Spatiotemporal Barcodes for Image Sequence Analysis

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Abstract. Taking as input a time-varying sequence of two-dimensional (2D) binary images, we develop an algorithm for computing a *spatiotemporal* 0–barcode encoding lifetime of connected components on the image sequence over time. This information may not coincide with the one provided by the 0–barcode encoding the 0–persistent homology, since the latter does not respect the principle that it is not possible to move backwards in time. A cell complex K is computed from the given sequence, being the cells of K classified as *spatial* or *temporal* depending on whether they connect two consecutive frames or not. A *spatiotemporal path* is defined as a sequence of edges of K forming a path such that two edges of the path cannot connect the same two consecutive frames. In our algorithm, for each vertex $v \in K$, a spatiotemporal path from v to the "oldest" *spatiotemporally-connected* vertex is computed and the corresponding *spatiotemporal* 0–bar is added to the *spatiotemporal* 0–barcode.

Keywords: Persistent homology · Barcodes · Spatiotemporal data · Digital image sequence analysis

1 Introduction

Persistent homology [3,5,12] and zigzag persistence [2] provides information about lifetime of homology classes along a filtration of cell complexes. Such a filtration might be determined by time in a set of spatiotemporal data. Our general aim is to compute the "spatiotemporal" topological information of such filtration, taking into account that it is not possible to move backwards in time (which is not obvious if we use the known algorithms for computing (zigzag) persistent homology).

In the context of mobile sensor networks, [4] is devoted to a problem related with the one posed here: can a moving intruder avoid being detected by the sensors? If the answer is yes, the path that describes the intruder over time is called an *evasion path*. In the study of evasion paths in [4], the region covered

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by sensors at time t is encoded using Rips complex. A single cell complex SR is computed by stacking the Rips complexes $R(t)$ for all times t. Theorem 7 of [4] proves that there is no evasion path in a given mobile sensor network under a "homological" criterion. Using zig-zag persistent homology, an equivalent condition is provided in [1]. Nevertheless, no general necessary and sufficient condition for the existence of an evasion path is given. The problem is how to capture in the cell complex SR, the idea that an intruder cannot move backwards in time. In [6], the authors analyze time-varying coverage properties in dynamic sensor networks by means of zigzag persistent homology. Coverage holes are tracked in the network by using representative cycles of 1–homology classes.

In this paper, we are concerned with the treatment of time-varying sequences of 2D binary images and the tracking of connected components over time inspired by persistent homology methods.

An overview of the main tools used in this paper: basics of persistent homology and AT-models are given in Sect. 2. We state the problem of computing the "correct" topological information of spatiotemporal data encoded in a single cell complex in Sect. 3, through two simple examples. Our method to solve the problem in dimension 0 is then introduced in Sect. 4. Cell complexes encoding spatiotemporal information of time-varying sequences of 2D binary images is given in Sect. 5. We conclude in Sect. 6 and describe possible directions for future work.

2 Persistent Homology Through AT-models

Roughly speaking, a cell complex K is a general topological structure by which a space is decomposed into basic elements (cells) of different dimensions that are glued together by their boundaries (see the definition of CW-complex in [10]). If the cells in K are p–dimensional cubes (vertices, edges, square faces, cubes, ...) then K is a *cubical complex*. The dimension of a cell $\sigma \in K$ is denoted by $dim(\sigma)$. A cell $\mu \in K$ is a *p–face* of a cell $\sigma \in K$ if μ lies in the boundary of σ and $p = dim(\mu) < dim(\sigma)$.

A *p–chain* is a formal sum of p–cells in K. Since we work with coefficients that are either 0 or 1, we can think of a p–chain as a set of p–cells, namely those with coefficients equals to 1. In set notation, the sum of two p–chains is their symmetric difference. The p–chains together with the addition operation form a group denoted as $C_p(K)$. Besides, the set $\{C_p(K)\}_p$ is denoted by $C(K)$. A set of homomorphism $\{f_p : C_p(K) \to C_p(K')\}_p$ is called a *chain map* and denoted by $f : C(K) \to C(K')$. Given two p–cells $\sigma \in K$ and $\sigma' \in K'$, we say that $\sigma' \in f(\sigma)$ if σ' belongs to the p–chain $f_p(\sigma)$ (in set notation). The *boundary map* $\partial : C(K) \to C(K)$ is defined on a p–cell σ as the sum of its $(p-1)$–faces. This way, for a p–chain, $c = \sum_{i \in I} \sigma_i$, the boundary of c is the sum of the boundaries of its cells, $\partial_p c = \sum_{i \in I} \partial_p \sigma_i$.

