Provably Total Primitive Recursive Functions: Theories with Induction

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Abstract. A natural example of a function algebra is $\mathcal{R}(\mathbf{T})$, the class of provably total computable functions (p.t.c.f.) of a theory $\mathbf{T}$ in the language of first order Arithmetic. In this paper a simple characterization of that kind of function algebras is obtained. This provides a useful tool for studying the class of primitive recursive functions in $\mathcal{R}(\mathbf{T})$. We prove that this is the class of p.t.c.f. of the theory axiomatized by the induction scheme restricted to (parameter free) $\Delta_1(\mathbf{T})$–formulas (i.e. Σ_1–formulas which are equivalent in $\mathbf{T}$ to Π_1–formulas).

Moreover, if $\mathbf{T}$ is a sound theory and proves that exponentiation is a total function, we characterize the class of primitive recursive functions in $\mathcal{R}(\mathbf{T})$ as a function algebra described in terms of bounded recursion (and composition). Extensions of this result are related to open problems on complexity classes. We also discuss an application to the problem on the equivalence between (parameter free) Σ_1–collection and (uniform) Δ_1–induction schemes in Arithmetic.

The proofs lean upon axiomatization and conservativeness properties of the scheme of $\Delta_1(\mathbf{T})$–induction and its parameter free version.

1 Introduction

A function algebra is a family of functions that can be described as the smallest class of functions that contains some initial functions and is closed under certain operators. Classical examples of function algebras include the class of primitive recursive functions, $\mathcal{PR}$, classes $\mathcal{E}^n$, $(n \geq 1)$, in the Grzegorczyk hierarchy and the class of Kalmár elementary functions, $\mathcal{E}$ (see [6, 13]). Another important example is given by $\mathcal{R}(\mathbf{T})$, the class of provably total computable functions (p.t.c.f.) of a theory $\mathbf{T}$ in the language of first order Arithmetic. The class $\mathcal{R}(\mathbf{T})$ can be used to obtain independence results for $\mathbf{T}$ and to separate it from other theories. On the other hand, if a function algebra, $\mathcal{C}$, is the class of p.t.c.f. of a theory, $\mathbf{T}$, then proof–theoretic and model–theoretic properties of $\mathbf{T}$ can be used to establish results on $\mathcal{C}$. This increases the methods available in the study of function algebras by adding to them techniques from Proof Theory and Model Theory. As surveyed in [6], function algebras provide machine–independent characterizations of many complexity classes and offer an alternative view of important open problems in Complexity Theory. In this way, classes of p.t.c.f. constitute a link among Complexity Theory, Proof Theory and Model Theory that has been exploited in the work on Bounded Arithmetic (see [12]).

In this paper we present a new example of the fruitful interactions among fragments of Arithmetic, function algebras and computational complexity. Given a function algebra, $\mathcal{C}$, we introduce the algebra $\mathcal{E}^{\mathcal{C}}$ defined as the smallest class containing the basic functions (zero, successor and projections) and closed under composition and $\mathcal{C}$–bounded recursion. We study the relationship between $\mathcal{C}$ and $\mathcal{E}^{\mathcal{C}}$ when $\mathcal{C}$ is the class of p.t.c.f. of a theory $\mathbf{T}$. If $\mathcal{C} = \mathcal{R}(\mathbf{T})$ then

– (Theorem 4) $\mathcal{C} \cap \mathcal{PR} \subseteq \mathcal{E}^{\mathcal{C}}$. Moreover, if $\mathcal{C}$ is closed under bounded minimization, $\mathcal{E}^{\mathcal{C}}$ is the closure of $\mathcal{C} \cap \mathcal{PR}$ under composition and bounded recursion.
– (Theorem 5) Assume that $\mathcal{C}$ is closed under bounded minimization. Then $\mathcal{C} \cap \mathcal{PR} = \mathcal{E}^{\mathcal{C}}$ if and only if there exists a theory $\mathbf{T}'$ such that $\mathcal{E}^{\mathcal{C}} = \mathcal{R}(\mathbf{T}')$.

For the proof of these results the concept of a Δ_0–generated function algebra is introduced. A function algebra, $\mathcal{C}$, is Δ_0–generated if (it contains Grzegorcyk's class $\mathcal{M}^2$ and) each function in $\mathcal{C}$ can be obtained as a composition of two functions in $\mathcal{C}$ with Δ_0-definable graph. We prove (see Theorem 1) that a function algebra is Δ_0–generated if and only if it is the class $\mathcal{R}(\mathbf{T})$ for some theory $\mathbf{T}$ (extending $\mathbf{I}\Delta_0$).

