symmetry

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Supplementary Material

Exploiting the Symmetry of Integral Transforms for Featuring Anuran Calls

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1. Derivation of integral transforms expressions

1.1. Integral transforms of non-symmetric functions

Let us consider an arbitrary example function $f(x)$, of which we know only one fragment in the interval $[x_0, x_0 + P]$. Now let us consider that this function is sampled, and the values only at specific points for $x = x_n$, separated at intervals Δx, are known. By denoting N as the total number of points (samples) in a period, we know that $\Delta x = P/N$. The sampled function will be called $\hat{f}(x_n) = f_n$ where the hat ($\hat{}$) above f represents a sampled function.

The usual way to obtain the spectrum of that function is to define a periodic function $f_p(x)$ of period P that coincides with the previous function in the known interval, and to proceed to compute the spectrum of that new function. The spectral representation of the function $f_p(x)$ is composed of the complex coefficients of the Fourier series expansion given by

$$c_k \equiv \frac{1}{P}\int_{x_0}^{x_0+P} f_p(x) e^{-j2\pi k \xi_0 x} dx, \tag{1}$$

where ξ_0 is the frequency of the periodic function that is given by $\xi_0 = 1/P$. Therefore,

$$c_k = \frac{1}{P}\int_{x_0}^{x_0+P} f_p(x) e^{-j\frac{2\pi k x}{P}} dx. \tag{2}$$

Since, in the interval $[x_0, x_0 + P]$, the function $f_p(x)$ coincides with $f(x)$, then the expression takes the form

$$c_k = \frac{1}{P}\int_{x_0}^{x_0+P} f(x) e^{-j\frac{2\pi k x}{P}} dx. \tag{3}$$

On the other hand, the sampled function, $\hat{f}(x_n) = f_n$, will have a spectral representation $\hat{c}_k$ that corresponds to c_k, when the sampling of the variable x is taken into account. Now let us call $I(x)$ the integrand of equation 3, that is,

$$I(x) = f(x) e^{-j\frac{2\pi k x}{P}}, \tag{4}$$

and hence the spectral representation of the non-sampled function $f_p(x)$ is featured by the coefficients

$$c_k = \frac{1}{P}\int_{x_0}^{x_0+P} I(x) dx. \tag{5}$$

in order to obtain the values $\hat{c}_k$ that take into account the sampling of the variable x, the continuous calculation of the area that supposes the integral of the previous expression is substituted with the sum of the rectangles corresponding to the discrete values (sum of Riemann). Therefore,

$$\hat{c}_k \equiv [c_k]_{x=x_n} = \left[\frac{1}{P}\int_{x_0}^{x_0+P} I(x)dx\right]_{x=x_n}. \tag{6}$$

$$\hat{c}_k = \frac{1}{P}\sum_{x_n=x_0}^{x_n=x_0+P-\Delta x} I(x_n)\,\Delta\, x. \tag{7}$$

$$\hat{c}_k = \frac{1}{P}\sum_{x_n=x_0}^{x_n=x_0+P-\Delta x} f_p(x_n)e^{-j\frac{2\pi k x_n}{P}}\,\Delta\, x. \tag{8}$$

In that equation, $x_n = x_0 + n\,\Delta\, x$, and $f_p(x) = f(x)$ in the interval $[x_0, x_0 + P]$, and hence $f_p(x_n) = f(x_n)$. At the sampling points, the sampled function is also $f_p(x_n) = f(x_n) = \hat{f}(x_n) = f_n$, and it can therefore be written that

$$\hat{c}_k = \frac{1}{P}\sum_{x_0+n\,\Delta\, x=x_0}^{x_0+n\,\Delta\, x=x_0+P-\Delta x} f_n\, e^{-j\frac{2\pi k(x_0+n\,\Delta\, x)}{P}}\,\Delta\, x. \tag{9}$$

$$\hat{c}_k = \frac{1}{N\,\Delta\, x}\sum_{n\,\Delta\, x=0}^{n\,\Delta\, x=N\,\Delta\, x-\Delta\, x} f_n\, e^{-j\frac{2\pi k x_0}{N\,\Delta\, x}}e^{-j\frac{2\pi k n\,\Delta\, x}{N\,\Delta\, x}}\,\Delta\, x. \tag{10}$$

$$\hat{c}_k = \frac{1}{N\,\Delta\, x}\sum_{n=0}^{N-1} f_n\, e^{-j\frac{2\pi k x_0}{N\,\Delta\, x}}e^{-j\frac{2\pi k n\,\Delta\, x}{N\,\Delta\, x}}\,\Delta\, x. \tag{11}$$