A *filtration* of K is an increasing sequence of cell complexes: $\emptyset = K_0 \subset K_1 \subset \cdots \subset K_n = K$. The partial ordering given by such a filtration can be extended to a total ordering of the cells of K: $\{\sigma_1, \ldots, \sigma_m\}$, satisfying that for each i, $1 \leq i \leq m$, the faces of σ_i lies in the set $\{\sigma_1, \ldots, \sigma_i\}$. Then, the map $index : K \to \mathbb{Z}$ is defined by $index(\sigma_i) := i$.

Informally, the p–th persistent homology groups [3,12] can be seen as a collection of p–homology classes (representing connected components when $p = 0$, holes when $p = 1$, cavities when $p = 2$, ...) that are born at or before we go from K_{i-1} to K_i and die after we go from K_i to K_{i+1}. A *p–barcode* [7] is a graphical representation of the p–th persistent homology groups as a collection of horizontal line segments (*bars*) in a plane. Axis corresponds to the indices of the cells in K. For example, if a p–homology class was born at time i (i.e. when σ_i is added) and died at time j ($1 \leq i < j \leq m$), then a bar with endpoints (i, i) and (j, i) is added to the p–barcode.

In [8] the authors establish a correspondence between the incremental algorithm for computing AT-models [9] and the one for computing persistent homology. The first approach provides a rich algebraic information encoded by a chain homotopy operator ϕ, that "connects" any p–cell to the corresponding surviving cell.

An *AT-model* for a cell complex K is a quintuple (f, g, ϕ, K, H), where:

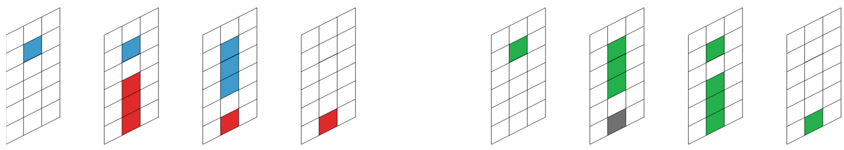
- K is the cell complex.
- $H \subseteq K$ describes the homology of K, in the sense that it contains a distinct p–cell for each p–homology class of a basis, for all p. The cells in H are called *surviving cells*. The set of all the surviving p–cells together with the addition operation form the group $C_p(H)$ for all p.
- $g : C(H) \to C(K)$ is a chain map that maps each p–cell h in H to one representative cycle $g_p(h)$ of the corresponding homology class $[g_p(h)]$.
- $f : C(K) \to C(H)$ is a chain map that maps each p–cell in K to a sum of surviving cells, satisfying that if $a, b \in C_p(K)$ are two homologous p–cycles then $f_p(a) = f_p(b)$.
- $\phi : C(K) \to C(K)$ is a *chain homotopy* (see [11]). Intuitively, for a p–cell σ, $\phi_p(\sigma)$ returns the $(p+1)$–cells needed to be contracted to "bring" σ to a surviving p–cell contained in $f_p(\sigma)$.

In the case of a 0–cell $v \in K$, $\phi_0(v)$ will provide a path in K (a sequence of edges of K connecting a sequence of different vertices) from the vertex v to the oldest (i.e., with lowest index) vertex in the same connected component.

3 Stating the Problem

Our general goal is to compute *spatiotemporal* p–barcodes for a time-varying sequence of nD binary images in the sense that they can represent evolution of homology classes *over time*. In this paper, we focus our effort in computing spatiotemporal 0–barcodes for time-varying sequences of 2D binary images.

In order to give some intuition about the problem we want to state, let us consider the simple examples given in Fig. 1, in which two sequences of a few 4–connected pixels appearing, moving and disappearing over time, are shown.



