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A NOTE ON UNIFORMLY DOMINATED SETS OF SUMMING OPERATORS

J. M. DELGADO and C. PIÑEIRO

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Let Y be a Banach space that has no finite cotype and p a real number satisfying $1 \leq p < \infty$. We prove that a set $\mathcal{M} \subset \Pi_p(X,Y)$ is uniformly dominated if and only if there exists a constant $C > 0$ such that, for every finite set $\{(x_i, T_i) : i = 1,\ldots,n\} \subset X \times \mathcal{M}$, there is an operator $T \in \Pi_p(X,Y)$ satisfying $\pi_p(T) \leq C$ and $\|T_i x_i\| \leq \|T x_i\|$ for $i = 1,\ldots,n$.

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1. Introduction. Let X and Y be Banach spaces and p a real number satisfying $1 \leq p < \infty$. A subset $\mathcal{M}$ of $\Pi_p(X,Y)$ is called *uniformly dominated* if there exists a positive Radon measure μ defined on the compact space $(B_{X^*}, \sigma(X^*,X)|_{B_{X^*}})$ such that

$$\|Tx\|^p \leq \int_{B_{X^*}} |\langle x^*, x\rangle|^p \, d\mu(x^*) \tag{1.1}$$

for all $x \in X$ and all $T \in \mathcal{M}$. Since the appearance of Grothendieck-Pietsch's domination theorem for p-summing operators, there is a great interest in finding out the structure of uniformly dominated sets. We will denote by $\mathcal{D}_p(\mu)$ the set of all operators $T \in \Pi_p(X,Y)$ satisfying (1.1) for all $x \in X$. It is easy to prove that $\mathcal{D}_p(\mu)$ is absolutely convex, closed, and bounded (for the p-summing norm).

In [4], the authors consider the case $p = 1$ and prove that $\mathcal{M} \subset \Pi_p(X,Y)$ is uniformly dominated if and only if $\bigcup_{T\in\mathcal{M}} T^*(B_{Y^*})$ lies in the range of a vector measure of bounded variation and valued in X^*.

In [3], the following sufficient condition is proved: "let $\mathcal{M} \subset \Pi_p(X,Y)$ and $1 \leq p < \infty$. Suppose that there is a positive constant $C > 0$ such that, for every finite set $\{x_1,\ldots,x_n\}$ of X, there exists $Q \in \mathcal{M}$ satisfying $\pi_p(Q) \leq C$ and

$$\sum_{i=1}^{n} ||Tx_i||^p \leq \sum_{i=1}^{n} ||Qx_i||^p \tag{1.2}$$

for all $T \in \mathcal{M}$. Then $\mathcal{M}$ is uniformly dominated." They also prove that this condition is necessary in the rather particular case that $\mathcal{M} \subset \Pi_p(c_0,c_0)$ and $\mathcal{M} = \mathcal{D}_p(\mu)$ for some positive Radon measure μ on B_{ℓ_1}.

In this note, we obtain a necessary and sufficient condition for a set $\mathcal{M} \subset \Pi_p(X,Y)$ to be uniformly dominated, with the only restriction that Y is a Banach space without finite cotype. We refer to [1] for our operator terminology. If X is a Banach space, B_X will denote its closed unit ball; $\ell_a^p(X)$ $(\ell_w^p(X))$ will be the Banach space of the strongly (weakly) p-summable sequences.

2. Main result. We need the following characterization of uniformly dominated sets.

Proposition 2.1. *Let $1 \le p < \infty$ and $\mathcal{M} \subset \Pi_p(X,Y)$. The following statements are equivalent:*

(a) *$\mathcal{M}$ is uniformly dominated.*

(b) *For every $\varepsilon > 0$ and $(x_n) \in \ell_w^p(X)$, there exists $n_0 \in \mathbb{N}$ such that*

$$\sum_{n \ge n_0} \|T_n x_n\|^p < \varepsilon \tag{2.1}$$

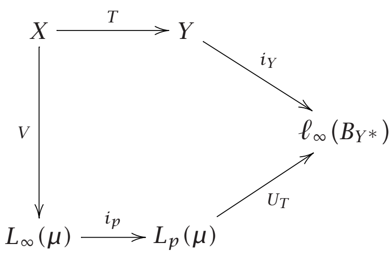
for all sequences (T_n) in $\mathcal{M}$.

