Some Results on $\mathbf{L}\,\Delta^{-}{}_{n+1}$

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Abstract. We study the quantifier complexity and the relative strength of some fragments of arithmetic axiomatized by induction and minimization schemes for Δ_{n+1}-formulas.

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1 Introduction and preliminaries

This work is connected with the Parameter Free Paris-Friedman's Conjecture:

$$\mathbf{L}\,\Delta^{-}_{n+1} \iff \mathbf{I}\,\Delta^{-}_{n+1}$$

The relationships among the schemas of induction, minimalization and collection for formulas in the classes of the Arithmetical Hierarchy (Σ_n, Π_n) has been studied by J. Paris and L. A. Kirby. The parameter free versions of these schemas have been studied by R. Kaye, J. Paris and C. Dimitracopoulos.

The aim of this paper is to study the quantifier complexity of $\mathbf{L}\,\Delta^{-}_{n+1}$ and some relations between this theory and the class of the Π_{n+1} true sentences. We prove that the following hold (see below for notation):

(a) $\mathbf{L}\,\Delta^{-}_{n+1}$ is Σ_{n+2}-axiomatized but is not Π_{n+2}-axiomatizable.

(b) $\mathbf{L}\,\Delta^{-}_{n+1}$ is not finitely axiomatizable.

(c) $\mathbf{L}\,\Delta^{-}_{1} \not\Longrightarrow \mathbf{I}\,\Delta_0$. For $n \geq 1$, $\mathbf{I}\,\Sigma_n \not\Longleftrightarrow \mathbf{L}\,\Delta^{-}_{n+1}$.

(d) Let T be an extension of $\mathbf{I}\,\Delta_0$ such that $\mathrm{Th}_{\Pi_2}(T) \Longrightarrow \mathbf{L}\,\Delta^{-}_{1}$ and $T + \exp$ is consistent. Then $\mathrm{Th}_{\Pi_1}(T + \exp) = \mathrm{Th}_{\Pi_1}(\mathcal{N})$.

(e) For $n \geq 1$, $\mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$ is (up to equivalence) the unique theory Π_{n+1}-axiomatized that is an extension of $\mathbf{L}\,\Delta^{-}_{n+1}$.

Part (e) is proved in [5] for $\mathbf{I}\,\Pi^{-}_{n+1}$. Since, $\mathbf{I}\,\Pi^{-}_{n+1}$ is a proper extension of $\mathbf{L}\,\Delta^{-}_{n+1}$, the property given in (e) improves that result. Properties (a) – (c) and weak versions of (d) and (e) are also true for $\mathbf{I}\,\Delta^{-}_{n+1}$.

Now we give the notation and the main results that we use through this paper.

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We work in the usual first-order language of Arithmetic, $\mathcal{L} = \{0, 1, +, \cdot, <\}$. We denote by $\mathcal{N}$ the standard model of $\mathcal{L}$ whose universe is the set of the natural numbers, ω, and the usual interpretation for nonlogical symbols of $\mathcal{L}$. Bounded quantifiers, denoted by $(\forall x \leq t)\, \varphi(x)$ and $(\exists x \leq t)\, \varphi(x)$, are, respectively, the formulas $\forall x\, [x \leq t \rightarrow \varphi(x)]$ and $\exists x\, [x \leq t \wedge \varphi(x)]$ (where x does not occur in t). The arithmetic hierarchy is the following classes of formulas of $\mathcal{L}$: $\Delta_0 = \Sigma_0 = \Pi_0$ is the class of all bounded formulas,

$$\Sigma_{n+1} = \{\exists \vec{x}\, \varphi(\vec{x}\,) : \varphi(\vec{x}\,) \in \Pi_n\} \text{ and } \Pi_{n+1} = \{\forall \vec{x}\, \varphi(\vec{x}\,) : \varphi(\vec{x}\,) \in \Sigma_n\}.$$

By P^- we denote a finite set of Π_1 axioms such that if $\mathfrak{A} \models \mathrm{P}^-$, then $\mathfrak{A}$ is the nonnegative part of a commutative discretely ordered ring (see [4] for details). Let $\varphi(x, \vec{v}\,)$ be a formula of $\mathcal{L}$. The *induction axiom* and the *least number principle axiom for* $\varphi(x, \vec{v}\,)$ *with respect to* x are, respectively, the formulas

$$\begin{aligned} \mathrm{I}_{\varphi,x}(\vec{v}\,) &\equiv \varphi(0, \vec{v}\,) \wedge \forall x\, [\varphi(x, \vec{v}\,) \rightarrow \varphi(x+1, \vec{v}\,)] \rightarrow \forall x\, \varphi(x, \vec{v}\,), \\ \mathrm{L}_{\varphi,x}(\vec{v}\,) &\equiv \exists x\, \varphi(x, \vec{v}\,) \rightarrow \exists x\, [\varphi(x, \vec{v}\,) \wedge (\forall y < x)\, \neg\varphi(y, \vec{v}\,)]. \end{aligned}$$

Let $\varphi(x, y, \vec{v}\,)$ be a formula of $\mathcal{L}$. The *collection axiom for* φ *with respect to* x, y is the formula $\mathrm{B}_{\varphi,x,y}(z, \vec{v}\,) \equiv (\forall x \leq z)\exists y\, \varphi(x, y, \vec{v}\,) \rightarrow \exists u (\forall x \leq z)(\exists y \leq u)\, \varphi(x, y, \vec{v}\,)$. As usual, we write I_φ instead of $\mathrm{I}_{\varphi,x}$ and similarly we use L_φ and B_φ. The axiom schemas $\mathbf{I}\,\Gamma$, $\mathbf{L}\,\Gamma$ and $\mathbf{B}\,\Gamma$, where Γ is a class of formulas of $\mathcal{L}$, are defined as follows:

$$\mathbf{I}\,\Gamma \equiv \mathrm{P}^- + \{\mathrm{I}_\varphi : \varphi \in \Gamma\},\ \mathbf{L}\,\Gamma \equiv \mathrm{P}^- + \{\mathrm{L}_\varphi : \varphi \in \Gamma\},\ \mathbf{B}\,\Gamma \equiv \mathbf{I}\,\Delta_0 + \{\mathrm{B}_\varphi : \varphi \in \Gamma\}.$$

Now we consider schemas for parameter free formulas. Let Γ be a class of formulas. We write $\varphi(x_1, \ldots, x_n) \in \Gamma^-$ if $\varphi \in \Gamma$ and $x_1, \ldots, x_n$ are all the variables that occurs free in φ. Schemas $\mathbf{I}\,\Gamma^-$ and $\mathbf{L}\,\Gamma^-$ are defined in a similar way. The schema $\mathbf{B}\,\Gamma^-$ is defined by $\mathbf{B}\,\Gamma^- = \mathbf{I}\,\Delta_0 + \{\mathrm{B}^-_{\varphi,x,y} : \varphi(x, y) \in \Gamma^-\}$, where

$$\mathrm{B}^-_{\varphi,x,y} \equiv \forall x \exists y\, \varphi(x, y) \rightarrow \forall z \exists u (\forall x \leq z)(\exists y \leq u)\, \varphi(x, y).$$

One of the basic functions used to describe metamathematical properties in the language of arithmetic, such as truth predicates, is the exponential function. We will denote by $\mathrm{E}(x, y, z)$ a Δ_0-formula which defines the exponential in the standard model, $\mathbf{I}\,\Delta_0$ proves that it verifies the elementary properties of the exponential function and $\mathbf{I}\,\Sigma_1$ proves that it is total (see [2] for details). We will usually write $x^y = z$ instead of $\mathrm{E}(x, y, z)$ and will denote by exp the Π_2 sentence $\forall x \forall y \exists z\, \mathrm{E}(x, y, z)$.