If $\mathcal{C} \subseteq \mathcal{PR}$ is closed under bounded minimization, then Theorem 5 states that $\mathcal{C} = \mathcal{E}^{\mathcal{C}}$ if and only if $\mathcal{E}^{\mathcal{C}}$ is Δ_0–generated. This fact has interesting applications to complexity classes as $\mathcal{F}\text{PH}$ (computable functions in the Polynomial Time Hierarchy, that is, $\bigcup_{i=1}^{\infty} \square_i^p$ in S. Buss' terminology, see [10]) and $\mathcal{F}\text{LTH}$ (computable functions in the Linear Time Hierarchy, see [6]). Both classes are contained in $\mathcal{PR}$ and are Δ_0–generated and closed under bounded minimization:

– $\mathcal{F}\text{LTH} = \mathcal{M}^2 = \mathcal{R}(\mathbf{I}\Delta_0)$ (see [6, 16]), and
– $\mathcal{F}\text{PH} = \mathcal{R}(\mathbf{I}\Delta_0 + \Omega_1)$ (see [10]), where $\mathbf{I}\Delta_0 + \Omega_1$ is the theory introduced by A. Wilkie and J. Paris in [17].

But $\mathcal{E}^{\mathcal{F}\text{LTH}} = \mathcal{E}^2 = \mathcal{F}\text{LINSPACE}$ (R.W. Ritchie, see [6]) and $\mathcal{E}^{\mathcal{F}\text{PH}} = \mathcal{F}\text{PSPACE}$ (D.B. Thompson, see [6]). Therefore, by Theorem 5 it follows that:

1. If $\mathcal{F}\text{PSPACE}$ is Δ_0–generated then $\mathcal{F}\text{PSPACE} = \mathcal{F}\text{PH}$.
2. If $\mathcal{F}\text{LINSPACE}$ is Δ_0–generated then $\mathcal{F}\text{LINSPACE} = \mathcal{F}\text{LTH}$. Or, equivalently, $\mathcal{E}^2 = \mathcal{M}^2$ if and only if $\mathcal{E}^2$ is Δ_0–generated.

These facts suggest that a deeper knowledge of structural properties of Δ_0–generated function algebras (specially, construction of non Δ_0–generated function algebras) could be relevant in the study of complexity classes. They also raise a natural question: if $\mathcal{C} = \mathcal{R}(\mathbf{T})$ and $\mathcal{E}^{\mathcal{C}}$ is Δ_0–generated, is there a natural theory $\mathbf{T}'$ such that $\mathcal{R}(\mathbf{T}') = \mathcal{E}^{\mathcal{C}}$? We obtain an answer to this question from the study of induction schemes for Δ_1–formulas. Let $\mathbf{I}\Delta_1(\mathbf{T})^-$ be the theory axiomatized by induction scheme restricted to parameter free $\Delta_1(\mathbf{T})$–formulas.

– (Theorem 2) $\mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T})^-) = \mathcal{C} \cap \mathcal{PR}$.

So, from Theorem 5 we get that, if $\mathcal{C}$ is closed under bounded minimization and $\mathcal{E}^{\mathcal{C}}$ is Δ_0–generated, then $\mathcal{E}^{\mathcal{C}} = \mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T})^-)$.

Next step is to find conditions ensuring $\mathcal{E}^{\mathcal{C}}$ is Δ_0–generated. Classes $\mathcal{E}^{\mathcal{C}}$ are a generalization of Grzegorcyzk's classes $\mathcal{E}^n$ and it is well–known that if exponential function is in $\mathcal{E}^n$, then bounded recursion can be reduced to bounded minimization (see [6, 13]). But bounded minimization has a straightforward formulation in the language of first order Arithmetic and as a consequence (for $n \geq 3$) $\mathcal{E}^n$ is Δ_0–generated. The key ingredients in the proof of this fact are exponential function (which allows for coding of sequences of arbitrary length) and Σ_1–collection principle (as a suitable formulation of the combinatorial principles involved). These arguments lead to a natural condition for $\mathcal{E}^{\mathcal{C}}$ to be a Δ_0–generated function algebra and relate the study of $\mathcal{E}^{\mathcal{C}}$ to the problem on the equivalence between the schemes of Σ_1–collection and Δ_1–induction in Arithmetic (see [7]). In [3, 5], L. Beklemishev obtains Π_2–axiomatized theories that are not closed under Σ_1–collection rule or Δ_1–induction rule. He proposes classes of p.t.c.f. as a tool to separate the fragments $\mathbf{I}\Delta_1$ and $\mathbf{B}\Sigma_1$. Recently (see [15]) T. Slaman has proved that $\mathbf{I}\Delta_1 + \mathbf{exp}$ is equivalent to $\mathbf{B}\Sigma_1 + \mathbf{exp}$ (where $\mathbf{exp}$ is a Π_2–sentence expressing that exponentiation defines a total function with Δ_0 definable graph (see [10])). So, Beklemishev's approach must fail. Nevertheless, as we shall show, classes of p.t.c.f. could be used to obtain positive results on fragments of Arithmetic. Motivated by Beklemishev's work in [3, 4, 5], we study the classes of p.t.c.f. of the theories $\mathbf{I}\Delta_1(\mathbf{T})$ and $\mathbf{L}\Delta_1(\mathbf{T})$ introduced in [9], and their relationship with the uniform counterpart of Slaman's result.