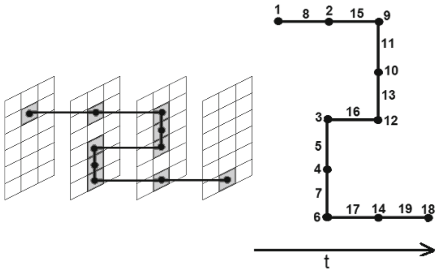
(a) (b) (c) (d) (e) (f) (g) (h)

Fig. 1. Pixels appearing, moving and disappearing over time.

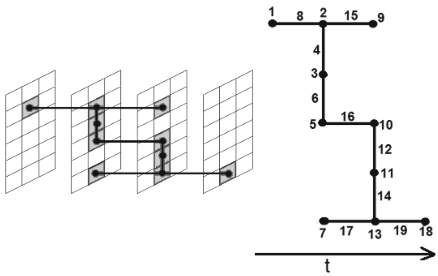
To encode the spatiotemporal information of the two sequences, we construct associated cell complexes by replacing each pixel by a vertex and adding an edge between two vertices if:

- The corresponding pixels are 4–connected (in the same frame).
- The vertices correspond to the same pixel at different times.

The resulting cell complexes K and K' are shown in Fig. 2.



(a) Cell complex K.



(b) Cell complex K'.

Fig. 2. Cell complexes K and K' obtained, respectively, from the sequence showed in Fig. 1(a)–(d) and (e)–(h).

Now, to compute 0–persistent homology on these two cell complexes K and K', we should select an appropriate filtration. Since we want to capture the variation of homology classes over time, we first classify the cells of K and K' in *spatial* and *temporal*:

- All vertices are spatial (since vertices represent pixels).
- An edge is spatial if its endpoints represent pixels of the same frame.
- If an edge is not spatial then it is temporal.

Therefore, we have the following *spatial subcomplexes* of K: $T_1 = \{1\}$, $T_2 = \{2, 3, 4, 5, 6, 7\}$, $T_4 = \{9, 10, 11, 12, 13, 14\}$, $T_6 = \{18\}$. And the following sets of temporal cells: $T_3 = \{8\}$, $T_5 = \{15, 16, 17\}$, $T_7 = \{19\}$, where numbers correspond to the labels of the cells showed in Fig. 2(a). The filtration $\emptyset = K_0 \subset K_1 \subset \cdots \subset K_7 = K$ is obtained by interleaving the temporal cells after the correspondent spatial subcomplexes. That is, $K_i = K_{i-1} \cup T_i$, $i = 1, \ldots, 7$.

Besides, the filtration on K' coincides with the filtration on K, where numbers now correspond to the labels of the cells showed in Fig. 2(b).

If we compute 0–persistent homology of K and K' using the above filtrations, we will obtain, in both cases, that a connected component (0–homology class) is born when cell 1 is added and survives until the end. So, in both cases, a bar with endpoints $(1, 1)$ and $(19, 1)$ is added to the 0–barcode.

However, we can observe that Fig. 1(a)–(d) cannot represent a connected component that is moving from the very beginning until the end while Fig. 1(e)–(h) can. So we wonder if we could modify the 0–barcode of the first sequence (Fig. 1(a)–(d)) so that it codifies the connected components that can survive along time. The idea is to replace the bar with endpoints $(1, 1)$ and $(19, 1)$ by respective bars from $(1, 1)$ to $(13, 1)$ and from $(3, 3)$ to $(19, 3)$, what will be formally described in next section.

4 Our Method

In this section, our aim is to design an algorithm to compute the spatiotemporal 0–barcode of a cell complex K encoding spatiotemporal data.

Suppose that K is composed by a stack of (*spatial*) complexes and a set of (*temporal*) cells such that each temporal cell connects two (consecutive) spatial complexes. Hence, our starting point is a (*spatiotemporal*) filtration of K, that is, a filtration $\emptyset = K_0 \subset K_1 \subset \cdots \subset K_n = K$ such that, for all i, $1 \leq i \leq n$, the set $T_i = K_i \setminus K_{i-1}$ is:

- a set of spatial cells if $i = 1$ or i is even;
- a set of temporal cells if $i > 1$ is odd.