(c) *There exists a constant $C > 0$ such that*

$$\sum_{i=1}^{n} \|T_i x_i\|^p \le C^p \sup_{x^* \in B_{X^*}} \sum_{i=1}^{n} |\langle x^*, x_i\rangle|^p \tag{2.2}$$

for all $\{x_1,\dots,x_n\} \subset X$ and $\{T_1,\dots,T_n\} \subset \mathcal{M}$.

Proof. (a)⇒(b). In a similar way as in the Pietsch factorization theorem [1], we can obtain, for all $T \in \mathcal{M}$, operators $U_T : L_p(\mu) \to \ell_\infty(B_{Y^*})$, $\|U_T\| \le \mu(B_{X^*})^{1/p}$, and an operator $V : X \to L_\infty(\mu)$ such that the following diagram is commutative:



(2.3)

Here i_p is the canonical injection from $L_\infty(\mu)$ into $L_p(\mu)$ and i_Y is the isometry from Y into $\ell_\infty(B_{Y^*})$ defined by $i_Y(y) = (\langle y^*, y\rangle)_{y^* \in B_{Y^*}}$. Given $\varepsilon > 0$ and $(x_n) \in \ell_w^p(X)$, we can choose $n_0 \in \mathbb{N}$ so that

$$\sum_{n \ge n_0} \|i_p \circ V(x_n)\|^p < \frac{\varepsilon}{\mu(B_{X^*})} \tag{2.4}$$

because $i_p \circ V$ is p-summing. Then, if (T_n) is a sequence in $\mathcal{M}$, we have

$$\begin{aligned} \sum_{n \ge n_0} \|T_n x_n\|^p &= \sum_{n \ge n_0} \|i_Y \circ T_n(x_n)\|^p \\ &= \sum_{n \ge n_0} \|U_{T_n} \circ i_p \circ V(x_n)\|^p \\ &\le \mu(B_{X^*}) \sum_{n \ge n_0} \|i_p \circ V(x_n)\|^p \le \varepsilon. \end{aligned} \tag{2.5}$$

(b)⇒(c). Using a standard argument, we can prove that $\mathcal{M}$ is bounded for the operator norm. Hence, given $\hat{x} = (x_n) \in \ell_w^p(X)$, there exists $M_{\hat{x}} > 0$ such that

$$\sum_{n=1}^{\infty} \|T_n x_n\|^p \le M_{\hat{x}} \tag{2.6}$$

for all (T_n) in $\mathcal{M}$. Then, we can consider the linear maps

$$\hat{T}:(x_n)\in \ell_w^p(X)\longmapsto (T_n x_n)\in \ell_a^p(Y) \tag{2.7}$$

for each $\hat{T}=(T_n)$ in $\mathcal{M}$. They have closed graph; so, by the uniform boundedness principle, there exists $M>0$ so that

$$\left(\sum_{n=1}^{\infty}\|T_n x_n\|^p\right)^{1/p}\leq M\epsilon_p(x_n) \tag{2.8}$$

for all $(x_n)\in \ell_w^p(X)$ and all (T_n) in $\mathcal{M}$ (we wrote ϵ_p for the norm in $\ell_w^p(X)$).

(c)⇒(a). Given $A=\{T_1,\dots,T_n\}\subset\mathcal{M}$ and $B=\{x_1,\dots,x_n\}\subset X$, we define $f_{A,B}:B_{X^*}\to\mathbb{R}$ by

$$f_{A,B}(x^*)=C^p\left(\sum_{i=1}^{n}|\langle x^*,x_i\rangle|^p\right)-\sum_{i=1}^{n}\|T_i x_i\|^p \tag{2.9}$$

for all $x^*\in X^*$. We denote by $\mathcal{P}$ the set of all functions $f_{A,B}$. It is clear that $\mathcal{P}$ is convex and disjoint from the cone $\mathcal{N}=\{f\in\mathscr{C}(B_{X^*}):f(x^*)<0,\ \text{for all } x^*\in B_{X^*}\}$. In a similar way as in the proof of Pietsch's domination theorem [1], we can show that there is a probability measure μ on B_{X^*} satisfying