Theorem 5 holds for $\mathcal{C}$ closed under bounded minimization. We prove that if Theorem 5 also holds under the (apparently weaker) following hypothesis:

(**IC**) $\quad \mathcal{C} = \mathcal{R}(\mathbf{T})$ and $\mathbf{T}$ extends $\mathbf{I}\Delta_1(\mathbf{T})$,

then a (weak) uniform counterpart of Slaman's result can be obtained, namely, theories $\mathbf{B}\Sigma_1^- + \mathbf{exp}$ and $\mathbf{UI}\Delta_1 + \mathbf{exp}$ are equivalent, modulo Π_1–true sentences (see Theorem 7). Last equivalence can be also obtained from Slaman's theorem and Σ_3–conservativeness between $\mathbf{I}\Delta_1 + \mathbf{exp}$ and $\mathbf{UI}\Delta_1 + \mathbf{exp}$ (see corollary 6 in [5]). However, we present an independent approach stressing the role of function algebras via classes of p.t.c.f.

Our main tools for the proofs are axiomatization and conservativeness results for $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and Herbrand analyses, essentially along the lines presented by W. Sieg in [14]; however, we work in a model–theoretic framework, following J. Avigad's work in [1].

2 Fragments of Arithmetic and Function Algebras

Through this paper we deal with classes of p.t.c.f. of a number of theories. We are mainly interested in characterizations of these classes as function algebras. So, first of all, we introduce the theories and classes of functions we are concerned with. These theories are axiomatized by axiom schemes expressing classical principles in Arithmetic as induction, minimization and collection.

Let $\mathcal{L} = \{0, 1, <, +, \cdot\}$ be the language of first order Arithmetic. The induction and minimization axioms for a formula $\varphi(x, \vec{v})$ with respect to x are, respectively,

$$\mathbf{I}_{\varphi,x}(\vec{v}) \equiv \varphi(0, \vec{v}) \wedge \forall x\, [\varphi(x, \vec{v}) \rightarrow \varphi(x+1, \vec{v})] \rightarrow \forall x\, \varphi(x, \vec{v}),$$
$$\mathbf{L}_{\varphi,x}(\vec{v}) \equiv \exists x\, \varphi(x, \vec{v}) \rightarrow \exists x\, (\varphi(x, \vec{v}) \wedge \forall z < x\, \neg\varphi(z, \vec{v})).$$

The collection axiom for a formula $\varphi(x, y, \vec{v})$ with respect to x, y is

$$\mathbf{B}_{\varphi,x,y}(z, \vec{v}) \equiv \forall x \leq z\, \exists y\, \varphi(x, y, \vec{v}) \rightarrow \exists u\, \forall x \leq z\, \exists y \leq u\, \varphi(x, y, \vec{v}).$$

As usual, we write $\mathbf{I}_\varphi$ instead of $\mathbf{I}_{\varphi,x}$ and similarly we use $\mathbf{L}_\varphi$ and $\mathbf{B}_\varphi$.

All theories considered in this paper are extensions of $\mathbf{P}^-$ a finite set of Π_1 formulas whose models are the nonnegative part of a discretely ordered commutative ring (see [11]). Other theories are defined by restricting the schemes just introduced to formulas in the classes Σ_n or Π_n in the Arithmetical Hierarchy. If Γ is a class of formulas of $\mathcal{L}$, then $\mathbf{I}\Gamma = \mathbf{P}^- + \{\mathbf{I}_\varphi :\ \varphi \in \Gamma\}$. The theory $\mathbf{L}\Gamma$ is similarly defined using $\mathbf{L}_\varphi$ instead of $\mathbf{I}_\varphi$. For collection, $\mathbf{B}\Gamma = \mathbf{I}\Delta_0 + \{\mathbf{B}_\varphi :\ \varphi \in \Gamma\}$, where Δ_0 denotes the class of bounded formulas of $\mathcal{L}$ (see [10, 11]).

Induction schemes for Δ_{n+1}–formulas will be also considered, $\mathbf{I}\Delta_{n+1}$ is the theory given by:

$$\mathbf{P}^- + \{\forall x\, (\varphi(x, \vec{v}) \leftrightarrow \psi(x, \vec{v})) \rightarrow \mathbf{I}_\varphi(\vec{v}) :\ \varphi(x, \vec{v}) \in \Sigma_{n+1},\ \psi(x, \vec{v}) \in \Pi_{n+1}\}.$$

If parameters, $\vec{v}$, are not allowed, then we obtain the theory $\mathbf{I}\Delta^-_{n+1}$. The uniform version of induction scheme, $\mathbf{UI}\Delta_{n+1}$, was introduced by R. Kaye. It is defined by considering the scheme $\forall \vec{v}\, \forall x\, (\varphi(x, \vec{v}) \leftrightarrow \psi(x, \vec{v})) \rightarrow \forall \vec{v}\, \mathbf{I}_\varphi(\vec{v})$. This theory is also studied by Beklemishev in [5], where it is denoted by $sI\Delta_1$.

Definition. Let $\mathbf{T}$ be a theory in the language $\mathcal{L}$. We say that $f : \omega^k \rightarrow \omega$ is a provably total computable function of $\mathbf{T}$ if there exists a formula $\varphi(\vec{x}, y) \in \Sigma_1$ such that

1. $\mathbf{T} \vdash \forall \vec{x}\, \exists! y\, \varphi(\vec{x}, y)$.
2. For all $a_1, \ldots, a_k, b \in \omega, \quad f(\vec{a}) = b \Longleftrightarrow \mathcal{N} \models \varphi(\vec{a}, b)$.