A *spatiotemporal path* c in K is a path in K such that $\#(c \cap T_i) \leq 1$, for any i odd, $1 < i \leq n$. That is, there are not two temporal edges connecting the same consecutive spatial complexes, which follows from the idea that it is not possible to move backwards in time. Two vertices are *spatiotemporally-connected* if there is a spatiotemporal path between them.

Algorithm 1 extends the incremental algorithm for computing AT-models given in [9]. The eleven last lines of Algorithm 1 are original in this paper.

Although Algorithm 1 follows the same idea behind the algorithm given in [9] (by which, for each cell σ, $\phi(\sigma)$ "connects" the cell σ to a surviving cell), the computation of the map ϕ' is new in this paper.

In Algorithm 1, we compute a path $\phi'(v)$ from v to a surviving cell and, if $\phi'(v)$ is not spatiotemporal, we break it in pieces that are spatiotemporal paths. Then, a spatiotemporal path $\phi'(v)$ is obtained from each vertex $v \in K$ to the

Algorithm 1. Spatiotemporal 0–barcode.

Input: An ordering of the cells of K extending the partial ordering imposed by a spatiotemporal filtration.
Output: An AT-model for K and a spatiotemporal 0–barcode $\mathcal{B}$.

$H := \emptyset$.
for $i = 1$ **to** m **do**
  $f(\sigma_i) := 0$, $\phi(\sigma_i) := 0$, $\phi'(\sigma_i) := 0$.
for $i = 1$ **to** m **do**
  if $f\partial(\sigma_i) = 0$ **then**
    $f(\sigma_i) := \sigma_i$, $g(\sigma_i) := \sigma_i + \phi\partial(\sigma_i)$, $H := H \cup \{\sigma_i\}$.
    if $\dim(\sigma_i) = 0$ **then**
      Add to $\mathcal{B}$ a point at (i, i).
  if $f\partial(\sigma_i) \neq 0$ **then**
    Let $\sigma_j \in f\partial(\sigma_i)$ s t. $j = \max\{index(\mu) : \mu \in f\partial(\sigma_i)\}$
    $H := H \setminus \{\sigma_j\}$
    foreach $x \in K$ *s.t.* $\sigma_j \in f(x)$ **do**
      $f(x) := f(x) + f\partial(\sigma_i)$, $\phi(x) := \phi(x) + \sigma_i + \phi\partial(\sigma_i)$.
    if $\dim(\sigma_i) = 1$ **then**
      Let $v, w, v', w' \in K$ s.t. $\partial(\sigma_i) = v + w$, $v' = \partial\phi'(v) + v$, $w' = \partial\phi'(w) + w$ and $index(v') < index(w')$.
      Add to $\mathcal{B}$ the bar with endpoints $\{(index(v'), index(v')), (i, index(v'))\}$ and the bar with endpoints $\{(index(w'), index(w')), (i, index(w'))\}$.
      if $v \in T_\ell$ *and* $w, w' \in T_{\ell'}$ *for some* ℓ, ℓ', *s.t.* $1 \leq \ell \leq \ell' \leq n$ **then**
        foreach $x \in K$, $x \neq w, w'$ *s.t.* $\partial\phi'(x) + x = w'$ **do**
          $\phi'(x) := \phi'(x) + \phi'(w) + \sigma_i + \phi'(v)$.
        $\phi'(w') := \phi'(w) + \sigma_i + \phi'(v)$;
        $\phi'(w) := \sigma_i + \phi'(v)$.

"oldest" spatiotemporally-connected vertex. Regarding the spatiotemporal 0–barcode, at time i, we elongate a bar only if $dim(\sigma_i) = 1$ and the connected component that represents the bar is spatiotemporally connected to some of the endpoints of the edge σ_i. Otherwise, we do not elongate the bar. This is different from classical barcodes in which, for example, the bar corresponding to a connected component that appear in time i and does not merge to other connected component later, is elongated until the very end.