Let T, T' be theories. In the following we shall write

$$\begin{array}{lll} T \implies T' & \text{if } T \text{ is an extension of } T'; \\ T \not\Longrightarrow T' & \text{if } T \text{ is not an extension of } T'; \\ T \not\Longleftrightarrow T' & \text{if } T \not\Longrightarrow T' \text{ and } T' \not\Longrightarrow T; \\ T \iff T' & \text{if } T \text{ and } T' \text{ are equivalent}; \\ T \models\!\!\!\Longrightarrow T' & \text{if } T \text{ is a proper extension of } T'. \end{array}$$

Let T be a theory and let Γ be a class of formulas. We denote by $\mathrm{Th}_\Gamma(T)$ the class of the sentences of Γ which are provable in T, that is, $\mathrm{Th}_\Gamma(T) = \{\varphi \in \Gamma \cap \mathrm{Sent} : T \vdash \varphi\}$. We say that T is Γ-axiomatizable if $T \iff \mathrm{Th}_\Gamma(T)$. If $\mathfrak{A}$ is a model we denote by $\mathrm{Th}(\mathfrak{A})$ the theory of $\mathfrak{A}$, that is, $\mathrm{Th}(\mathfrak{A}) = \{\varphi \in \mathrm{Sent} : \mathfrak{A} \models \varphi\}$. We write $\mathrm{Th}_\Gamma(\mathfrak{A})$ instead of $\mathrm{Th}_\Gamma(\mathrm{Th}(\mathfrak{A}))$.

The collection axioms show how we can deal with bounded and unbounded quantifiers. The relationships among the axiom schemas of induction, least number principle and collection for the class of formulas Σ_n and Π_n were studied by J. Paris and L. Kirby (see [2] and [4]). The parameter free versions of these schemas were studied by R. Kaye, J. Paris and C. Dimitracopoulos (see [3] and [5]). We now give some results on these theories.

Theorem 1.1.

(a) $\mathbf{I}\,\Pi_1^- \mid\!\!\Longrightarrow \mathbf{L}\,\Delta_0^- \Longleftrightarrow \mathbf{I}\,\Delta_0^- \Longleftrightarrow \mathbf{I}\,\Delta_0$.

(b) *For all $n \in \omega$,*

(1)

$$\begin{array}{ccccccccccc}
 & & \mathbf{I}\,\Sigma_{n+1} & \mid\!\!\Longrightarrow & \mathbf{B}\,\Sigma_{n+1} & \mid\!\!\Longrightarrow & \mathbf{B}\,\Sigma_{n+1}^- & & \mid\!\!\Longrightarrow & & \mathbf{I}\,\Sigma_n \\
 & & \overline{\Downarrow} & & & & \underline{\Uparrow} & & & & \\
\mathbf{I}\,\Pi_{n+2}^- & \mid\!\!\Longrightarrow & \mathbf{I}\,\Sigma_{n+1}^- & & \Longleftrightarrow & & \mathbf{L}\,\Pi_{n+1}^- & \mid\!\!\Longrightarrow & \mathbf{I}\,\Pi_{n+1}^- & \Longleftrightarrow & \mathbf{L}\,\Sigma_{n+1}^-
\end{array}$$

(2) $\mathbf{B}\,\Sigma_{n+1} \nLeftrightarrow \mathbf{I}\,\Pi_{n+2}^- \nLeftrightarrow \mathbf{I}\,\Sigma_{n+1}$;

(3) $\mathbf{I}\,\Sigma_{n+1}^- \nLeftrightarrow \mathbf{B}\,\Sigma_{n+1} \nLeftrightarrow \mathbf{I}\,\Pi_{n+1}^- \nLeftrightarrow \mathbf{B}\,\Sigma_{n+1}^-$.

Let $\mathfrak{A} \vDash \mathrm{P}^-$ and $n \in \omega$. Then $\mathcal{K}_n(\mathfrak{A})$ is the substructure of $\mathfrak{A}$ whose universe is the set $\{b \in \mathfrak{A} : b \text{ is } \Sigma_n\text{-definable in } \mathfrak{A}\}$.

Theorem 1.2.

(a) *If $\mathfrak{A} \vDash \mathbf{I}\,\Sigma_n^-$ is nonstandard, then $\mathcal{K}_{n+1}(\mathfrak{A}) \vDash \mathbf{I}\,\Sigma_n$ and $\mathcal{K}_{n+1}(\mathfrak{A}) \prec_{n+1} \mathfrak{A}$.*

(b) *Let $\mathfrak{A} \vDash \mathbf{I}\,\Sigma_{n+1}$ such that $\mathcal{K}_{n+1}(\mathfrak{A})$ is nonstandard. Then $\mathcal{K}_{n+1}(\mathfrak{A}) \nvDash \mathbf{B}\,\Sigma_{n+1}$.*

Let us also introduce the following scheme: $\mathbf{B}_{\mathrm{s}}\Gamma^- = \mathbf{I}\,\Delta_0 + \{\mathrm{B}_{\varphi,x,y} : \varphi(x,y) \in \Gamma^-\}$. From [5, Proposition 1.7] we have

Lemma 1.3. $\mathbf{I}\,\Sigma_{n+1}^- \Longrightarrow \mathbf{B}_{\mathrm{s}}\,\Sigma_{n+1}^- \Longleftrightarrow \mathbf{B}_{\mathrm{s}}\,\Pi_n^- \Longrightarrow \mathbf{B}\,\Sigma_{n+1}^-$.

Now we introduce the axiom schemas for Δ_{n+1} formulas.

$$\begin{aligned}
\mathbf{I}\,\Delta_{n+1} &= \mathrm{P}^- + \{\forall x\,[\varphi(x,\vec{v}\,) \leftrightarrow \psi(x,\vec{v}\,)] \rightarrow \mathrm{I}_{\varphi,x}(\vec{v}\,) :\ \varphi \in \Sigma_{n+1},\ \psi \in \Pi_{n+1}\},\\
\mathbf{L}\,\Delta_{n+1} &= \mathrm{P}^- + \{\forall x\,[\varphi(x,\vec{v}\,) \leftrightarrow \psi(x,\vec{v}\,)] \rightarrow \mathrm{L}_{\varphi,x}(\vec{v}\,) :\ \varphi \in \Sigma_{n+1},\ \psi \in \Pi_{n+1}\},\\
\mathbf{B}\,\Delta_{n+1} &= \mathbf{I}\,\Delta_0 + \{\forall x\forall y\,[\varphi(x,y,\vec{v}\,) \leftrightarrow \psi(x,y,\vec{v}\,)] \rightarrow \mathrm{B}_{\varphi,x,y}(z,\vec{v}\,) :\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \varphi \in \Sigma_{n+1},\ \psi \in \Pi_{n+1}\}.
\end{aligned}$$

The parameter free schemas, $\mathbf{I}\,\Delta_{n+1}^-$ and $\mathbf{L}\,\Delta_{n+1}^-$, are defined similarly. It is easy to see that $\mathbf{B}\,\Sigma_{n+1} \Longleftrightarrow \mathbf{B}\,\Delta_{n+1}$. We have the following result.