Where $\mathcal{N}$ denotes the standard model of Arithmetic whose universe is the set of natural numbers, ω. In such a case, we say that $\varphi(\vec{x}, y)$ defines f in $\mathbf{T}$.

This definition is sensitive to changes in the language of the theory. If $\mathbf{T}$ is a theory in a language $\mathcal{L}'$ extending $\mathcal{L}$, then $\mathcal{R}(\mathbf{T})$ will denote the class obtained by considering $\Sigma_1(\mathcal{L}')$–formulas instead of Σ_1–formulas.

The class $\mathcal{R}(\mathbf{T})$ has turned out to be a natural object, its closure properties (under certain operators) reflecting axiom schemes (or inference rules) provable in $\mathbf{T}$. Thus, closure under primitive recursion corresponds to Σ_1–induction and bounded minimization to Σ_1–collection (see [2]). In particular, $\mathcal{R}(\mathbf{I}\Sigma_1) = \mathcal{PR}$ and $\mathcal{R}(\mathbf{I}\Delta_0 + \mathbf{exp}) = \mathcal{E}$.

It is easy to check that if $\Phi \subseteq \mathbf{Th}_{\Pi_1}(\mathcal{N})$ and $\mathbf{T}$ is a sound theory (that is, $\mathcal{N} \models \mathbf{T}$) then $\mathcal{R}(\mathbf{T}) = \mathcal{R}(\mathbf{T} + \Phi)$ (see [14]). The class $\mathcal{R}(\mathbf{T})$ is determined by $\mathbf{Th}_{\Pi_2}(\mathbf{T})$ (the set of Π_2–sentences provable in $\mathbf{T}$). The converse also holds modulo Π_1–true sentences.

Proposition 1. *Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be Π_2–axiomatized sound extensions of $\mathbf{I}\Delta_0$. The following conditions are equivalent:*

1. $\mathcal{R}(\mathbf{T}_1) = \mathcal{R}(\mathbf{T}_2)$.
2. $\mathbf{T}_1 + \mathbf{Th}_{\Pi_1}(\mathcal{N}) \Longleftrightarrow \mathbf{T}_2 + \mathbf{Th}_{\Pi_1}(\mathcal{N})$.

Proof. We only prove $(\mathbf{1}) \Longrightarrow (\mathbf{2})$. By symmetry, it is enough to show that $\mathbf{T}_1 + \mathbf{Th}_{\Pi_1}(\mathcal{N}) \Longrightarrow \mathbf{T}_2$. Let $\theta(x, y)$ be a Δ_0–formula such that $\mathbf{T}_2 \vdash \forall x\, \exists y\, \theta(x, y)$. Let $\theta'(x, y)$ be the formula $\theta(x, y) \wedge \forall z < y\, \neg\theta(x, z)$. Since $\mathbf{T}_2 \Longrightarrow \mathbf{I}\Delta_0$, $\mathbf{T}_2 \vdash \forall x\, \exists! y\, \theta'(x, y)$. Let f be the computable function defined by θ' in $\mathcal{N}$. Then, $f \in \mathcal{R}(\mathbf{T}_2)$; so, by $(\mathbf{1})$, $f \in \mathcal{R}(\mathbf{T}_1)$. Hence, there is $\varphi(x, y) \in \Sigma_1$ defining f in $\mathbf{T}_1$. Thus, $\mathcal{N} \models \varphi(x, y) \leftrightarrow \theta'(x, y)$. In particular, $\mathcal{N} \models \forall x, y\, (\varphi(x, y) \rightarrow \theta'(x, y))$; so, since this last formula is a Π_1 sentence, $\mathbf{T}_1 + \mathbf{Th}_{\Pi_1}(\mathcal{N}) \vdash \forall x\, \exists y\, \theta(x, y)$. □

Functions with a Δ_0–definable graph will play a prominent role throughout this work. Let us introduce the following notation.

We denote by Δ_0^0 the class of sets Δ_0 definable in the standard model. The graph of a function f is denoted by $Gr(f) = \{(\vec{a}, b) \in \omega^{k+1} :\ f(\vec{a}) = b\}$. If $\mathcal{C}$ is a class of functions, then $\mathcal{C}_*$ denotes the class of subsets of ω^k whose characteristic functions are in $\mathcal{C}$. Finally, given $f, g : \omega^k \to \omega$, we write $f \leq g$ to mean that for each $\vec{a} \in \omega^k$, $f(\vec{a}) \leq g(\vec{a})$.