Proposition 1. *If v is a vertex in K then, $\phi'(v)$ is a spatiotemporal path.*

Proof. Let us prove the proposition by construction. At the beginning of the algorithm, $\phi'(v) = 0$ for every vertex $v \in K$. Suppose the algorithm is running and we are in step i, $1 \leq i \leq m$. Suppose that σ_i is an edge of K. Then, $\partial(\sigma_i) = v + w$ being v and w two vertices of K. Besides, by induction, $\phi'(v)$ and

$\phi'(w)$ are spatiotemporal paths. Then, $\partial\phi'(v) + v = v'$ and $\partial\phi'(w) + w = w'$ for some vertices $v' \in T_\ell$ and $w' \in T_{\ell'}$ being $1 \leq \ell, \ell' \leq n$.

We can assume that $index(v') < index(w')$. The case $index(v') = index(w')$ can only occur when $v' = w'$, what means that $f\partial(\sigma_i) = 0$ (a 1-cycle is being closed) and neither $\mathcal{B}$ nor ϕ' are modified in this case.

Now, let $c_w = \sigma + \phi'(v)$, $c_{w'} = \phi'(w) + \sigma + \phi'(v)$ and $c_x = \phi'(x) + \phi'(w) + \sigma + \phi'(v)$, for any $x \in K$ such that $\partial\phi'(x) + x = w'$. Then, $\partial(c_w) = w + v'$, $\partial(c_{w'}) = w' + v'$ and $\partial(c_x) = x + v'$. We have to consider the following cases:

- If σ_i is spatial, then $\sigma_i, v, w \in T_j$ for some j, $1 \leq j \leq n$. We have to consider the following cases:
  - If $\ell' < j$ then ϕ' is not updated.
  - If $\ell' = j$ then $\phi'(x) \subseteq T_j$ for any $x \in K$ s.t. $\partial\phi'(x) + x = w'$ and, therefore, c_w, $c_{w'}$ and c_x are spatiotemporal paths.
- If σ_i is temporal, then $v \in T_j$ and $w \in T_{j'}$ for some $j \neq j'$, $1 \leq j, j' \leq n$.
  - If $j < j'$. We consider two cases:
    - If $\ell' = j'$ then $\phi'(x) \subseteq T_{j'}$ for any $x \in K$ s.t. $\partial\phi'(x) + x = w'$ and, therefore, c_w, $c_{w'}$ and c_x are spatiotemporal paths.
    - If $\ell' < j'$ then ϕ' is not updated.
  - If $j' < j$ then ϕ' is not updated. □

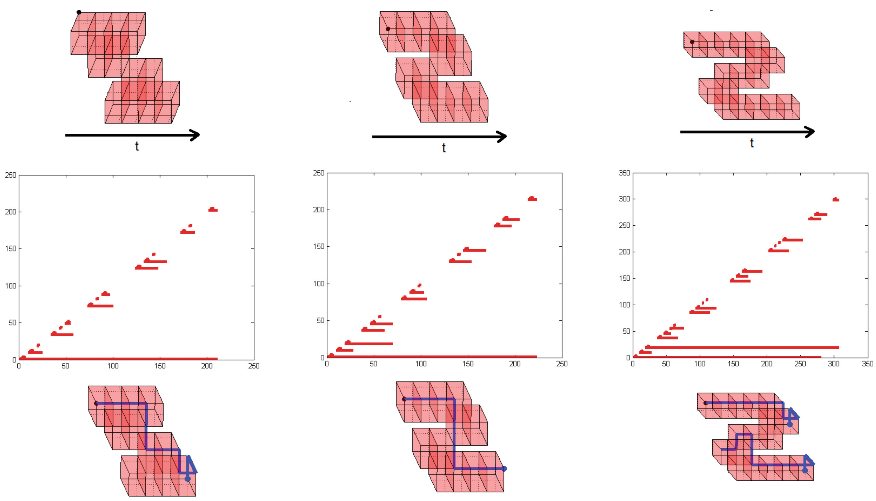


Fig. 3. Top: Three simple examples of stacked cubical complexes (t being the temporal dimension). Middle: The associated spatiotemporal barcodes obtained by applying Algorithm 1. Bottom: The spatiotemporal paths of the longest-lived 0–homology classes (in blue) (Color figure online).

