\tau = \tau_j^0$ (resp. $\tau = \tau_j^\pm$) and the following transversality conditions hold*

$$\left[\frac{d\Re(\lambda)}{d\tau}\right]_{\tau=\tau_j^0,\omega=\omega_0} > 0, \left(resp. \quad \left[\frac{d\Re(\lambda)}{d\tau}\right]_{\tau=\tau_j^+,\omega=\omega_+} > 0 \quad and \quad \left[\frac{d\Re(\lambda)}{d\tau}\right]_{\tau=\tau_j^-,\omega=\omega_-} < 0\right). \tag{16}$$

Proof. From (6), taking the derivative with respect to τ, we have

$$\left\{3\lambda^2 + 2a_2\lambda + a_1 + \left[b_1 - \tau\left(b_1\lambda + b_0\right)\right]e^{-\lambda\tau}\right\}\frac{d\lambda}{d\tau} = \left(b_1\lambda + b_0\right)\lambda e^{-\lambda\tau}. \tag{17}$$

Let $\lambda = i\omega$, with $\omega \in \{\omega_0, \omega_+, \omega_-\}$, be a root of (6) that is not simple. Then this leads us to conclude from (17) that $(b_1 i\omega_k + b_0)\,e^{-i\omega_k\tau_j^k} = 0$. By separating real and imaginary parts,

$$b_1\omega_k\cos\omega_k\tau_j^k - b_0\sin\omega_k\tau_j^k = 0, \qquad b_1\omega_k\sin\omega_k\tau_j^k + b_0\cos\omega_k\tau_j^k = 0. \tag{18}$$

From (8) and as $\omega_k > 0$, we see that (18) reduces to $\omega_k^2 - a_1 = 0$, $a_2\omega_k^2 - a_0 = 0$. Thus, $a_1a_2 = a_0$ and so the contradiction $0 < [2a\,(\delta + r)\,V_\infty + \delta r]\,(\delta + r) + 4a^2\,(\delta + r)\,V_\infty^2 = 0$. This proves the first part of the statement. Next, from (17)

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{3\lambda^2 + 2a_2\lambda + a_1}{(b_1\lambda + b_0)\,\lambda e^{-\lambda\tau}} + \frac{b_1}{(b_1\lambda + b_0)\,\lambda} - \frac{\tau}{\lambda}.$$

Using (6), a direct calculation yields

$$\begin{aligned}
sign\left\{\left.\frac{d(\Re\lambda)}{d\tau}\right|_{\tau=\tau_j^k,\omega=\omega_k}\right\} &= sign\left\{\Re\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_j^k,\omega=\omega_k}\right\}\\
&= sign\left\{\Re\left[\frac{3\lambda^2+2a_2\lambda+a_1}{(b_1\lambda+b_0)\lambda e^{-\lambda\tau}}+\frac{b_1}{(b_1\lambda+b_0)\lambda}-\frac{\tau}{\lambda}\right]_{\tau=\tau_j^k,\omega=\omega_k}\right\}\\
&= sign\left\{\frac{3\omega_k^6+2\left(a_2^2-2a_1\right)\omega_k^4+\left(a_1^2-2a_0a_2-b_1^2\right)\omega_k^2}{b_1^2\omega_k^4+b_0^2\omega_k^2}\right\}\\
&= sign\left\{3\omega_k^4+2p\omega_k^2+q\right\}\\
&= sign\left\{h'(z_k)\right\}.
\end{aligned}$$

Finally, notice that $sign\{h'(z_k)\} = +1$ if $\omega = \omega_0$ or $\omega = \omega_+$, while $sign\{h'(z_k)\} = -1$ if $\omega = \omega_-$. This completes the proof. □

Remark 6. If Eq. (9) has the unique positive root $\omega_* = \sqrt{z_*}$, then $sign\{h'(z_*)\} = 0$ and the transversality condition does not hold.