One of the aims of this work is to obtain descriptions of $\mathcal{R}(\mathbf{T})$ as a function algebra generated by means of some operators from a small set of basic functions. The considered classes of basic functions will always contain the set

$$\mathcal{B} = \{S,\, O\} \cup \{\Pi_i^n :\ 1 \leq i \leq n\}$$

where S, $O : \omega \to \omega$ are given by $S(a) = a + 1$ and $O(a) = 0$, and $\Pi_i^n : \omega^n \to \omega$, by $\Pi_i^n(a_1, \ldots, a_n) = a_i$. As operators, beside *composition*, we consider:

Bounded minimization, $\mu_{\leq}$: If $g : \omega^{m+1} \to \omega$, then $f = \mu_{\leq}(g)$ is the function $f : \omega^{m+1} \to \omega$ defined by

$$f(a_1, \ldots, a_m, b) = \begin{cases} \min(\{z :\ g(\vec{a}, z) = 0\}), & \text{if } \exists z \leq b\, (g(\vec{a}, z) = 0); \\ 0, & \text{otherwise.} \end{cases}$$

Bounded recursion, **BR**: A function $f : \omega^{n+1} \to \omega$ is defined from $g : \omega^n \to \omega$, $h : \omega^{n+2} \to \omega$ and $C : \omega^{n+1} \to \omega$ by bounded recursion, if $f \leq C$ and

$$f(\vec{x}, 0) = g(\vec{x}); \quad f(\vec{x}, y + 1) = h(\vec{x}, y, f(\vec{x}, y)).$$

In this case we write, $f = \mathbf{BR}_C(g, h)$ and we shall say that f is defined by *C–bounded recursion* from g and h.

Let $\mathcal{F}$ be a class containing $\mathcal{B}$. In this paper, $\mathbf{C}(\mathcal{F})$ will denote the closure of $\mathcal{F}$ under composition and $\mathbf{E}(\mathcal{F})$ the closure of $\mathcal{F}$ under composition and bounded recursion. We also consider the following slight (but crucial) modification of closure under bounded recursion.

Definition. $\mathcal{E}^{\mathcal{F}}$ is the smallest class of functions containing $\mathcal{B}$ and closed under composition and $\mathcal{F}$–bounded recursion; that is, closed under C–bounded recursion for every $C \in \mathcal{F}$.

Let us observe that $\mathcal{E}^{\mathcal{F}} \subseteq \mathcal{PR}$ and if $\mathcal{F} \subseteq \mathcal{E}^{\mathcal{F}}$, then $\mathbf{C}(\mathcal{F}) \subseteq \mathcal{E}^{\mathcal{F}} \subseteq \mathbf{E}(\mathcal{F})$.

Grzegorczyk's classes, $\mathcal{E}^n$, can be described in the form $\mathcal{E}^{\mathcal{F}}$. For instance, let $\mathcal{P}_0$, $\mathcal{P}_1$ and $\mathcal{P}_2$ be, respectively, the classes of functions $\mathbf{C}(\mathcal{B})$, $\mathbf{C}(\mathcal{B} \cup \{+\})$ and $\mathbf{C}(\mathcal{B} \cup \{+, \times\})$, then, for $j = 0, 1, 2$, it holds that $\mathcal{E}^{\mathcal{P}_j} = \mathcal{E}^j$ (see [13]).

The basic function algebra in this paper will be Grzegorczyk's class $\mathcal{M}^2$: the closure of $\mathcal{B} \cup \{+, \times\}$ under composition and $\mu_{\leq}$ (see [6, 13]). As we shall see (Proposition 3), $\mathcal{M}^2$ is the class $\mathcal{R}(\mathbf{I}\Delta_0)$ and, therefore, all function algebras considered in this paper contain it. This motivates the following definition.

Definition. An F–algebra is a family, $\mathcal{C}$, of computable functions containing $\mathcal{B}$ and closed under composition. We shall say that $\mathcal{C}$ is rudimentary if $\mathcal{M}^2 \subseteq \mathcal{C}$.

A pairing function is available in $\mathcal{M}^2$. Let $J : \omega^2 \to \omega$ be Cantor's function:

$$J(a, b) = \frac{(a + b)(a + b + 1)}{2} + a.$$

Its lateral inverses K, L are given by $K(a) = (\mu z)_{\leq a}(\exists y \leq a\,(J(z, y) = a))$ and $L(a) = (\mu z)_{\leq a}(\exists x \leq a\,(J(x, z) = a))$. Then $J, K, L \in \mathcal{M}^2$. We shall write $\langle x, y\rangle = J(x, y)$ and $(z)_0 = K(z)$, $(z)_1 = L(z)$.

Basic properties of $\mathcal{M}^2$ are summed up in next proposition (see [6] or [13]).

Proposition 2. *1.* $\Delta^0_0 = \mathcal{M}^2_*$.
2. For each $f : \omega^k \to \omega$, the following conditions are equivalent:
(a) $f \in \mathcal{M}^2$.
(b) $Gr(f) \in \Delta^0_0$ and there exists a term t of $\mathcal{L}$ such that $f \leq t$.

As a consequence a characterization of $\mathcal{R}(\mathbf{I}\Delta_0)$ can be obtained. A proof–theoretic proof of this result was obtained by G. Takeuti (see [16]).

Proposition 3. $\mathcal{M}^2 = \mathcal{R}(\mathbf{I}\Delta_0)$.