Theorem 1.4. *For all $n \in \omega$, $\mathbf{B}\,\Sigma_{n+1} \Longleftrightarrow \mathbf{L}\,\Delta_{n+1} \Longrightarrow \mathbf{I}\,\Delta_{n+1} \mid\!\!\Longrightarrow \mathbf{I}\,\Sigma_n$.*

R. O. Gandy (see [2]) proved that $\mathbf{B}\,\Sigma_{n+1} \Longleftrightarrow \mathbf{L}\,\Delta_{n+1}$. For $\mathbf{I}\,\Delta_{n+1} \mid\!\!\Longrightarrow \mathbf{I}\,\Sigma_n$ see Corollary 4.5(a). The uniform version of the above fragments for Δ_{n+1} formulas are the following theories

$$\begin{aligned}
\mathbf{UB}\,\Delta_{n+1} &= \mathbf{I}\,\Delta_0 + \{\forall x\forall\vec{v}\,[\exists y\,\varphi(x,y,\vec{v}\,) \leftrightarrow \forall w\,\psi(x,w,\vec{v}\,)] \rightarrow \forall z\forall\vec{v}\,\mathrm{B}_{\varphi,x,y}(z,\vec{v}\,) :\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \varphi \in \Pi_n,\ \psi \in \Sigma_n\},\\
\mathbf{UI}\,\Delta_{n+1} &= \mathrm{P}^- + \{\forall x\forall\vec{v}\,[\varphi(x,\vec{v}\,) \leftrightarrow \psi(x,\vec{v}\,)] \rightarrow \forall\vec{v}\,\mathrm{I}_{\varphi,x}(\vec{v}\,) : \varphi \in \Sigma_{n+1},\ \psi \in \Pi_{n+1}\},\\
\mathbf{UL}\,\Delta_{n+1} &= \mathrm{P}^- + \{\forall x\forall\vec{v}\,[\varphi(x,\vec{v}\,) \leftrightarrow \psi(x,\vec{v}\,)] \rightarrow \forall\vec{v}\,\mathrm{L}_{\varphi,x}(\vec{v}\,) : \varphi \in \Sigma_{n+1},\ \psi \in \Pi_{n+1}\}.
\end{aligned}$$

Theses schemas were introduced by R. Kaye in [3]. It holds the following result (see [3]).

Theorem 1.5.

$$\begin{array}{ccccccc}
 & & \mathbf{I}\,\Pi^-_{n+1} & \models\!\!\!\Longrightarrow & \mathbf{L}\,\Delta^-_{n+1} & \Longrightarrow & \mathbf{I}\,\Delta^-_{n+1} \\
 & & & & \underline{\Uparrow} & & \Uparrow \\
\mathbf{UB}\,\Delta_{n+1} & \Longleftrightarrow & \mathbf{B}\,\Sigma^-_{n+1} & \Longleftrightarrow & \mathbf{UL}\,\Delta_{n+1} & \Longrightarrow & \mathbf{UI}\,\Delta_{n+1} \\
 & & \underline{\Uparrow} & & \underline{\Uparrow} & & \Uparrow \\
 & & \mathbf{B}\,\Sigma_{n+1} & \Longleftrightarrow & \mathbf{L}\,\Delta_{n+1} & \Longrightarrow & \mathbf{I}\,\Delta_{n+1}
\end{array}$$

We also have

(a) *For* $n > 0$, $\mathbf{UI}\,\Delta_{n+1} \models\!\!\!\Longrightarrow \mathbf{I}\,\Delta^-_{n+1}$. (b) *For* $n > 0$, $\mathbf{L}\,\Delta^-_{n+1} \not\Longrightarrow \mathbf{UI}\,\Delta_{n+1}$.

(c) $\mathbf{I}\,\Delta^-_{n+1} \not\Longrightarrow \mathbf{UL}\,\Delta_{n+1}$.

In a preprint, H. Friedman claimed (about 1985) that $\mathbf{L}\,\Delta_{n+1}$ and $\mathbf{I}\,\Delta_{n+1}$ are equivalent (see [2, p. 398]). In [1] this equivalence appears as an open problem (Problem 34) and it is credited to J. Paris. We have the following open problems.

- The Paris-Friedman's Conjecture: $\mathbf{L}\,\Delta_{n+1} \Longleftrightarrow \mathbf{I}\,\Delta_{n+1}$.
- The Parameter Free Paris-Friedman's Conjecture: $\mathbf{L}\,\Delta^-_{n+1} \Longleftrightarrow \mathbf{I}\,\Delta^-_{n+1}$.
- The Uniform Paris-Friedman's Conjecture: $\mathbf{UL}\,\Delta_{n+1} \Longleftrightarrow \mathbf{UI}\,\Delta_{n+1}$.

The remainder of this paper is organized as follows. In Section 2 we study the quantifier complexity and finite axiomatizability of $\mathbf{L}\,\Delta^-_{n+1}$. In Section 3 we study the relationship between $\mathbf{L}\,\Delta^-_{n+1}$ and the Π_{n+1}-theory of $\mathcal{N}$. Section 4 is devoted to obtain for $\mathbf{I}\,\Delta^-_{n+1}$ similar results to those obtained for $\mathbf{L}\,\Delta^-_{n+1}$. Finally, we close this paper with some open problems.

2 Quantifier complexity of $\mathbf{L}\,\boldsymbol{\Delta}^-_{n+1}$

First we prove some properties that will be useful in the following.

Lemma 2.1.

(a) *Let* T *be a consistent theory such that* $\mathrm{Th}_{\Pi_{n+1}}(\mathcal{N}) \subseteq \mathrm{Th}_{\Pi_{n+1}}(T)$. *Then we have* $\mathrm{Th}_{\Pi_{n+1}}(\mathcal{N}) = \mathrm{Th}_{\Pi_{n+1}}(T)$.

(b) $\mathrm{Th}_{\Pi_{n+1}}(\mathcal{N}) \models\!\!\!\Longrightarrow \mathbf{I}\,\Pi^-_{n+1}$.

Proof. The crucial fact for the proof is that for all model $\mathfrak{A}$,

$(*)$ $\quad \mathfrak{A} \models \mathrm{Th}_{\Pi_{n+1}}(\mathcal{N}) \quad$ iff $\quad \mathcal{N} \prec_{n+1} \mathfrak{A}$.

(a). Let T be a theory such that $\mathrm{Th}_{\Pi_{n+1}}(\mathcal{N}) \subseteq \mathrm{Th}_{\Pi_{n+1}}(T)$, let $\varphi \in \mathrm{Th}_{\Pi_{n+1}}(T)$ and let $\mathfrak{A} \models T$. Then $\mathfrak{A} \models \mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$. Hence, by $(*)$, $\mathcal{N} \prec_{n+1} \mathfrak{A}$. Since $\mathfrak{A} \models \varphi$, from this follows that $\mathcal{N} \models \varphi$, that is, $\varphi \in \mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$.

(b). Since $\mathbf{I}\,\Pi^-_{n+1}$ has a recursive set of Σ_{n+2} axioms and $\mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$ is a Π^0_{n+1}-complete set, the result follows from $(*)$. $\square$

The following result is essentially Proposition 1.9 of [5].

Lemma 2.2. *Let* $\mathfrak{A}$ *be a model in which every element is* Σ_{n+1}*-definable. Then*

(a) $\mathfrak{A} \models \mathbf{B}\,\Sigma_{n+1}$ *iff* $\mathfrak{A} \models \mathbf{B}\,\Sigma^-_{n+1}$; (b) $\mathfrak{A} \models \mathbf{L}\,\Delta_{n+1}$ *iff* $\mathfrak{A} \models \mathbf{L}\,\Delta^-_{n+1}$.

Proposition 2.3. *Let T be a consistent extension of* $\mathbf{I}\,\Sigma_{n+1}$. *The following conditions are equivalent:*

(a) $\mathrm{Th}_{\Pi_{n+1}}(T) = \mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$.