In case Eq. (6) has two positive roots ω_-, ω_+, with $\omega_- < \omega_+$, then crossing from left to right with increasing τ occurs whenever τ assumes a value corresponding to ω_+, and crossing from right to left occurs for values of τ corresponding to ω_-. This implies that there exists a finite number of delayed intervals in which the equilibrium point is locally asymptotically stable, while unstable for the outside of the delayed ranges. Hence, the system dynamics switches from stable to unstable, and then back to stable when delay increases and crosses the critical delayed values.

Based on the transversality conditions (16) and the Hopf bifurcation theorem (see [12]), we have the following results on the stability of the equilibrium of system (1).

Theorem 7. *Let $b - c\eta > 0$ and assume condition (S) holds. Let $h(z), p, q, z_*, \tau_0^0$ and $\tau_0^\pm$ be defined as in* (10), (11), (12), (13), (14) *and* (15), *respectively.*

1) *If $q \geq 0$ and $s \geq 0$ or if $q < 0, s > 0$ and $h(z_*) > 0$, then the fixed point $(V_\infty, E_\infty, K_\infty)$ of system* (1) *is locally asymptotically stable for all $\tau \geq 0$.*
2) *If $q \geq 0$ and $s < 0$ or if $q < 0$ and $s \leq 0$, then the fixed point $(V_\infty, E_\infty, K_\infty)$ of system* (1) *is locally asymptotically stable for $\tau < \tau_0^0$ and unstable for $\tau > \tau_0^0$. Furthermore, system* (1) *undergoes a Hopf bifurcation at $(V_\infty, E_\infty, K_\infty)$ when $\tau = \tau_0^0$.*
3) *If $q < 0, s > 0$ and $h(z_*) < 0$, then there exist stability switches for $\tau > 0$. Furthermore, system* (1) *undergoes a Hopf bifurcation at $(V_\infty, E_\infty, K_\infty)$ for a certain values of $\tau = \tau_j^\pm$ $(j = 0, 1, 2, ...)$ where a stability switch occurs.*

Proof. The proof of the theorem follows from Lemma 4 and Proposition 5. The expressions of the bifurcating values for the delays is given by (14) and (15). □

Remark 8. *We note that the bifurcation analysis (together with the study of roots of eq. (10)) we propose here is more detailed and different with respect to that contained in* [15].

4. **Numerical Simulations.** In this section we will provide some numerical experiments to illustrate the results of the previous sections.

4.1. **A case of instability.** We start with an example in which the fixed point is unstable. The parameters are fixed as follows:

$$a = 0.1; \qquad b = 0.85; \qquad c = 1; \qquad \delta = 1; \qquad \eta = 0.75; \qquad m_1 = 0.5;$$
$$m_2 = 2; \qquad \bar{P} = 25; \qquad r = 0.001,$$

while the initial point is $V(0) = E(0) = K(0) = 1$. This choice satisfies the condition:

$$b - c\eta = 0.1 > 0,$$

while does not satisfy the condition of positiveness of E_∞:

$$\bar{P}a\delta r - m_2(1-\eta)(b - c\eta) = -0.0475 < 0.$$

The fixed point is:

$$V_\infty = 0.2524, \qquad E_\infty = -0.2397, \qquad K_\infty = 0.0631.$$

In order to check the stability we need to compute the following quantities:

$$A = 1.0515, \qquad B = -0.3985, \qquad C = 0.0496.$$

Then, the stability condition is not fulfilled because

$$AB - C = -0.4685 < 0.$$

The solution is represented in Fig. 1 below: the fixed point is not stable, the solution diverges to $-\infty$.