$$\int_{B_{X^*}}\left(\|Tx\|^p-C^p|\langle x^*,x\rangle|^p\right)d\mu\leq 0 \tag{2.10}$$

for all $T\in\mathcal{M}$ and all $x\in X$. □

As an application of this result, we can show a relatively compact set for the p-summing norm which is not uniformly dominated. Put $T_n=(1/n)e_n^*\otimes e_n$, $n\in\mathbb{N}$, where (e_n) and (e_n^*) are the unit basis of c_0 and ℓ_1, respectively. As $\pi_1(T_n)=1/n$, (T_n) is a null sequence in $\Pi_1(c_0,c_0)$, so (T_n) is relatively compact. To see that it is not uniformly dominated, we will use Proposition 2.1: the sequence (e_n) is weakly summable but, for all $n\in\mathbb{N}$, we have

$$\sum_{k\geq n}\|T_k e_k\|_\infty=\sum_{k\geq n}\frac{1}{k}. \tag{2.11}$$

We are now ready to introduce our main result.

Theorem 2.2. *Let Y be a Banach space that has no finite cotype, $\mathcal{M}\subset\Pi_p(X,Y)$, and $1\leq p<\infty$. The following statements are equivalent:*

(a) *$\mathcal{M}$ is uniformly dominated.*

(b) *There is a constant $C>0$ such that, for every $\{x_1,\dots,x_n\}\subset X$ and $\{T_1,\dots,T_n\}\subset\mathcal{M}$, there exists an operator $T\in\Pi_p(X,Y)$ satisfying $\pi_p(T)\leq C$ and*

$$\|T_i x_i\|\leq\|Tx_i\|,\quad i=1,\dots,n. \tag{2.12}$$

Proof. (a)⇒(b). By hypothesis, there exists a positive Radon measure μ on B_{X^*} such that

$$\|Tx\|\leq\left(\int_{B_{X^*}}|\langle x^*,x\rangle|^p d\mu(x^*)\right)^{1/p} \tag{2.13}$$

for all $T \in \mathcal{M}$ and all $x \in X$. Since Y has no finite cotype, Y contains ℓ_∞^n's uniformly. By [2], for every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is an isomorphism J_n from ℓ_∞^n onto a subspace of Y satisfying $\|J_n^{-1}\| = 1$ and $\|J_n\| \le 1+\varepsilon$ for all $n \in \mathbb{N}$.

Given $\{x_1,\dots,x_n\} \subset X$ and $\{T_1,\dots,T_n\} \subset \mathcal{M}$, by (2.13) we have

$$\|T_i x_i\| \le \left(\int_{B_{X^*}} |\langle x^*, x_i\rangle|^p d\mu(x^*)\right)^{1/p}, \quad i = 1,\dots,n. \tag{2.14}$$

For every $i = 1,\dots,n$, take $g_i \in L_q(\mu)$ such that $\|g_i\|_q = 1$ and

$$\left(\int_{B_{X^*}} |\langle x^*, x_i\rangle|^p d\mu(x^*)\right)^{1/p} = \int_{B_{X^*}} \langle x^*, x_i\rangle g_i(x^*) d\mu(x^*). \tag{2.15}$$

From (2.14) and (2.15), we obtain

$$\|T_i x_i\| \le \int_{B_{X^*}} \langle x^*, x_i\rangle g_i(x^*) d\mu(x^*), \quad i = 1,\dots,n. \tag{2.16}$$

Put $y_i = J_n e_i$, being $(e_i)_{i=1}^n$ the unit basis of ℓ_∞^n. We define an operator $T : X \to Y$ by

$$Tx = \sum_{i=1}^n \left(\int_{B_{X^*}} \langle x^*, x\rangle g_i(x^*) d\mu(x^*)\right) y_i. \tag{2.17}$$