(b) $\mathrm{Th}_{\Pi_{n+1}}(T) \Longrightarrow \mathbf{I}\,\Pi^-_{n+1}$.

(c) $\mathrm{Th}_{\Pi_{n+1}}(T) \Longrightarrow \mathbf{L}\,\Delta^-_{n+1}$.

(d) $\mathrm{Th}_{\Pi_{n+2}}(T) \Longrightarrow \mathbf{L}\,\Delta^-_{n+1}$.

Proof. The implication (a) $\Rightarrow$ (b) follows from Lemma 2.1(b). The implication (b) $\Rightarrow$ (c) follows from $\mathbf{I}\,\Pi^-_{n+1} \Longleftrightarrow \mathbf{L}\,\Sigma^-_{n+1} \Longrightarrow \mathbf{L}\,\Delta^-_{n+1}$. The implication (c) $\Rightarrow$ (d) is trivial. To prove the implication (d) $\Rightarrow$ (a) by way of contradiction suppose that $\mathrm{Th}_{\Pi_{n+1}}(T) \neq \mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$. Then, by Lemma 2.1(a), there exists a nonstandard model $\mathfrak{A}$ of T such that $\mathfrak{A} \nvDash \mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$. Let $\varphi(x) \in \Sigma_n$ such that $\mathcal{N} \vDash \forall x\, \varphi(x)$ and $\mathfrak{A} \vDash \exists x\, \neg\varphi(x)$. Then $\mathcal{K}_{n+1}(\mathfrak{A})$ is nonstandard. So, by Theorem 1.2(b) and Theorem 1.4, $\mathcal{K}_{n+1}(\mathfrak{A}) \nvDash \mathbf{L}\,\Delta_{n+1}$. Since every element of $\mathcal{K}_{n+1}(\mathfrak{A})$ is Σ_{n+1}-definable, then, by Lemma 2.2,

$$(**) \quad \mathcal{K}_{n+1}(\mathfrak{A}) \nvDash \mathbf{L}\,\Delta^-_{n+1}.$$

Since $\mathfrak{A} \vDash \mathbf{I}\,\Sigma_{n+1}$, by Theorem 1.2(a), $\mathcal{K}_{n+1}(\mathfrak{A}) \prec_{n+1} \mathfrak{A}$. Thus, $\mathcal{K}_{n+1}(\mathfrak{A}) \vDash \mathrm{Th}_{\Pi_{n+2}}(T)$. So, from (d) and $(**)$ we get the desired contradiction. □

Theorem 2.4. (a) $\mathbf{L}\,\Delta^-_{n+1}$ *is* Σ_{n+2}*-axiomatized but is not* Π_{n+2}*-axiomatizable.*
(b) $\mathbf{L}\,\Delta^-_{n+1}$ *is not finitely axiomatizable.*

Proof.

(a). It is clear that $\mathbf{L}\,\Delta^-_{n+1}$ is Σ_{n+2}-axiomatized. We also have that $\mathbf{I}\,\Sigma_{n+1} \Longrightarrow \mathbf{L}\,\Delta^-_{n+1}$ and, by Proposition 2.3, $\mathrm{Th}_{\Pi_{n+2}}(\mathbf{I}\,\Sigma_{n+1}) \not\Longrightarrow \mathbf{L}\,\Delta^-_{n+1}$. Hence, $\mathbf{L}\,\Delta^-_{n+1}$ is not Π_{n+2}-axiomatizable.

(b). Suppose that $\mathbf{L}\,\Delta^-_{n+1}$ is finitely axiomatizable. Since $\mathrm{Th}_{\Pi_{n+1}}(\mathcal{N}) \Longrightarrow \mathbf{L}\,\Delta^-_{n+1}$, then there exists a recursively axiomatized consistent extension T of $\mathbf{I}\,\Sigma_{n+1}$ such that $\mathrm{Th}_{\Pi_{n+2}}(T) \Longrightarrow \mathbf{L}\,\Delta^-_{n+1}$. Then, by Proposition 2.3(a), $\mathrm{Th}_{\Pi_{n+1}}(T) = \mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$. But $\mathrm{Th}_{\Pi_{n+1}}(T)$ is recursively enumerable, and $\mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$ is Π^0_{n+1}-complete, which provides the desired contradiction. □

3 $\mathbf{L}\,\boldsymbol{\Delta}^-_{n+1}$ and $\boldsymbol{\Pi}_{n+1}$ true sentences

In the following we answer the following questions.

Let T be a theory such that $T \Longrightarrow \mathbf{L}\,\Delta^-_{n+1}$.

- Suppose T is Π_{n+1}-axiomatized. Does it hold that $T \Longleftrightarrow \mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$?
- Suppose T is Π_{n+2}-axiomatized. Does it hold that $\mathrm{Th}_{\Pi_{n+1}}(T) = \mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$?

First we discuss the case $n = 0$ and then we analyse the case $n > 0$.

Lemma 3.1. *Let* $\mathfrak{A} \vDash \mathbf{I}\,\Delta_0$. *Then* $\mathcal{K}_0(\mathfrak{A})$ *is cofinal in* $\mathcal{K}_1(\mathfrak{A})$.

Proof. Let $a \in \mathcal{K}_1(\mathfrak{A})$ and let $\psi(x, y) \in \Delta_0$ such that

$$\mathfrak{A} \vDash \exists y\, \psi(a, y) \,\wedge\, \exists! x\, \exists y\, \psi(x, y).$$

Since $\mathcal{K}_1(\mathfrak{A}) \prec_1 \mathfrak{A}$, there exists $b \in \mathcal{K}_1(\mathfrak{A})$ such that $\mathcal{K}_1(\mathfrak{A}) \vDash \psi(a, b)$. So, (where J is Cantor's pairing function) $\mathcal{K}_1(\mathfrak{A}) \vDash \exists z (\exists x \leq z)(\exists y \leq z)\, [z = J(x, y) \,\wedge\, \psi(x, y)]$. This formula is also true in $\mathfrak{A}$. Let $\theta(z) \in \Delta_0$ be $(\exists x \leq z)(\exists y \leq z)\, [z = J(x, y) \,\wedge\, \psi(x, y)]$. So, there is $c \in \mathfrak{A}$ such that $\mathfrak{A} \vDash \theta(c) \,\wedge\, (\forall z < c)\, \neg\theta(z)$. So, $c \in \mathcal{K}_0(\mathfrak{A})$ and $a \leq c$. □

T h e o r e m 3.2. *Let T be an extension of* $\mathbf{I}\,\Delta_0$ *such that* (i) $T+\exp$ *is consistent and* (ii) $\mathrm{Th}_{\Pi_2}(T) \Longrightarrow \mathbf{L}\,\Delta_1^-$. *Then* $\mathrm{Th}_{\Pi_1}(T+\exp) = \mathrm{Th}_{\Pi_1}(\mathcal{N})$.

P r o o f. By way of contradiction suppose that the theorem's conclusion is false. Then, by (i) and Lemma 2.1, there exists $\varphi(x) \in \Delta_0$ such that $\mathcal{N} \models \forall x\, \neg\varphi(x)$ and $T+\exp \not\models \forall x\, \neg\varphi(x)$. Let $\mathfrak{A} \models T+\exp+\exists x\, \varphi(x)$. It is clear that $\mathfrak{A}$ and $\mathcal{K}_1(\mathfrak{A})$ are nonstandard, and $\mathcal{K}_1(\mathfrak{A}) \models \mathrm{Th}_{\Pi_2}(T+\exp+\exists x\, \varphi(x))$. We also have

C l a i m. $\mathcal{K}_1(\mathfrak{A}) \not\models \mathbf{L}\,\Delta_1^-$.