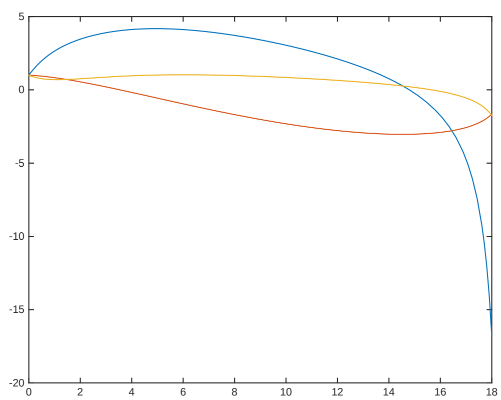

FIGURE 1. Figure of experiment 1: the fixed point is not stable, the solution diverges

4.2. **A limit cycle.** We now consider an example in which there exists a limit cycle. In this experiment the parameters are set as follows:

$$a = 0.1; \qquad b = 0.85; \qquad c = 1; \qquad \delta = 0.2; \qquad \eta = 0.75; \qquad m_1 = 5;$$
$$m_2 = 0.001; \qquad \bar{P} = 25; \qquad r = 0.000001,$$

while the initial point is $V(0) = E(0) = K(0) = 0.1$. This choice satisfies the condition

$$b - c\eta = 0.1 > 0,$$

while does not satisfy the positiveness of E_∞:

$$\bar{P}a\delta r - m_2(1-\eta)(b-c\eta) = -2.45 \cdot 10^{-5} < 0.$$

The fixed point is:

$$V_\infty = 2.5 \cdot 10^{-4}, \qquad E_\infty = -4.0978 \cdot 10^{-6}, \qquad K_\infty = 3.125 \cdot 10^{-4}$$

In order to check the stability we need to compute the following quantities:

$$A = 0.2001, \qquad B = 0.4998, \qquad C = 0.1.$$

Then, the stability condition is not fulfilled either because

$$AB - C = -2.2472 \cdot 10^{-5}.$$

The solution is represented in figures 2 below: the fixed point is not stable, there exists an attractive limit cycle.

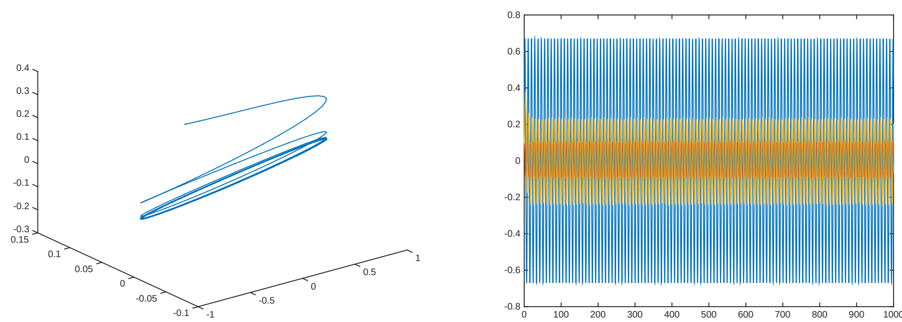

Figure 2. Figures for Experiment 2: stable limit cycle.

4.3. **A case of stability with $r < \delta$.** We consider an example in which the interior fixed point is stable. The parameters are fixed as follows:

$$a = 1; \qquad b = 0.73; \qquad c = 0.8; \qquad \delta = 0.2; \qquad \eta = 0.9; \qquad m_1 = 5;$$
$$m_2 = 0.001; \qquad \bar{P} = 1; \qquad r = 0.1,$$

and the initial point is $V(0) = E(0) = K(0) = 1$. This choice satisfies the conditions of existence of a positive fixed point:

$$b - c\eta = 0.01 > 0, \qquad \bar{P}a\delta r - m_2(1-\eta)(b-c\eta) = 0.02 > 0,$$

and the stability condition (S). This simulation suggests that stability is possible also for $r < \delta$. In this case the fixed point is stable (see figure 3 below):

$$V_\infty = 2.2361, \qquad E_\infty = 0.7764, \qquad K_\infty = 1.1180.$$

and the solutions converges to it very fast.