We first prove that $\|Tx\|^p \le (\int_{B_{X^*}} |\langle x^*, x\rangle|^p d\mu(x^*))(1+\varepsilon)$ for all $x \in X$:

$$\begin{aligned}
\|Tx\| &= \sup_{y^* \in B_{Y^*}} \left| \left\langle y^*, \sum_{i=1}^n \left(\int_{B_{X^*}} \langle x^*, x\rangle g_i(x^*) d\mu(x^*)\right) y_i \right\rangle \right| \\
&\le \sup_{y^* \in B_{Y^*}} \sum_{i=1}^n \left(\int_{B_{X^*}} |\langle x^*, x\rangle| |g_i(x^*)| d\mu(x^*)\right) |\langle y^*, y_i\rangle| \\
&\le \sup_{y^* \in B_{Y^*}} \sum_{i=1}^n \left(\int_{B_{X^*}} |\langle x^*, x\rangle|^p d\mu(x^*)\right)^{1/p} \left(\int_{B_{X^*}} |g_i(x^*)|^q d\mu(x^*)\right)^{1/q} |\langle y^*, y_i\rangle| \\
&\le \left(\int_{B_{X^*}} |\langle x^*, x\rangle|^p d\mu(x^*)\right)^{1/p} \sup_{y^* \in B_{Y^*}} \sum_{i=1}^n |\langle y^*, y_i\rangle| \\
&\le \left(\int_{B_{X^*}} |\langle x^*, x\rangle|^p d\mu(x^*)\right)^{1/p} \|J_n^*\| \\
&\le \left(\int_{B_{X^*}} |\langle x^*, x\rangle|^p d\mu(x^*)\right)^{1/p} (1+\varepsilon).
\end{aligned} \tag{2.18}$$

Finally, we need to prove that $\|T_i x_i\| \le \|T x_i\|$ for $i = 1,\dots,n$. Put $y_i^* = e_i^* \circ J_n^{-1}$, $(e_i^*)_{i=1}^n$ being the unit basis of $(\ell_\infty^n)^* \simeq \ell_1^n$. Notice that $\|y_i^*\| \le 1$ for $i = 1,\dots,n$. We also denote by y_i^* a Hahn-Banach extension of $e_i^* \circ J_n^{-1}$ to Y. We have

$$\begin{aligned}
\|Tx_i\| &\geq |\langle y_i^*, Tx_i\rangle| \\
&= \left|\left\langle y_i^*, \sum_{j=1}^{n}\left(\int_{B_{X^*}}\langle x^*, x_i\rangle g_j(x^*)\,d\mu(x^*)\right)y_j\right\rangle\right| \\
&= \left|\sum_{j=1}^{n}\left(\int_{B_{X^*}}\langle x^*, x_i\rangle g_j(x^*)\,d\mu(x^*)\right)\langle y_i^*, y_j\rangle\right| \\
&= \left|\sum_{j=1}^{n}\left(\int_{B_{X^*}}\langle x^*, x_i\rangle g_j(x^*)\,d\mu(x^*)\right)\langle e_i^* \circ J_n^{-1}, J_n e_j\rangle\right| \qquad (2.19)\\
&= \left|\sum_{j=1}^{n}\left(\int_{B_{X^*}}\langle x^*, x_i\rangle g_j(x^*)\,d\mu(x^*)\right)\langle e_i^*, e_j\rangle\right| \\
&= \int_{B_{X^*}}\langle x^*, x_i\rangle g_i(x^*)\,d\mu(x^*) \\
&\geq \|T_i x_i\|,
\end{aligned}$$

the last inequality is due to (2.16).

(b)⇒(a). It follows easily using Proposition 2.1(c). □

Remarks. (1) It is interesting to give an example of a uniformly dominated set $\mathcal{M}$ for which there is no operator $T \in \mathcal{M}$ satisfying $\|T_i x_i\| \leq \|Tx_i\|$, $i = 1, \ldots, n$, for some finite set $\{(x_i, T_i) : i = 1, \ldots, n\} \subset X \times \mathcal{M}$. Let $X = \ell_1$ and $Y = \ell_\infty$ and consider the set $\mathcal{M} = \{T_\beta : \beta \in B_{\ell_2}\}$, $T_\beta : \ell_1 \to \ell_\infty$ being defined by $T_\beta(\alpha) = (\alpha_n \beta_n)$ for all $\alpha = (\alpha_n) \in \ell_1$. Obviously, $\mathcal{M}$ is a uniformly dominated subset of $\Pi_1(\ell_1, \ell_\infty)$.