P r o o f. Every element of $\mathcal{K}_1(\mathfrak{A})$ is Σ_1-definable in $\mathcal{K}_1(\mathfrak{A})$. So, by Lemma 2.2 and Theorem 1.4, it is enough to see that $\mathcal{K}_1(\mathfrak{A}) \not\models \mathbf{B}\,\Sigma_1$. Let $a,\, d \in \mathcal{K}_1(\mathfrak{A})$ nonstandard. Since $\mathcal{K}_1(\mathfrak{A}) \models \mathbf{I}\,\Delta_0+\exp$, we have that

$$(*) \qquad \begin{aligned} \mathcal{K}_1(\mathfrak{A}) \models (\forall u \le d+1)(\exists w \le d)\,\exists x\,[(\forall z < x)\,\neg\mathcal{V}_0(w, \langle z\rangle, 2^{(z+2)^a}) \\ \wedge\, \mathcal{V}_0(w, \langle x\rangle, 2^{(x+2)^a}) \,\wedge\, u = (x)_0], \end{aligned}$$

where $\mathcal{V}_0(v_1, v_2, v_3) \in \Delta_0$ is a truth definition for Δ_0 formulas whose properties are provable in $\mathbf{I}\,\Delta_0+\exp$ (see [2] for details). [This follows from the proof of Lemma 3.1. Let $u \in \mathcal{K}_1(\mathfrak{A})$ such that $u \le d+1$ and let $\exists y\, \psi(x,y) \in \Sigma_1$ be a formula that defines u in $\mathfrak{A}$. Let $w \in \omega$ be the Gödel number of $(\exists x \le z)(\exists y \le z)\,[z = J(x,y) \,\wedge\, \psi(x,y)]$. Since d is nonstandard, $w \le d$ and satisfies $(*)$.] Now suppose that $\mathcal{K}_1(\mathfrak{A}) \models \mathbf{B}\,\Sigma_1$. Then there exists $c \in \mathcal{K}_1(\mathfrak{A})$ such that

$$\begin{aligned} \mathcal{K}_1(\mathfrak{A}) \models (\forall u \le d+1)(\exists w \le d)(\exists x < c)[(\forall z < x)\,\neg\mathcal{V}_0(w, \langle z\rangle, 2^{(z+2)^a}) \\ \wedge\, \mathcal{V}_0(w, \langle x\rangle, 2^{(x+2)^a}) \,\wedge\, u = (x)_0]. \end{aligned}$$

This gives an injective Δ_0-map from $(\le d+1)$ to $(\le d)$. Since $\mathcal{K}_1(\mathfrak{A}) \models \mathbf{I}\,\Delta_0+\exp$, this contradicts the Pigeon-hole principle for (coded) Δ_0-functions in $\mathbf{I}\,\Delta_0+\exp$ (see [2]). ☐ Claim

Since $\mathcal{K}_1(\mathfrak{A}) \models \mathrm{Th}_{\Pi_{n+2}}(T)$, the claim and (ii) provide the desired contradiction. ☐

R e m a r k 3.3. We have that for all $\mathfrak{A} \models \mathbf{I}\,\Pi_1^-$, $\mathcal{K}_1(\mathfrak{A}) \models \mathbf{I}\,\Delta_0+\exp$ (see [5, Theorem 2.9]). So, with a proof similar to the one given for Theorem 3.2, we have

C l a i m 3.3.1. *If T is a consistent theory such that* $\mathrm{Th}_{\Pi_2}(T) \Longrightarrow \mathbf{I}\,\Pi_1^-$, *then* $\mathrm{Th}_{\Pi_1}(T) = \mathrm{Th}_{\Pi_1}(\mathcal{N})$.

This improves Theorem 3.2 for $\mathbf{I}\,\Pi_1^-$.

C o r o l l a r y 3.4.

(a) $\mathbf{I}\,\Delta_0 \not\Rightarrow \mathbf{L}\,\Delta_1^-$, *hence,* $\mathbf{L}\,\Delta_1^- \mathrel{|}\!\!\Longrightarrow \mathbf{I}\,\Delta_0$. (b) *If $n>0$, then* $\mathbf{I}\,\Sigma_n \not\Leftrightarrow \mathbf{L}\,\Delta_{n+1}^-$.

(c) *If $n>0$, then* $\mathbf{L}\,\Delta_{n+1}^- \not\Rightarrow \mathbf{UI}\,\Delta_{n+1} \mathrel{|}\!\!\Longrightarrow \mathbf{I}\,\Delta_{n+1}^- \not\Rightarrow \mathbf{I}\,\Sigma_n$.

P r o o f.

(a). Let us consider $\mathbf{I}\,\Delta_0+\exp$. By Theorem 3.2, $\mathrm{Th}_{\Pi_1}(\mathbf{I}\,\Delta_0+\exp) \not\Rightarrow \mathbf{L}\,\Delta_1^-$. Since $\mathbf{I}\,\Delta_0$ is Π_1-axiomatized, this proves (a).

(b). Since $\mathbf{I}\,\Sigma_n$ is Π_{n+2}-axiomatizable, from Proposition 2.3 for $T = \mathbf{I}\,\Sigma_{n+1}$ follows $\mathbf{I}\,\Sigma_n \not\Rightarrow \mathbf{L}\,\Delta_{n+1}^-$. If $n>0$, then $\mathbf{I}\,\Pi_{n+1}^- \not\Rightarrow \mathbf{I}\,\Sigma_n$ and $\mathbf{I}\,\Pi_{n+1}^- \Longrightarrow \mathbf{L}\,\Delta_{n+1}^-$. Hence, $\mathbf{L}\,\Delta_{n+1}^- \not\Rightarrow \mathbf{I}\,\Sigma_n$.

(c). The assertion $\mathbf{I}\,\Delta_{n+1}^- \not\Rightarrow \mathbf{I}\,\Sigma_n$ follows from (b) and $\mathbf{L}\,\Delta_{n+1}^- \Longrightarrow \mathbf{I}\,\Delta_{n+1}^-$. The assertion $\mathbf{UI}\,\Delta_{n+1} \mathrel{|}\!\!\Longrightarrow \mathbf{I}\,\Delta_{n+1}^-$ follows from $\mathbf{UI}\,\Delta_{n+1} \Longrightarrow \mathbf{I}\,\Sigma_n + \mathbf{I}\,\Delta_{n+1}^-$ and the above property. Finally, $\mathbf{L}\,\Delta_{n+1}^- \not\Rightarrow \mathbf{UI}\,\Delta_{n+1}$ follows from (b), since $\mathbf{UI}\,\Delta_{n+1} \Longrightarrow \mathbf{I}\,\Sigma_n$. ☐

Corollary 3.5. *Let T be a consistent Π_2-axiomatized extension of $\mathbf{I}\Delta_0$ such that $T \vdash \exp$. Then the following conditions are equivalent:*

(a) $\mathrm{Th}_{\Pi_1}(T) = \mathrm{Th}_{\Pi_1}(\mathcal{N})$; (b) $T \Longrightarrow \mathbf{I}\,\Pi_1^-$; (c) $T \Longrightarrow \mathbf{L}\,\Delta_1^-$.