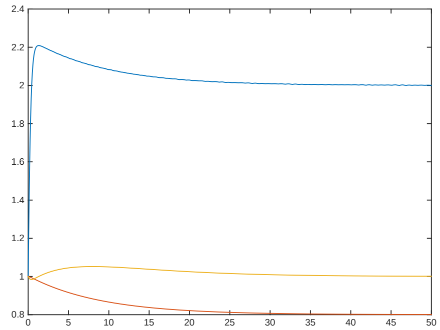

FIGURE 3. Stability of the fixed point for $r < \delta$.

4.4. **Delay model: stability change.** In this example we illustrate the case in which the delay affects the stability of the fixed point. We consider the following values of the parameters:

$$a = 0.5; \qquad b = 1; \qquad c = 0.8; \qquad \delta = 0.2; \qquad \eta = 0.9; \qquad m_1 = 1;$$
$$m_2 = 0.2; \qquad \bar{P} = 4; \qquad r = 0.1;$$

thus, the fixed point is

$$V_\infty = 1.2102, \qquad E_\infty = 0.6113, \qquad K_\infty = 0.6051.$$

The initial data has been chosen as $E, V, K = (1, 1, 1)$ for $t \in [-\tau, 0]$, while the critical value of the delay parameter for stability is $\tau_0^0 = 9.7592$. The hypotheses of 2) in theorem 7 are fulfilled, then we expect that the fixed point $(V_\infty, E_\infty, K_\infty)$ is stable for $\tau < \tau_0^0$ and unstable otherwise. In Fig. 4 below we represent the solutions for $\tau = 8.6$ and $\tau = 9.8$ respectively: the fixed point looses stability and the solution diverges to $-\infty$.

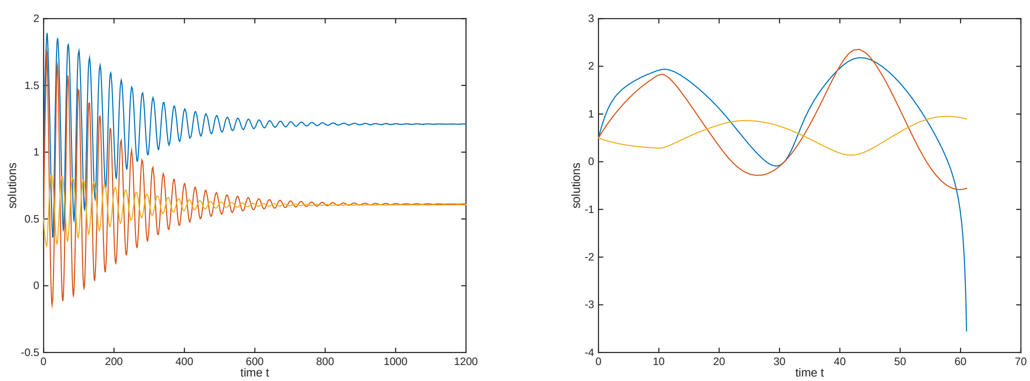

FIGURE 4. The solution of the system with delay and for $\tau = 8.6$ and $\tau = 9.8$ respectively.

4.5. **Stability switches.** We consider an example in which the hypotheses of 3) of theorem 7 are fulfilled. The parameters are chosen as follows:

$$a = 0.1; \qquad b = 1; \qquad c = 1; \qquad \delta = 0.1; \qquad \eta = 0.001;$$
$$m_1 = 0.05; \qquad m_2 = 0.1; \qquad \bar{P} = 5; \qquad r = 0.05;$$

and the fixed point is

$$V_\infty = 1.5811; \qquad E_\infty = -26.5912; \qquad K_\infty = 15.7956.$$

The other relevant quantities are

$$z_* = 0.0028, \quad z_+ = 0.0073, \quad z_- = 0.0029, \quad \omega_- = 0.0529, \quad \omega_+ = 0.0854$$

Since the sign of $M_\pm$ are $M_- > 0$ and $M_+ < 0$ respectively, then we deduce, for $j = 0$, the following bifurcation values:

$$\tau_0^- = 2.1263, \qquad \tau_0^+ = 68.3574, \qquad \tau_1^- = 120.8674, \qquad \tau_1^+ = 141.8965.$$

In FIG. 5 we represent the solution for the following values of the delay parameter: $\tau = 2.1, 2.3, 117, 121$....... We observe changes of stability of the fixed point.