Hence, for every extension, $\mathbf{T}$, of $\mathbf{I}\Delta_0$ the class $\mathcal{R}(\mathbf{T})$ is a rudimentary F–algebra. Now we introduce a necessary and sufficient condition under which a rudimentary F–algebra is the class of p.t.c.f. of some theory. The following results seem to be folklore and have appeared more or less explicitly in the literature (see proposition 4.1 in [2] and previous remarks in that paper). However, Theorem 1 below does not seem to be known. As it was remarked in the Introduction, it provides interesting insights on open problems in Complexity Theory.

Lemma 1. *Let $\mathcal{C}$ be a rudimentary F–algebra and $f : \omega^k \to \omega$ such that $Gr(f) \in \Delta^0_0$. If there exists $g \in \mathcal{C}$ such that $f \leq g$, then $f \in \mathcal{C}$.*

Proof. Let $h : \omega^{k+1} \to \omega$ given by $h(\vec{a}, b) = (\mu z)_{\leq b}[f(\vec{a}) = z]$. Since $Gr(f) \in \Delta^0_0$, then $Gr(h) \in \Delta^0_0$. By Proposition 2–(2), $h \in \mathcal{M}^2 \subseteq \mathcal{C}$. Let $g \in \mathcal{C}$ such that $f \leq g$. Then $f(\vec{a}) = h(\vec{a}, g(\vec{a}))$. Since $\mathcal{C}$ is closed under composition, $f \in \mathcal{C}$. □

Lemma 2. *Let $\mathbf{T}$ be an extension of $\mathbf{I}\Delta_0$. Then for each $f \in \mathcal{R}(\mathbf{T})$ there exists $g \in \mathcal{R}(\mathbf{T})$ such that $Gr(g) \in \Delta^0_0$ and $f = K \circ g$.*

Proof. For each $f \in \mathcal{R}(\mathbf{T})$ and $\varphi(\vec{x}, y, z) \in \Delta_0$ such that $\exists z\, \varphi(\vec{x}, y, z)$ defines f in $\mathbf{T}$, let $\psi(\vec{x}, v) \in \Delta_0$ be the formula

$$\exists y, z \leq v\, [\langle y, z\rangle = v \wedge \varphi(\vec{x}, y, z) \wedge \forall z' < z\, \neg\varphi(\vec{x}, y, z')].$$

Then $\mathbf{T} \vdash \forall \vec{x}\, \exists! y\, \psi(\vec{x}, y)$. Let g be the function defined in $\mathcal{N}$ by $\psi(\vec{x}, v)$. Then $g \in \mathcal{R}(\mathbf{T})$, $Gr(g) \in \Delta_0^0$ and $f(\vec{a}) = K(g(\vec{a}))$. □

The above lemma motivates the following definition.

Definition. Let $\mathcal{C}$ be a rudimentary F–algebra. We say that $\mathcal{C}$ is a Δ_0–generated F–algebra if for each $f \in \mathcal{C}$ there exist $g_1, g_2 \in \mathcal{C}_0$ such that $f = g_1 \circ g_2$.

The following result shows that Δ_0–generated F–algebras correspond to classes of p.t.c.f. of extensions of $\mathbf{I}\Delta_0$.

Theorem 1. *The following conditions are equivalent:*

1. *$\mathcal{C}$ is a Δ_0–generated F–algebra.*
2. *There exists a (sound) $\mathcal{L}$–theory, $\mathbf{T}$, extending $\mathbf{I}\Delta_0$ such that $\mathcal{R}(\mathbf{T}) = \mathcal{C}$.*

Proof. **(2)⟹(1)**: It follows from Proposition 3 and Lemma 2.
(1)⟹(2): For each $f \in \mathcal{C}_0 = \{h \in \mathcal{C} :\ Gr(h) \in \Delta_0^0\}$, let $\theta_f(x, y)$ be a Δ_0–formula defining f in $\mathcal{N}$. Let $\Gamma = \{\forall x\, \exists y\, \theta_f(x, y) \ :\ f \in \mathcal{C}_0,\ f : \omega \to \omega\}$. Next claim is a slight generalization of proposition 4.2 in [2] and it can also be proved along the lines sketched there.

Claim: $\mathcal{R}(\mathbf{I}\Delta_0 + \Gamma) = \mathbf{C}(\mathcal{M}^2 \cup \mathcal{C}_0)$.

Thus, $\mathcal{R}(\mathbf{I}\Delta_0 + \Gamma) = \mathbf{C}(\mathcal{M}^2 \cup \mathcal{C}_0) = \mathcal{C}$, last equality since $\mathcal{C}$ is Δ_0– generated. □

3 Axiomatizing $\boldsymbol{\Delta_{n+1}}$(T)–Induction

The aim of this section is to characterize the class of primitive recursive functions in $\mathcal{R}(\mathbf{T})$, where $\mathbf{T}$ is an extension of $\mathbf{I}\Delta_0$, as the class of p.t.c.f. of a suitable theory. To this end, we consider the class of $\Delta_{n+1}(\mathbf{T})$–formulas:

$$\Delta_{n+1}(\mathbf{T}) = \{\varphi(x, \vec{v}) \in \Sigma_{n+1} :\ \text{there exists } \psi(x, \vec{v}) \in \Pi_{n+1},\ \mathbf{T} \vdash \varphi \leftrightarrow \psi\}.$$