Remark 3.6. Now we generalize Proposition 2.3 and Theorem 3.2 to extensions of $\mathbf{I}\,\Sigma_n^-$ for $n \geq 1$. In the following let $\mathfrak{A} \models \mathbf{I}\,\Sigma_n^-$ with $n \geq 1$. We first characterize a subset of $\mathcal{K}_{n+1}(\mathfrak{A})$ that plays the same role with respect of $\mathcal{K}_{n+1}(\mathfrak{A})$ that $\mathcal{K}_0(\mathfrak{A})$ plays with respect to $\mathcal{K}_1(\mathfrak{A})$.

Definition (McAloon). We say that $a \in \mathfrak{A}$ is Π_n-*minimal in* $\mathfrak{A}$ if there is $\varphi(x) \in \Pi_n^-$ such that $\mathfrak{A} \models \varphi(a) \wedge (\forall x < a)\, \neg\varphi(x)$. Let

$$\mathcal{M}_n(\mathfrak{A}) = \{a \in \mathfrak{A} : a \text{ is } \Pi_n\text{-minimal in } \mathfrak{A}\}.$$

Claim 3.6.1. (a) $\mathcal{M}_n(\mathfrak{A}) \subseteq \mathcal{K}_{n+1}(\mathfrak{A})$. (b) $\mathcal{M}_n(\mathfrak{A})$ *is cofinal in* $\mathcal{K}_{n+1}(\mathfrak{A})$.

Proof.

(i). By Lemma 1.3, if $\varphi(x) \in \Pi_n^-$, then the formula $\varphi(x) \wedge (\forall y < x)\, \neg\varphi(y)$ is Δ_{n+1} in $\mathbf{I}\,\Sigma_n^-$. So, (i) holds.

(ii). Let $a \in \mathcal{K}_{n+1}(\mathfrak{A})$ and let $\psi(x, y) \in \Pi_n^-$ such that

$$\mathfrak{A} \models \exists y\, \psi(a, y) \wedge \exists! x\, \exists y\, \psi(x, y), \quad \text{that is, } \exists y\, \psi(x, y) \text{ defines } a \text{ in } \mathfrak{A}.$$

Since $\mathcal{K}_{n+1}(\mathfrak{A}) \prec_{n+1} \mathfrak{A}$, $\mathcal{K}_{n+1}(\mathfrak{A}) \models \exists y\, \psi(a, y)$. Let $b \in \mathfrak{A}$ such that $\mathcal{K}_{n+1}(\mathfrak{A}) \models \psi(a, b)$. Then $\mathcal{K}_{n+1}(\mathfrak{A}) \models \exists z (\forall x, y \leq z)\, [z = J(x, y) \rightarrow \psi(x, y)]$. So, from $\mathcal{K}_{n+1}(\mathfrak{A}) \prec_{n+1} \mathfrak{A}$ follows that $\mathfrak{A} \models \psi(a, b)$ and $\mathfrak{A} \models \exists z (\forall x, y \leq z)\, [z = J(x, y) \rightarrow \psi(x, y)]$. Let $\theta(z) \in \Pi_n^-$ be the formula $(\forall x, y \leq z)\, [z = J(x, y) \rightarrow \psi(x, y)]$. By Theorem 1.1, $\mathfrak{A} \models \mathbf{L}\,\Pi_n^-$. Then there is $c \in \mathfrak{A}$ such that $\mathfrak{A} \models \theta(c) \wedge (\forall z < c)\, \neg\theta(z)$. Hence, $c \in \mathcal{M}_n(\mathfrak{A})$ and $a \leq c$. □

Now we establish the promised result:

Theorem 3.7. *Let $n \geq 1$ and let T be a consistent theory. Then the following conditions are equivalent:*

(a) $\mathrm{Th}_{\Pi_{n+1}}(T) = \mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$.

(b) $\mathrm{Th}_{\Pi_{n+1}}(T) \Longrightarrow \mathbf{I}\,\Pi_{n+1}^-$.

(c) $\mathrm{Th}_{\Pi_{n+1}}(T) \Longrightarrow \mathbf{L}\,\Delta_{n+1}^-$.

(d) $\mathrm{Th}_{\Pi_{n+2}}(T) \Longrightarrow \mathbf{L}\,\Delta_{n+1}^-$.

Proof. It is enough to prove that (d) implies (a). By way of contradiction suppose that (a) is false. Since T is consistent, by Lemma 2.1, there exists $\varphi(x) \in \Pi_n^-$ such that $\mathcal{N} \models \forall x\, \neg\varphi(x)$ and $T \nvdash \forall x\, \neg\varphi(x)$. Let $\mathfrak{A} \models T + \exists x\, \varphi(x)$. It is clear that $\mathfrak{A}$ is nonstandard. We also have

Claim 3.7.1. *$\mathcal{K}_{n+1}(\mathfrak{A})$ is nonstandard.*

Proof. First observe that for every $c \in \mathfrak{A}$

(∗) if $\mathfrak{A} \models \varphi(c)$, then c is nonstandard.

Since $\mathfrak{A} \models \exists x\, \varphi(x)$ and $\mathfrak{A} \models \mathbf{I}\,\Sigma_n^-$, then there exists $a \in \mathfrak{A}$ such that

$$\mathfrak{A} \models \varphi(a) \wedge (\forall z < a)\, \neg\varphi(z).$$

So, $a \in \mathcal{M}_n(\mathfrak{A})$. Then, by Claim 3.6.1(i), $a \in \mathcal{K}_{n+1}(\mathfrak{A})$. Hence, by (∗), $\mathcal{K}_{n+1}(\mathfrak{A})$ is nonstandard. □ Claim

Since $\mathcal{K}_{n+1}(\mathfrak{A}) \prec_{n+1} \mathfrak{A}$, then $\mathcal{K}_{n+1}(\mathfrak{A}) \models \mathrm{Th}_{\Pi_{n+2}}(T + \exists x\, \varphi(x))$. Since

$$\mathrm{Th}_{\Pi_{n+2}}(T) \Longrightarrow \mathbf{L}\,\Delta_{n+1}^- \Longrightarrow \mathbf{I}\,\Sigma_n^-,$$

then $\mathcal{K}_{n+1}(\mathfrak{A}) \models \mathbf{I}\,\Sigma_n$. So, it is enough to prove the following

Claim 3.7.2. $\mathcal{K}_{n+1}(\mathfrak{A}) \nvDash \mathbf{L}\,\Delta^-_{n+1}$.

Proof. Since every element of $\mathcal{K}_{n+1}(\mathfrak{A})$ is Σ_{n+1}-definable, it is enough to see that $\mathcal{K}_{n+1}(\mathfrak{A}) \nvDash \mathbf{B}\,\Sigma_{n+1}$. Let $d \in \mathcal{K}_{n+1}(\mathfrak{A})$ nonstandard. Since $\mathcal{K}_{n+1}(\mathfrak{A}) \vDash \mathbf{I}\,\Sigma_n$ and $n \geq 1$ we have that

$$(**) \quad \mathcal{K}_{n+1}(\mathfrak{A}) \vDash (\forall u \leq d+1)(\exists w \leq d)\exists x\,[(\forall z < x)\,\neg \mathrm{Sat}_{\Pi_n}(w(\dot{z})) \\ \wedge \mathrm{Sat}_{\Pi_n}(w(\dot{x})) \,\wedge\, u = (x)_0],$$

where $\mathrm{Sat}_{\Pi_n}(v) \in \Pi_n$ is a truth definition for Π_n formulas whose properties are provable in $\mathbf{I}\,\Sigma_1$ (see [2] or [4] for details). [This follows from the proof of Claim 3.6.1(ii). Let $u \in \mathcal{K}_{n+1}(\mathfrak{A})$ such that $u \leq d+1$ and let $\exists y\, \psi(x, y) \in \Sigma_{n+1}$ be a formula that defines u in $\mathfrak{A}$. Let $w \in \omega$ be the Gödel number of the formula

$$(\forall x, y \leq z)\,[z = J(x, y) \rightarrow \psi(x, y)].$$

Since d is nonstandard, then $w \leq d$ and satisfies $(**)$.] Now suppose that $\mathcal{K}_{n+1}(\mathfrak{A}) \vDash \mathbf{B}\,\Sigma_{n+1}$. Then there is $c \in \mathcal{K}_{n+1}(\mathfrak{A})$ such that

$$\mathcal{K}_{n+1}(\mathfrak{A}) \vDash (\forall u \leq d+1)(\exists w \leq d)(\exists x < c)\,[(\forall z < x)\neg \mathrm{Sat}_{\Pi_n}(w(\dot{z})) \\ \wedge \mathrm{Sat}_{\Pi_n}(w(\dot{x})) \,\wedge\, u = (x)_0].$$