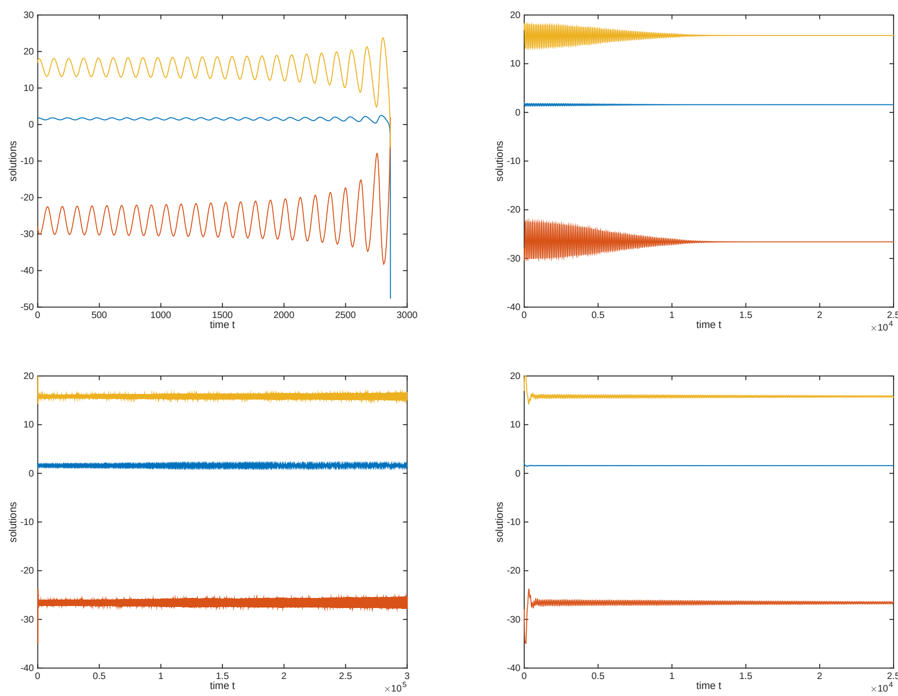

FIGURE 5. The solution of the system with delay and for $\tau = 2.1, 2.3, 117, 121$.

5. **Comments on the model.** In this section we would like to point out some features of the model that could be improved. In particular, the quantities V, E, K are positive and the system should preserve their sign. The system is well posed if for any positive initial conditions the solution remains positive, moreover by the modelisation strategy it is required that $E \in [0, \bar{P}]$.
Suppose that we start with a positive initial condition $V_0, K_0, E_0 > 0$. Then if V or K become zero then $\dot{V} > 0$ (resp $\dot{K} > 0$). A contradiction of the requirement that $E \in [0, \bar{P}]$ may occur when E reaches the value 0 or $\bar{P}$.
In fact if there exists t_1 such that $E(t_1) = \bar{P}$, then $\dot{E}(t_1) > 0$ if $b - c\eta < 0$ and E takes values bigger than $\bar{P}$. Then we need:

$$b - c\eta > 0.$$

Conversely, if there exists t_2 such that $E(t_2) = 0$ then

$$\dot{E}(t_2) = r\bar{P} - (b - c\eta)V_d(t_2),$$

and if

$$V_d(t_2) > \frac{r\bar{P}}{(b - c\eta)},$$

we have that $\dot{E}(t_2) < 0$ and E takes values less than 0. A sufficient condition (not necessary but that can be considered in the modelisation process) to avoid this should be

$$b - c\eta < 0.$$

Of course these two conditions are not compatible.
In order to preserve the positiveness of the system (and of its solutions) we consider a modified version of the second equation:

$$\left\{ \begin{array}{lcl} \dot{V} & = & m_1 E + m_2 K - aV^2, \\ \dot{E} & = & r\left(\bar{P} - E\right) - (b - c\eta)V_d E, \\ \dot{K} & = & (1 - \eta)\, V_d - \delta K, \end{array} \right. \tag{19}$$