5 Spatiotemporal Representation of Image Sequences

In this section, we explain how to compute a spatiotemporal filtration representing a time-varying sequence of 2D binary images, inspired by the stack complexes described in [1,4].

Consider $\mathbb{Z}^2$ as the set of points with integer coordinates in 2D space $\mathbb{R}^2$. A *2D binary image* is a set $I = (\mathbb{Z}^2, 8, 4, B)$, where $B \subset \mathbb{Z}^2$ is the *foreground*, $B^c = \mathbb{Z}^2 \backslash B$ the *background*, and $(8, 4)$ is the adjacency relation for the foreground and background, respectively. A point $p \in \mathbb{Z}^2$ can be interpreted as a unit closed square (called *pixel*) in $\mathbb{R}^2$ centered at p with edges parallel to the coordinate axes. The set of pixels centered at the points of B together with their faces (edges and vertices) constitute a cubical complex denoted by $Q(I)$. A p–cell in I can be identified by its barycentric coordinates $(x_\sigma, y_\sigma) \in \mathbb{R}^2$.

Following the construction given in [4], a cubical complex in which consecutive images are stacked to include a third, temporal dimension, is defined.

Definition 1. *Consider a sequence of 2D binary images $S = \{I_1, \ldots, I_n\}$ and the associated (2D) cubical complexes $Q(I_1), \ldots, Q(I_n)$. The stacked (3D) cubical complex $SQ[S]$ is obtained as follows. Let $Q(I_i) \times \{i\}$, $1 \le i \le n$, be the cubical complex obtained by adding a third coordinate i to the barycentric coordinates of the cells of $Q(I)$. Initially, $SQ[S] = \sqcup_{i=1}^{n}(Q(I_i) \times \{i\})$. Now, if a p-cell σ with barycentric coordinates (x_σ, y_σ) belongs to $Q(I_i) \cap Q(I_{i+1})$ for some i, $1 \le i < n$, add to $SQ[S]$ the $(p + 1)$–cell $\tau = \sigma \times [i, i + 1]$. This way, the barycentric coordinates of τ are $(x_\sigma, y_\sigma, i + \frac{1}{2})$.*

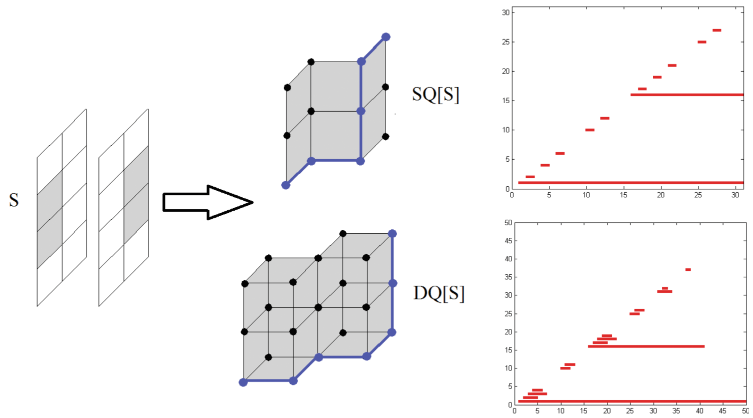


Fig. 4. A sequence S of two 2D images, the associated 3D cubical complexes $SQ[S]$ and $DQ[S]$ and the corresponding spatiotemporal 0–barcodes.

Since each cell $\sigma \in SQ[S]$ can be identified by its barycentric coordinates $(x_\sigma, y_\sigma, t_\sigma) \in \mathbb{R}^3$, then σ is spatial if, for some $i \in \mathbb{Z}$, $t_\sigma = i$; and it is temporal otherwise. For example, a cube $\tau \in SQ[S]$ is always temporal, and with respect to its faces we find: 6 spatial vertices; 8 spatial and 4 temporal edges; and 2 spatial and 4 temporal squares.

Let us denote by $Q(I_i, I_{i+1})$ the set of temporal cells with faces in $Q(I_i)$ and $Q(I_{i+1})$. The spatiotemporal filtration $\emptyset \subset SQ_0 \subset SQ_1 \subset \cdots \subset SQ_n = SQ[S]$ is given by: $SQ_i = Q(I_1)$, if $i = 1$; $SQ_i = SQ_{i-1} \cup Q(I_{j+1})$, if $i = 2j$ and $j > 0$; and $SQ_i = SQ_{i-1} \cup Q(I_j, I_{j+1})$, if $i = 2j + 1$ and $j > 0$.