From this we obtain an injective $\Sigma_0(\Sigma_n)$-map from $(\leq d+1)$ to $(\leq d)$. This contradicts the Pigeon-hole principle for coded $\Sigma_0(\Sigma_n)$-maps. □ Claim

Since $\mathfrak{A} \vDash \mathrm{Th}_{\Pi_{n+2}}(T)$, from Claim 3.7.2 and (d) we obtain the desired contradiction. □

An immediate consequence of Theorem 3.7 is

Theorem 3.8. *If* $n \geq 1$, *then* $\mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$ *is* (*up to equivalence*) *the unique* Π_{n+1}*-axiomatized theory that is an extension of* $\mathbf{L}\,\Delta^-_{n+1}$. □

4 The theory $\mathbf{I}\boldsymbol{\Delta}^-_{n+1}$

Here we shall see that the properties proved above for $\mathbf{L}\,\Delta^-_{n+1}$ (or weak versions of them) are also true for $\mathbf{I}\,\Delta^-_{n+1}$. In the following we will use results of K. McAloon [7] and H. Lessan [6] whose proofs lean upon the Arithmetized Completeness Theorem.

Let $\mathfrak{A}$ be a model of PA (Peano Arithmetic). The standard system of $\mathfrak{A}$, denoted by $\mathrm{SSy}(\mathfrak{A})$, is the collection of subsets of ω which are definable in $\mathfrak{A}$, that is, $X \in \mathrm{SSy}(\mathfrak{A})$ if there exist a formula $\varphi(x, v)$ and $b \in \mathfrak{A}$ such that $X = \{k \in \omega : \mathfrak{A} \vDash \varphi(k, b)\}$. Let T be a theory and let Γ be a class of formulas. We say that T *is coded in* $\mathfrak{A}$ (denoted by $T \in \mathrm{SSy}(\mathfrak{A})$) if $\{\ulcorner \psi \urcorner : \psi \text{ is an axiom of } T\} \in \mathrm{SSy}(\mathfrak{A})$. We say that T is Γ*-definable in* $\mathcal{N}$ if there is $\varphi(x) \in \Gamma$ with $\{\ulcorner \psi \urcorner : \psi \text{ is an axiom of } T\} = \{k \in \omega : \mathcal{N} \vDash \varphi(k)\}$.

We have the following results.

Theorem 4.1 (McAloon). *Let* T *be an extension of* PA *and let* $\mathfrak{A} \vDash T$ *such that* $T \in \mathrm{SSy}(\mathfrak{A})$. *Then for every* $n \in \omega$ *there exists* $\mathfrak{B} \vDash \mathrm{Th}_{\Pi_{n+2}}(T)$ *such that*

(a) $\mathfrak{B}$ *is an* n*-elementary final extension of* $\mathfrak{A}$, $\mathfrak{A} \prec^{\mathrm{e}}_n \mathfrak{B}$;

(b) *there exist* $\varphi(x, \vec{v}\,) \in \Delta_{n+1}(\mathfrak{B})$ *and* $b \in \mathfrak{B}$ *such that* $\omega = \{c \in \mathfrak{B} : \mathfrak{B} \vDash \varphi(c, b)\}$.

Theorem 4.2 (McAloon). *Let* T *be an extension of* PA *consistent with* $\mathrm{Th}_{\Pi_n}(\mathcal{N})$ *and* Σ_{n+1}*-definable in* $\mathcal{N}$. *Then there are a nonstandard model* $\mathfrak{A}$ *of* $\mathrm{Th}_{\Pi_{n+2}}(T) + \mathrm{Th}_{\Pi_n}(\mathcal{N})$ *and* $\varphi(x) \in \Delta^-_{n+1}(\mathfrak{A})$ *such that* $\omega = \{a \in \mathfrak{A} : \mathfrak{A} \vDash \varphi(a)\}$.

Theorem 4.3 (Lessan). *Let $\mathfrak{A} \models \mathrm{PA}$ and let $X \subseteq \omega$ nonrecursive. Then for every $k \geq 1$ there exists $\mathfrak{B} \models \mathrm{Th}_{\Pi_k}(\mathfrak{A}) + \mathrm{PA}$ such that $X \in \mathrm{SSy}(\mathfrak{B})$.*

Now we get a weak version of Theorem 3.7 for $\mathbf{I}\,\Delta_{n+1}$.

Proposition 4.4. *Let T be a consistent extension of* PA *such that* (i) *T is recursively axiomatized, or* (ii) *there exists $\Gamma \subseteq \Pi_k$ such that $T \iff \mathrm{PA} + \Gamma$. Then for all $n \in \omega$, $\mathrm{Th}_{\Pi_{n+2}}(T) \not\Longrightarrow \mathbf{I}\,\Delta_{n+1}$.*

Proof. First we assume (i). Let $\mathfrak{A} \models T$ be nonstandard. Since T is recursively axiomatized, T is coded in $\mathfrak{A}$. Then, by Theorem 4.1, there exists $\mathfrak{B} \models \mathrm{Th}_{\Pi_{n+2}}(T)$ such that $\mathfrak{B} \not\models \mathbf{I}\,\Delta_{n+1}$, as required.

Now we assume (ii). Let $\Gamma \subseteq \Pi_k$ be such that $T \iff \mathrm{PA} + \Gamma$ and let $\mathfrak{A} \models T$. By Theorem 4.3, there exists $\mathfrak{B} \models \mathrm{PA} + \mathrm{Th}_{\Pi_k}(\mathfrak{A})$ such that $T \in \mathrm{SSy}(\mathfrak{B})$. So, by Theorem 4.1, there exists $\mathfrak{C} \models \mathrm{Th}_{\Pi_{n+2}}(T)$ such that $\mathfrak{C} \not\models \mathbf{I}\,\Delta_{n+1}$, as required. □

Corollary 4.5. *For all $n \in \omega$,* (a) $\mathbf{I}\,\Delta_{n+1} \models\!\!\Longrightarrow \mathbf{I}\,\Sigma_n$, (b) $\mathrm{Th}_{\Pi_{n+2}}(\mathcal{N}) \not\Longrightarrow \mathbf{I}\,\Delta_{n+1}$.

Proof. Since $\mathbf{I}\,\Sigma_n$ is Π_{n+2}-axiomatizable, the assertion (a) follows from Proposition 5.4(i), and (b) is a consequence of Proposition 4.4(ii). □

Now we consider the theory $\mathbf{I}\,\Delta^-_{n+1}$. From Theorem 4.2 we get

Corollary 4.6.