It is worth highlighting that, for the original model, even if the condition of existence of the stable positive fixed point is fulfilled, the solutions can become negative. We provide a simulation of this problem in figure 6 below for which the parameters of the original model are set as follows:

$$a = 1.5; \qquad b = 1; \qquad c = 0.95; \qquad \delta = 0.027; \qquad \eta = 0.95;$$
$$m_1 = 0.1; \qquad m_2 = 0.2; \qquad \bar{P} = 0.99; \qquad r = 0.025,$$

and the initial point is $V(0) = E(0) = K(0) = 0.8$. The fixed point in this case is

$$V_\infty = 0.2504, \qquad E_\infty = 0.0133, \qquad K_\infty = 0.4638$$

and satisfies the stability condition (S) : $AB - C = 0.0312 > 0$. A similar problem has been observed in the case with delay.
This example suggests that a modification of the model should be necessary, in particular our suggested modification solves this problem.
Further research concerning the analysis of system (19) and a comparison with the results about the original system will be carried out somewhere else.

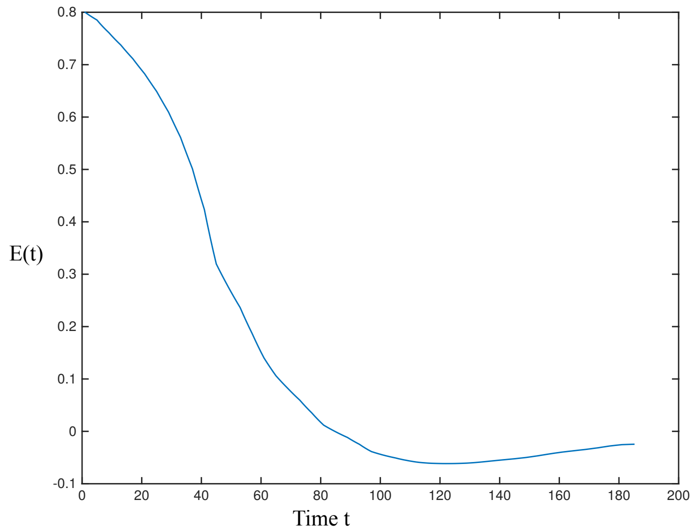

FIGURE 6. There exists a positive stable fixed point and the second component of the solution takes negative values.

6. **Further development: non autonomous dynamics and some open directions.** We have seen that the introduction of a delay may affect the stability of the interior fixed point. Moreover, Theorem 7 provides a very complex and varied picture. We have already observed that the stability switches introduce a serious difficulty for the managers of PAs: if they act on the delay and let it change, then it can bring both stabilizing and destabilizing effects. We consider that is very important to deal with this problem. However, in many situations it seems more realistic to consider a non-constant delay, in other words, instead of a constant delay $\tau > 0$, we can consider a time dependent delay $\tau(t)$. More precisely, one could consider the following variation of (1), and study its asymptotic dynamics:

$$\begin{cases} \dot{V}(t) &= m_1 E(t) + m_2 K(t) - aV^2(t), \\ \dot{E}(t) &= r\left(\bar{P} - E(t)\right) - (b - c\eta)V(t - \tau(t)), \\ \dot{K}(t) &= (1-\eta)\, V(t - \tau(t)) - \delta K(t), \end{cases} \tag{20}$$

where $\tau(\cdot)$ is a continuous function (although in many real applications it uses to be continuously differentiable) from $\mathbb{R}$ to $[0, h]$ where $h > 0$ is the maximum admissible delay.