By pulling out the terms that do not depend on n, we obtain

$$\hat{c}_k = \frac{1}{N\,\Delta\, x}e^{-j\frac{2\pi k x_0}{N\,\Delta\, x}}\,\Delta\, x\sum_{n=0}^{N-1} f_n\, e^{-j\frac{2\pi k n}{N}}. \tag{12}$$

$$\hat{c}_k = \frac{1}{N}e^{-j\frac{2\pi k x_0}{N\,\Delta\, x}}\sum_{n=0}^{N-1} f_n\, e^{-j\frac{2\pi k n}{N}}. \tag{13}$$

It can be observed that the spectral representation $\hat{c}_k$ depends on the point x_0 selected as the origin of coordinates, due to the factor $e^{-j\frac{2\pi k x_0}{N\,\Delta\, x}}$. This factor does not affect the amplitude spectrum (since its modulus is 1), but it does affect the phase spectrum, corresponding to the known time-shift property of the Fourier Transform. For practical purposes, the origin of coordinates is usually considered to be the starting point of the sequence, that is, at $x_0 = 0$, and hence the spectral representation finally becomes

$$\hat{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n\, e^{-j\frac{2\pi k n}{N}}. \tag{14}$$

This expression coincides with the usual definition of the Discrete Fourier Transform (DFT). In other words: the Discrete Fourier Transform of a known fragment of a function presupposes the periodic repetition of that fragment.

1.2. Integral transforms of symmetric functions

Let us now again consider the function $f(x)$ of which we know only sampled values of a fragment f_n in the interval $[x_0, x_0 + P]$. An alternative way of representing its spectrum to that of periodically repeating the values f_n, lies in defining a sequence of values g_n of length $2P$ that coincides with f_n in the interval $[x_0, x_0 + P]$, which is its symmetric in the interval $[x_0 - P, x_0]$. It can be observed that

$$\begin{gathered} g_n = f_n \; \forall n \in [0, N-1] \\ g_n = f_{-n-1} \; \forall n \in [-1, -N] \end{gathered} \quad (15)$$

Subsequently, a sequence of periodic values h_n of period $P' = 2P$ is defined that coincides with g_n in the interval $[x_0 - P, x_0 + P]$.

In order to obtain the spectrum of the sequence of values h_n in accordance with equation (8), it can be written that

$$\hat{c}_k = \frac{1}{P'} \sum_{x_n = x_0 - P}^{x_n = x_0 + P - \Delta x} h_n \, e^{-j\frac{2\pi k x_n}{P'}} \Delta x. \quad (16)$$

$$\hat{c}_k = \frac{1}{2P} \sum_{x_n = x_0 - P}^{x_n = x_0 + P - \Delta x} h_n \, e^{-j\frac{2\pi k x_n}{2P}} \Delta x. \quad (17)$$

In the interval $[x_0 - P, x_0 + P]$, it is true that $h_n = g_n$, and hence we can write

$$\hat{c}_k = \frac{1}{2P} \sum_{x_n = x_0 - P}^{x_n = x_0 + P - \Delta x} g_n \, e^{-j\frac{2\pi k x_n}{2P}} \Delta x. \quad (18)$$

$$\hat{c}_k = \frac{\Delta x}{2P} \left[\sum_{x_n = x_0 - P}^{x_n = x_0 - \Delta x} g_n \, e^{-j\frac{\pi k x_n}{P}} + \sum_{x_n = x_0}^{x_n = x_0 + P - \Delta x} g_n \, e^{-j\frac{\pi k x_n}{P}} \right]. \quad (19)$$