When the schemes of induction and minimization are restricted to these classes of formulas we obtain the theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{L}\Delta_{n+1}(\mathbf{T})$ introduced in [9]. There the following version of the collection scheme is also considered

$$\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) = \mathbf{I}\Delta_0 + \{\mathbf{B}_{\varphi,x,y}(z, \vec{v}) :\ \varphi \in \Pi_n,\ \exists y\, \varphi(x, y, \vec{v}) \in \Delta_{n+1}(\mathbf{T})\}.$$

Let us state here some basic properties of these theories, for details and proofs see [9]. If $\varphi \in \Sigma_{n+1}$ and $\psi \in \Pi_{n+1}$ then $\varphi \leftrightarrow \psi$ is a Π_{n+2}–formula. Therefore,

– If $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}')$ then $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T}')$.

A similar result holds for minimization and collection. The following basic properties will be used without explicit mention.

– $\mathbf{L}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{I}\Sigma_n$.
– If $\mathbf{T}$ is an extension of $\mathbf{I}\Sigma_n$ then $\mathbf{L}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$.

As noticed in [9], last property follows by an argument that mimics the proof of Gandy's Theorem, $\mathbf{L}\Delta_1 \Longrightarrow \mathbf{B}\Sigma_1$, given in [10], lemma I.2.17. A variation of that argument, considering also lemma I.2.16 in [10], gives us that

Lemma 3. $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) + \mathbf{L}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) + \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$.

The following notion, introduced in [9], has turned out to be useful for the study of $\Delta_{n+1}(\mathbf{T})$–induction.

Definition. We say that $\mathbf{T}$ has Δ_{n+1}–induction if $\mathbf{T} \Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$.

Theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{L}\Delta_{n+1}(\mathbf{T})$ are Π_{n+3}–axiomatizable. But adding to them $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$, their quantifier complexity is reduced to Π_{n+2}.

Lemma 4. $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) + \mathbf{I}\Delta_{n+1}(\mathbf{T})$ *and* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) + \mathbf{L}\Delta_{n+1}(\mathbf{T})$ *are* Π_{n+2}*–axiomatizable.*

In this section we shall obtain a useful axiomatization of $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ in terms of $\mathbf{I}\Sigma_{n+1}$ and $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$. To this end we introduce the disjunction of two theories, which corresponds to intersection between classes of p.t.c.f.

If $\mathbf{T}_1$ and $\mathbf{T}_2$ are theories in the language $\mathcal{L}$, then $\mathbf{T}_1 \vee \mathbf{T}_2$ denotes the theory axiomatized by the set of formulas $\{\varphi_1 \vee \varphi_2 : \ \varphi_1 \in \mathbf{T}_1 \text{ and } \varphi_2 \in \mathbf{T}_2\}$.

Lemma 5. *If* $\mathbf{T}_1, \mathbf{T}_2 \Longrightarrow \mathbf{I}\Delta_0$ *then* $\mathcal{R}(\mathbf{T}_1 \vee \mathbf{T}_2) = \mathcal{R}(\mathbf{T}_1) \cap \mathcal{R}(\mathbf{T}_2)$.

Proof. Since $\mathbf{T}_1$, $\mathbf{T}_2 \Longrightarrow \mathbf{T}_1 \vee \mathbf{T}_2$, it holds that $\mathcal{R}(\mathbf{T}_1 \vee \mathbf{T}_2) \subseteq \mathcal{R}(\mathbf{T}_1) \cap \mathcal{R}(\mathbf{T}_2)$.

Conversely, if $f \in \mathcal{R}(\mathbf{T}_1) \cap \mathcal{R}(\mathbf{T}_2)$, then there exist $\psi_1(\vec{x}, y, z)$, $\psi_2(\vec{x}, y, z) \in \Delta_0$ such that $\exists z\, \psi_i(\vec{x}, y, z)$ defines f in $\mathbf{T}_i$. Let $\theta_0(\vec{x}, u) \in \Delta_0$ the formula $(\psi_1(\vec{x}, (u)_0, (u)_1) \vee \psi_2(\vec{x}, (u)_0, (u)_1))$ and let $\theta(\vec{x}, y)$ be the formula

$$\exists z\ [\theta_0(\vec{x}, \langle y, z\rangle) \wedge \forall w < \langle y, z\rangle\, \neg\theta_0(\vec{x}, w)]\,.$$

Then $\theta(\vec{x}, y)$ defines f in $\mathbf{T}_1 \vee \mathbf{T}_2$. So, $f \in \mathcal{R}(\mathbf{T}_1 \vee \mathbf{T}_2)$. □

Next proposition is theorem 2.2 in [8]. Now it can be rephrased as follows.

Proposition 4. *If* $\mathfrak{A} \not\models \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ *and* $\mathfrak{A} \models \mathbf{I}\Delta_{n+1}(\mathbf{T})$, *then* $\mathfrak{A} \models \mathbf{I}\Sigma_{n+1}$. *Hence,*

$$\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{I}\Sigma_{n+1} \vee \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}).$$

Moreover, if $\mathbf{T}$ *has* Δ_{n+1}*–induction then* $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Sigma_{n+1} \vee \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$.