The variable delay case in (20) possesses the same equilibria as the constant delay case, however the problem becomes automatically non-autonomous and in order to analyze the non-autonomous dynamical system generated by (20), it first needs to be stated as an ordinary differential system of equations in an infinite-dimensional space. Indeed, let us denote by $\mathcal{C}$ the space Banach space $C^0([-h, 0]; \mathbb{R}^3)$, and for any continuous function $g : \mathbb{R} \to \mathbb{R}^3$ and $t \in \mathbb{R}$ we denote by g_t the function in $\mathcal{C}$ defined by

$$g_t(s) = g(t + s), \quad \text{for all } s \in [-h, 0].$$

Then, problem (20) can be rewritten as follows. For a given initial time $t_0 \in \mathbb{R}$ and an initial function $\phi \in \mathcal{C}$, we consider

$$\begin{cases} \dfrac{d}{dt} y(t) = f(t, y_t), \ t > t_0, \\ \\ y(t) = \phi(t - t_0), \quad t \in [t_0 - h, t_0], \end{cases} \tag{21}$$

where $y(t) = (E(t), K(t), V(t))$ and $f(\cdot, \cdot)$ is the function defined from $\mathbb{R} \times \mathcal{C}$ by

$$f(t, \phi) = \begin{pmatrix} m_1\phi_1(0) + m_2\phi_2(0) - a\phi_3^2(0), \\ r\left(\bar{P} - \phi_1(0)\right) - (b - c\eta)\phi_3(-\tau(t)), \\ (1 - \eta)\,\phi_3(-\tau(t)) - \delta\phi_2(0), \end{pmatrix}.$$

Then, when we particularize $f(t, \phi)$ for $\phi = y_t$, we obtain

$$\begin{aligned} f(t, y_t) &= \begin{pmatrix} m_1E_t(0) + m_2K_t(0) - aV_t^2(0), \\ r\left(\bar{P} - E_t(0)\right) - (b - c\eta)V_t(-\tau(t)), \\ (1 - \eta)\,V_t(-\tau(t)) - \delta K_t(0), \end{pmatrix} \\ &= \begin{pmatrix} m_1E(t) + m_2K(t) - aV^2(t), \\ r\left(\bar{P} - E(t)\right) - (b - c\eta)V(t - \tau(t)), \\ (1 - \eta)\,V(t - \tau(t)) - \delta K(t), \end{pmatrix} \end{aligned}$$

which is our delay system.

The asymptotic behavior of this delay system can be carried out in a local or in a global way. For the local analysis, there exists several well-known methods based, for instance, in the construction of certain Lyapunov functionals (see, Kolmanovskii and Shaikhet [16], Caraballo et al. [3], Kuang [14] amongst many others) as well as the use of the Razumikhin-Lyapunov theory which allows for more generality on the variable delay functions (see Hale and Lunel [11], Caraballo et al. [4], etc). We plan to investigate this in a future paper. However, concerning the global asymptotic behavior, as our problem will generate a non-autonomous dynamical system, then we need to choose an appropriate framework for our problem. To this respect we have several options, either the skew-product flow formalism (see e.g. Sell [17]), or the theory of uniform attractor (see Carvalho et al. [5], Kloeden and Rasmussen [13] and the references therein), or the theory of pullback attractor which usually allows for more generality on the delay terms. This problem will be analyzed in a subsequent paper.

We think that this should be the right context to study the consequence of modifying the delay and should be of a real interest for management of PAs to completely clarify this problem. However, we will leave this topic for a future work.

Finally, we would like to mention that there are some other interesting questions which we have not analyzed in this paper as, for instance, the global stability of the positive fixed point and the case $sign\,\{h'(z_*)\} = 0$ for which the transversality condition does not hold. Another question is to investigate the possible presence of a double Hopf bifurcations that may occur when two critical delays τ_j^- and τ_j^+ are identical. We plan to investigate these issues in future.

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