In that expression, the second summand extends in the interval $[x_0, x_0 + P]$, where it is true that $g_n = f_n$, while the first summand extends in the interval $[x_0 - P, x_0]$ where $g_n = f_{-n-1}$, and therefore we can write

$$\hat{c}_k = \frac{\Delta x}{2P} \left[\sum_{x_n = x_0 - P}^{x_n = x_0 - \Delta x} f_{-n-1} \, e^{-j\frac{\pi k x_n}{P}} + \sum_{x_n = x_0}^{x_n = x_0 + P - \Delta x} f_n \, e^{-j\frac{\pi k x_n}{P}} \right]. \quad (20)$$

Recalling that $x_n = x_0 + n \Delta x$ and $P = N \Delta x$, it can be written that

$$\hat{c}_k = \frac{\Delta x}{2N \Delta x} \left[\sum_{x_0 + n\Delta x = x_0 - P}^{x_0 + n \Delta x = x_0 - \Delta x} f_{-n-1} \, e^{-j\frac{\pi k (x_0 + n \Delta x)}{N \Delta x}} + \sum_{x_0 + n \Delta x = x_0}^{x_0 + n \Delta x = x_0 + P - \Delta x} f_n \, e^{-j\frac{\pi k (x_0 + n \Delta x)}{N \Delta x}} \right]. \quad (21)$$

$$\hat{c}_k = \frac{1}{2N} \left[\sum_{n \Delta x = -P}^{n \Delta x = -\Delta x} f_{-n-1} \, e^{-j\frac{\pi k x_0}{N \Delta x}} e^{-j\frac{\pi k n}{N}} + \sum_{n \Delta x = 0}^{n \Delta x = P - \Delta x} f_n \, e^{-j\frac{\pi k x_0}{N \Delta x}} e^{-j\frac{\pi k n}{N}} \right]. \quad (22)$$

$$\hat{c}_k = \frac{1}{2N} \left[\sum_{n=-N}^{n=-1} f_{-n-1} \, e^{-j\frac{\pi k x_0}{N \Delta x}} e^{-j\frac{\pi k n}{N}} + \sum_{n=0}^{n=N-1} f_n \, e^{-j\frac{\pi k x_0}{N \Delta x}} e^{-j\frac{\pi k n}{N}} \right]. \quad (23)$$

Pulling out the terms that do not depend on n, it can be written that

$$\hat{c}_k = \frac{1}{2N} e^{-j\frac{\pi k x_0}{N \Delta x}} \left[\sum_{n=-N}^{n=-1} f_{-n-1} \, e^{-j\frac{\pi k n}{N}} + \sum_{n=0}^{n=N-1} f_n \, e^{-j\frac{\pi k n}{N}} \right]. \quad (24)$$

By calling $\nu \equiv -n - 1$, that is, $n = -\nu - 1$, we obtain

$$\hat{c}_k = \frac{1}{2N} e^{-j\frac{\pi k x_0}{N \Delta x}} \left[\sum_{-\nu-1=-N}^{-\nu-1=-1} f_{-(-\nu-1)-1} \, e^{-j\frac{\pi k(-\nu-1)}{N}} + \sum_{n=0}^{n=N-1} f_n \, e^{-j\frac{\pi k n}{N}} \right]. \quad (25)$$

$$\hat{c}_k = \frac{1}{2N} e^{-j\frac{\pi k x_0}{N \Delta x}} \left[\sum_{\nu=N-1}^{\nu=0} f_\nu \, e^{j\frac{\pi k(\nu+1)}{N}} + \sum_{n=0}^{n=N-1} f_n \, e^{-j\frac{\pi k n}{N}} \right]. \tag{26}$$

$$\hat{c}_k = \frac{1}{2N} e^{-j\frac{\pi k x_0}{N \Delta x}} \left[\sum_{\nu=0}^{N-1} f_\nu \, e^{j\frac{\pi k \nu}{N}} e^{j\frac{\pi k}{N}} + \sum_{n=0}^{N-1} f_n \, e^{-j\frac{\pi k n}{N}} \right]. \tag{27}$$

$$\hat{c}_k = \frac{1}{2N} e^{-j\frac{\pi k x_0}{N \Delta x}} \left[e^{j\frac{\pi k}{N}} \sum_{\nu=0}^{N-1} f_\nu \, e^{j\frac{\pi k \nu}{N}} + \sum_{n=0}^{N-1} f_n \, e^{-j\frac{\pi k n}{N}} \right]. \tag{28}$$