Figure 3 shows three simple examples of stacked cubical complexes. The associated spatiotemporal 0–barcodes are computed using Algorithm 1. From left to right, the first and second spatiotemporal 0–barcodes have only one long bar, while third one has two. Notice that in this last case, the classical 0–barcode would produce only one long bar.

Observe that we could construct the 3D cubical complex $DQ[S]$, just considering every pixel centered at point (x, y, t) as a voxel (unit cube with faces parallel to the coordinate planes) centered at point (x, y, t). We have the following result:

Proposition 2. *Given a sequence of 2D binary images $S = \{I_1, \ldots, I_n\}$, the 3D cubical complexes $SQ[S]$ and $DQ[S]$ are homotopy equivalent.*

Proof. In our approach, to construct the 3D cubical complex $SQ[S]$, we build a cube only when two pixels in same spatial locations (i.e., with identical barycentric coordinates) belong to two consecutive frames; the other approach is to consider pixels as voxels (cubes) to directly obtain a 3D cubical complex $DQ[S]$ (see Fig. 4). To prove that $SQ[S]$ and $DQ[S]$ are homotopy equivalent, we describe how to collapse one complex, $DQ[S]$, to the other one, $SQ[S]$. For this aim, we first apply the translation $\tau(x, y, t) = (x, y, t + 1/2)$ to cells in $DQ[S]$. Consider a pixel (square cell) $\sigma \in SQ[S]$ centered at $(x_\sigma, y_\sigma, t_\sigma)$ that belongs to a cube c centered at $(x_\sigma, y_\sigma, t_\sigma + 1/2)$ in $SQ[S]$. Let c_σ be the voxel in $DQ[S]$ centered at $(x_\sigma, y_\sigma, t_\sigma)$. Then clearly $\tau(c_\sigma) = c \in SQ[S]$. Now, the idea is to successively collapse all the cells that are in $\tau(DQ[S])$ but not in $SQ[S]$. First, if σ does not belong to any cube in $SQ[S]$ centered at (x, y, t) with $t = t_\sigma + 1/2$, then collapse the square face centered at $(x_\sigma, y_\sigma, t_\sigma + 1)$ in $DQ[S]$. Similarly if an edge e of σ centered at (x_e, y_e, t_σ) does not belong to any cube in $SQ[S]$ centered at (x, y, t) with $t = t_\sigma + 1/2$, then collapse the edge centered at $(x_e, y_e, t_\sigma + 1)$. Finally, if a vertex v of σ with coordinates (x_v, y_v, t_σ) does not belong to any cube in $SQ[I]$

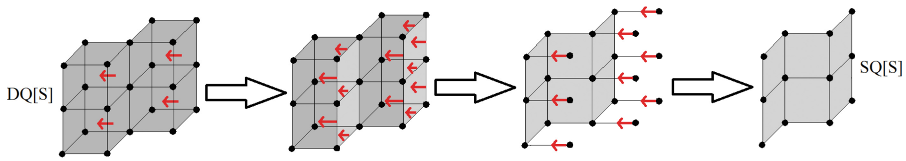


Fig. 5. A sequences of collapses starting from the complex $DQ[S]$ and ending at the complex $SQ[S]$. First, 4 square faces collapse, then 12 edges collapse and finally, 9 vertices collapse.

centered at (x, y, t) with $t = t_\sigma + 1/2$, then collapse the vertex with coordinates $(x_v, y_v, t_\sigma + 1)$. See Fig. 5. □

In this paper, we use the construction $SQ[S]$ instead of $DQ[S]$ because we considered that, in $SQ[S]$, the notion of spatial and temporal cells is more intuitive.

6 Conclusions and Future Work

In this paper, we have computed a modified 0–barcode for a temporal sequence of 2D binary images respecting the time nature of the data. This is part of an ongoing project to define and compute spatiotemporal p–barcodes for sequences of nD binary images.

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