By contradiction, suppose the following condition holds: "there is a constant $C > 0$ such that, for every finite set $\{(x_i, T_i) : i = 1, \ldots, n\} \subset X \times \mathcal{M}$, there exists $T \in \mathcal{M}$ satisfying $\|T_i x_i\| \leq C\|Tx_i\|$, $i = 1, \ldots, n$." Put $x_i = e_i$ and $T_i = T_{\beta_i}$ for $i = 1, \ldots, n$, where $(e_i)_{i=1}^{\infty}$ is the unit basis of ℓ_1 and $\beta_i = (1/\sqrt{i}, \overset{(i)}{\ldots}, 1/\sqrt{i}, 0, \ldots)$. Take $T_\gamma \in \mathcal{M}$ such that

$$\|T_i x_i\| \leq C\|T_\gamma x_i\|, \quad i = 1, \ldots, n; \tag{2.20}$$

this yields

$$\frac{1}{\sqrt{i}} \leq C|\gamma_i|, \quad i = 1, \ldots, n. \tag{2.21}$$

Then we have

$$1 \geq \sum_{i=1}^{\infty}|\gamma_i|^2 \geq \sum_{i=1}^{n}|\gamma_i|^2 \geq \frac{1}{C^2}\sum_{i=1}^{n}\frac{1}{i}. \tag{2.22}$$

So, we have obtained the inequality $\sum_{i=1}^{n} 1/i \leq C^2$ for all $n \in \mathbb{N}$ which allows us to state that such an operator T cannot exist.

(2) Notice that, in the above example, $\mathcal{M}$ is absolutely convex and weakly compact in $\Pi_1(\ell_1, \ell_\infty)$. Then, $\mathcal{M}$ is absolutely convex, closed, and uniformly dominated but $\mathcal{M} \neq \mathfrak{D}_1(\mu)$ for every admissible positive Radon measure μ.

(3) Finally, we give an example of a bounded set $\mathcal{M}$ of 2-summing operators that does not have property (b) in Theorem 2.2. Consider the set $\mathcal{M}$ of all 2-summing operators $T_\beta : c_0 \to \ell_\infty$ defined by $T_\beta(\alpha) = (\alpha_n \beta_n)$ for all $\alpha = (\alpha_n) \in c_0$, where $\beta = (\beta_n)$ runs over the unit ball of ℓ_2. We have $T_\beta = i \circ S_\beta$, i being the identity map from ℓ_2 into ℓ_∞ and $S_\beta : c_0 \to \ell_2$ defined by $S_\beta(\alpha) = (\alpha_n \beta_n)$. Since ℓ_2 has cotype 2, it follows that S_β is 2-summing [1]. Nevertheless, $\mathcal{M}$ does not satisfy property (b) in the above theorem. By contradiction, suppose that there is a constant $C > 0$ such that (b) holds. Again, we take $\tilde{\beta}_i = (1/\sqrt{i}, \overset{(i)}{\ldots}, 1/\sqrt{i}, 0, \ldots)$ for all $i \in \mathbb{N}$. By hypothesis, there exists $T \in \Pi_2(c_0, \ell_\infty)$ such that $\pi_2(T) \le C$ and $\|T_{\tilde{\beta}_i} e_i\| \le \|Te_i\|$ for $i = 1, \ldots, n$. Then we have

$$\sum_{i=1}^{n} \frac{1}{i} = \sum_{i=1}^{n} \left\|T_{\tilde{\beta}_i} e_i\right\|^2 \le \sum_{i=1}^{n} \left\|Te_i\right\|^2 \le C^2 \tag{2.23}$$

for all $n \in \mathbb{N}$. Hence, $\mathcal{M}$ does not have property (b) in Theorem 2.2.

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J. M. Delgado: Departamento de Matemáticas, Escuela Politécnica Superior, Universidad de Huelva, La Rábida 21819, Huelva, Spain

E-mail address: jmdelga@uhu.es

C. Piñeiro: Departamento de Matemáticas, Escuela Politécnica Superior, Universidad de Huelva, La Rábida 21819, Huelva, Spain

E-mail address: candido@uhu.es