In the first summation of this equation, the variable ν can be substituted with any other symbol, for instance by n, which yields

$$\hat{c}_k = \frac{1}{2N} e^{-j\frac{\pi k x_0}{N \Delta x}} \left[e^{j\frac{\pi k}{N}} \sum_{n=0}^{N-1} f_n \, e^{j\frac{\pi k n}{N}} + \sum_{n=0}^{N-1} f_n \, e^{-j\frac{\pi k n}{N}} \right]. \tag{29}$$

As can be observed, due to the factor $e^{-j\frac{2\pi k x_0}{N \Delta x}}$, the spectral representation $\hat{c}_k$ depends on the point x_0 where the origin of coordinates is defined. This factor does not affect the amplitude spectrum (since its modulus is 1), but it does affect the phase spectrum, which corresponds to the known time-shifting property of the Fourier transform. For practical purposes, the origin of coordinates is usually considered to be located the midpoint of the symmetric sequence g_n, that is, $x_0 = \Delta x/2$.

Finally the spectral representation is

$$\hat{c}_k = \frac{1}{2N} e^{-j\frac{\pi k \Delta x/2}{N \Delta x}} \left[e^{j\frac{\pi k}{N}} \sum_{n=0}^{N-1} f_n \, e^{j\frac{\pi k n}{N}} + \sum_{n=0}^{N-1} f_n \, e^{-j\frac{\pi k n}{N}} \right]. \tag{30}$$

$$\hat{c}_k = \frac{1}{2N} e^{-j\frac{\pi k}{2N}} \left[e^{j\frac{\pi k}{N}} \sum_{n=0}^{N-1} f_n \, e^{j\frac{\pi k n}{N}} + \sum_{n=0}^{N-1} f_n \, e^{-j\frac{\pi k n}{N}} \right]. \tag{31}$$

$$\hat{c}_k = \frac{1}{2N} \left[\sum_{n=0}^{N-1} f_n \, e^{j\frac{\pi k n}{N}} e^{j\frac{\pi k}{N}} e^{-j\frac{\pi k}{2N}} + \sum_{n=0}^{N-1} f_n \, e^{-j\frac{\pi k n}{N}} e^{-j\frac{\pi k}{2N}} \right]. \tag{32}$$

$$\hat{c}_k = \frac{1}{2N} \left[\sum_{n=0}^{N-1} f_n \, e^{j\frac{\pi k}{N}\left(n+1-\frac{1}{2}\right)} + \sum_{n=0}^{N-1} f_n \, e^{-j\frac{\pi k}{N}\left(n+\frac{1}{2}\right)} \right]. \tag{33}$$

$$\hat{c}_k = \frac{1}{2N} \left[\sum_{n=0}^{N-1} f_n \, e^{j\frac{\pi k}{N}\left(n+\frac{1}{2}\right)} + \sum_{n=0}^{N-1} f_n \, e^{-j\frac{\pi k}{N}\left(n+\frac{1}{2}\right)} \right]. \tag{34}$$

$$\hat{c}_k = \frac{1}{2N} \sum_{n=0}^{N-1} \left[f_n \, e^{j\frac{\pi k}{N}\left(n+\frac{1}{2}\right)} + f_n \, e^{-j\frac{\pi k}{N}\left(n+\frac{1}{2}\right)} \right]. \tag{35}$$

$$\hat{c}_k = \frac{1}{2N} \sum_{n=0}^{N-1} f_n \left[e^{j\frac{\pi k}{N}\left(n+\frac{1}{2}\right)} + e^{-j\frac{\pi k}{N}\left(n+\frac{1}{2}\right)} \right]. \tag{36}$$

$$\hat{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n \cos\left[\frac{\pi k}{N}\left(n + \frac{1}{2}\right)\right]. \tag{37}$$

This expression coincides with the usual definition of the Discrete Cosine Transform (DCT). In other words, the Discrete Cosine Transform of a known fragment of a function presupposes the periodic repetition of that fragment and its symmetric.