(a) *Let T be an extension of* PA *consistent with $\mathrm{Th}_{\Pi_n}(\mathcal{N})$ and Σ_{n+1}-definable in $\mathcal{N}$. Then $\mathrm{Th}_{\Pi_{n+2}}(T) \not\Longrightarrow \mathbf{I}\,\Delta^-_{n+1}$.*

(b) $\mathrm{Th}_{\Pi_{n+2}}(\mathrm{PA}) \not\Longrightarrow \mathbf{I}\,\Delta^-_{n+1}$.

We now study the quantifier complexity of $\mathbf{I}\,\Delta^-_{n+1}$.

Theorem 4.7. *For all $n \in \omega$ we have:* (a) *$\mathbf{I}\,\Delta^-_{n+1}$ is Σ_{n+2}-axiomatized but is not Π_{n+2}-axiomatizable.* (b) *$\mathbf{I}\,\Delta^-_{n+1}$ is not finitely axiomatizable.*

Proof. It is clear that $\mathbf{I}\,\Delta^-_{n+1}$ is Σ_{n+2}-axiomatized and then from Corollary 4.6(b) we get (a). For (b) let us assume that $\mathbf{I}\,\Delta^-_{n+1}$ is finitely axiomatizable. Since $\mathrm{Th}_{\Pi_{n+1}}(\mathcal{N}) \Longrightarrow \mathbf{I}\,\Delta^-_{n+1}$, then there exists a recursively axiomatized, and so in $\mathcal{N}$ Σ_1-definable, extension T of PA such that $\mathrm{Th}_{\Pi_{n+2}}(T) \Longrightarrow \mathbf{I}\,\Delta^-_{n+1}$. This contradicts Corollary 4.6(a), which proves (b). □

Corollary 4.8. (a) $\mathbf{I}\,\Delta^-_1 \models\!\!\Longrightarrow \mathbf{I}\,\Delta_0$. (b) *For all $n > 0$, $\mathbf{I}\,\Sigma_n \not\Longleftrightarrow \mathbf{I}\,\Delta^-_{n+1}$.*

Proof. Assertion (a) follows from Corollary 4.6(a). Since $\mathbf{I}\,\Sigma_n$ is Π_{n+2}-axiomatizable, by Corollary 4.6(b), $\mathbf{I}\,\Sigma_n \not\Longrightarrow \mathbf{I}\,\Delta^-_{n+1}$. Since $\mathbf{L}\,\Delta^-_{n+1} \Longrightarrow \mathbf{I}\,\Delta^-_{n+1}$, by Corollary 3.4(b) we get that $\mathbf{I}\,\Delta^-_{n+1} \not\Longrightarrow \mathbf{I}\,\Sigma_n$. This proves (b). □

In [5] it is proved that $\mathbf{B}\,\Sigma^-_{n+1}$ is neither Σ_{n+2}- nor Π_{n+2}-axiomatizable. By Theorem 1.5, this is also true for $\mathbf{UL}\,\Delta_{n+1}$. Now we prove that $\mathbf{UI}\,\Delta_{n+1}$ also satisfies this property.

Corollary 4.9. *$\mathbf{UI}\,\Delta_{n+1}$ is not Π_{n+2}-axiomatizable, and for $n > 0$, $\mathbf{UI}\,\Delta_{n+1}$ is not Σ_{n+2}-axiomatizable.*

Proof. Since $\mathbf{UI}\,\Delta_{n+1} \Longrightarrow \mathbf{I}\,\Delta^-_{n+1}$, from Corollary 4.6(b) it follows that $\mathbf{UI}\,\Delta_{n+1}$ is not Π_{n+2}-axiomatizable. Let us suppose that $n > 0$.

Claim 4.9.1. *Let T be a Σ_{n+2}-axiomatizable theory such that $\mathcal{N} \models T$. Then $T \not\Longrightarrow \mathbf{I}\,\Sigma_n$.*

Proof. Let $\mathfrak{A} \models \mathrm{Th}(\mathcal{N})$ and let $a \in \mathfrak{A}$ nonstandard. Then $\mathcal{K}_n(\mathfrak{A}, a) \prec_n \mathfrak{A}$ and $\mathcal{N} \prec \mathfrak{A}$. So, $\mathcal{N} \prec_{n+1} \mathcal{K}_n(\mathfrak{A}, a)$. Hence, $\mathcal{K}_n(\mathfrak{A}, a) \models \mathrm{Th}_{\Sigma_{n+2}}(\mathcal{N})$ and so, $\mathcal{K}_n(\mathfrak{A}, a) \models T$. On the other hand, $\mathcal{K}_n(\mathfrak{A}, a) \not\models \mathbf{I}\,\Sigma_n$, as required. □ Claim

Since $\mathbf{UI}\,\Delta_{n+1} \Longrightarrow \mathbf{I}\,\Sigma_n$, by the claim it follows that $\mathbf{UI}\,\Delta_{n+1}$ is not Σ_{n+2}-axiomatizable. □

5 Remarks and open questions

In [5] the following question appears as an open problem:

Problem 5.1. $\mathbf{B}_{\mathrm{s}}\,\Sigma^-_{n+1} \Longleftrightarrow \mathbf{B}\,\Sigma^-_{n+1}$.

The results proved in Theorem 3.2 and Theorem 3.7 seems to suggest that $\mathbf{UI}\,\Delta_{n+1} + \exp \not\Longrightarrow \mathbf{L}\,\Delta^-_{n+1}$.

Problem 5.2. Can we obtain this property from the above refered results?

In Theorem 3.2 we have proved a weak version of Theorem 3.8 for $n = 0$. The following question ask if the exponential function can be eliminated in Theorem 3.2.

Problem 5.3. Is $\mathrm{Th}_{\Pi_1}(\mathcal{N})$ (up to equivalence) the unique Π_1-axiomatized theory that is an extension of $\mathbf{L}\,\Delta^-_1$? In [3] it is proved that this is true for $\mathbf{I}\,\Pi^-_1$ (see Claim 3.3.1).

Since $\mathbf{I}\,\Delta^-_{n+1}$ is Σ_{n+2}-axiomatized, from Corollary 4.9 we have that for $n > 0$, $\mathbf{UI}\,\Delta_{n+1} \not\Longrightarrow \mathbf{I}\,\Delta^-_{n+1}$. Let us consider the following question:

Problem 5.4. (a) $\mathbf{UI}\,\Delta_1 \not\Longrightarrow \mathbf{I}\,\Delta^-_1$? (b) Is $\mathbf{UI}\,\Delta_1$ a Σ_2-axiomatizable theory?

In this paper we have studied properties for $\mathbf{L}\,\Delta^-_{n+1}$ and $\mathbf{I}\,\Delta^-_{n+1}$. But in some cases we have only proved that $\mathbf{I}\,\Delta^-_{n+1}$ satisfies a weak version of the property that satisfies $\mathbf{L}\,\Delta^-_{n+1}$. Let us consider the following question.

Problem 5.5. Are the Theorems 3.2 and 3.7 true for $\mathbf{I}\,\Delta^-_{n+1}$?

It is easy to see that $\mathcal{K}_0(\mathfrak{A})$ is a substructure of $\mathfrak{A}$, and if $\mathcal{K}_0(\mathfrak{A})$ is nonstandard, then $\mathcal{K}_0(\mathfrak{A}) \not\models \mathbf{I}\,\mathbf{E}_0$, where $\mathbf{E}_0$ is the class of open formulas of $\mathcal{L}$.

Let us consider the following question.

Problem 5.6. $\mathcal{K}_0(\mathfrak{A}) \models \mathrm{P}^-$? In other words, $\mathcal{K}_0(\mathfrak{A}) \models \forall x (\forall y \leq x) \exists z\, [x = y + z]$?

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