From this proposition, using Lemma 5, a first result on $\mathcal{R}(\mathbf{T}) \cap \mathcal{PR}$ is obtained for theories with Δ_1–induction.

Corollary 1. *1.* $\mathcal{PR} \cap \mathcal{R}(\mathbf{T}) \subseteq \mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T}))$.

2. If $\mathbf{T}$ *has* Δ_1*–induction then* $\mathcal{PR} \cap \mathcal{R}(\mathbf{T}) = \mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T}))$.

By Lemma 5, $\mathcal{R}(\mathbf{T}) \cap \mathcal{PR}$ is always a Δ_0–generated F–algebra. To get a natural theory $\mathbf{T}'$ such that $\mathcal{R}(\mathbf{T}') = \mathcal{R}(\mathbf{T}) \cap \mathcal{PR}$ without assuming that $\mathbf{T}$ has Δ_1–induction, we consider parameter free versions of $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{I}\Sigma_{n+1}$, denoted by $\mathbf{I}\Delta_{n+1}(\mathbf{T})^-$ and $\mathbf{I}\Sigma_{n+1}^-$, respectively.

Theorem 2. *For each sound theory,* $\mathbf{T}$*,* $\mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T})^-) = \mathcal{R}(\mathbf{T}) \cap \mathcal{PR}$.

Proof. By a similar argument to that of theorem 2.2 in [8], it is shown that:

$$\mathbf{I}\Delta_{n+1}(\mathbf{T})^- \Longrightarrow \mathbf{I}\Sigma_{n+1}^- \vee \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}).$$

Since $\mathcal{PR} = \mathcal{R}(\mathbf{I}\Sigma_1^-)$, by Lemma 5, $\mathcal{PR} \cap \mathcal{R}(\mathbf{T}) = \mathcal{R}(\mathbf{I}\Sigma_1^- \vee \mathbf{T}) \subseteq \mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T})^-)$.

Let us prove $\mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T})^-) \subseteq \mathcal{R}(\mathbf{T}) \cap \mathcal{PR}$. Obviously, $\mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T})^-) \subseteq \mathcal{PR}$. Now, let us observe that $\mathbf{I}\Delta_1^-$ is Σ_2–axiomatizable and, therefore,

$$\mathbf{T} + \mathbf{Th}_{\Pi_1}(\mathcal{N}) \Longrightarrow \mathbf{T} + \mathbf{I}\Delta_1^- \Longrightarrow \mathbf{T} + \mathbf{I}\Delta_1(\mathbf{T})^- \Longrightarrow \mathbf{I}\Delta_1(\mathbf{T})^-.$$

So, by Proposition 1, $\mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T})^-) \subseteq \mathcal{R}(\mathbf{T} + \mathbf{Th}_{\Pi_1}(\mathcal{N})) = \mathcal{R}(\mathbf{T})$. □

4 $\mathcal{C}$–Bounded Recursive Arithmetic: $\mathcal{C}$–BRA

In this section we characterize $\mathcal{R}(\mathbf{T}) \cap \mathcal{PR}$ in terms of bounded recursion. Our main tool will be a version of the well–known system **PRA** (Primitive Recursive Arithmetic). Our analysis of this theory follows the lines sketched in [1].

Definition. Let $\mathcal{C}$ be a rudimentary F–algebra. The theory $\mathcal{C}$–**BRA**, $\mathcal{C}$–Bounded Recursive Arithmetic, is given by:

- Language: $\mathcal{L}_{pr}^{\mathcal{C}} = \bigcup_{i \in \omega} \mathbf{L}_i$, where
  - $\mathbf{L}_0 = \mathcal{L}$ plus a function symbol B_f for each basic function, $f \in \mathcal{B}$.
  - $\mathbf{L}_{j+1} = \mathbf{L}_j$ plus a function symbol, $\mathbf{f}_t$ for each term of $\mathbf{L}_j$, and a function symbol $\mathbf{f}_{t_1,t_2,g}$ for each function $g \in \mathcal{C}$ and terms $t_1(\vec{x})$, $t_2(\vec{x}, y, z)$ of $\mathbf{L}_j$ such that the function defined from t_1 and t_2 by primitive recursion is bounded by g, i. e., $h \le g$, where $h : \omega^{n+1} \to \omega$ is the function given by

$$h(\vec{x}, 0) = t_1(\vec{x}), \qquad h(\vec{x}, y+1) = t_2(\vec{x}, y, h(\vec{x}, y)).$$

- Axioms:
  (1) $\mathbf{P}^-$.
  (2) $B_S(x) = x + 1, \quad B_{\Pi_i^n}(x_1, \ldots, x_n) = x_i, \quad B_O(x) = 0.$
  (3) $\mathbf{f}_t(\vec{x}) = t(\vec{x})$.
  (4) $\mathbf{f}_{t_1,t_2,g}(\vec{x}, 0) = t_1(\vec{x}), \quad \mathbf{f}_{t_1,t_2,g}(\vec{x}, y+1) = t_2(\vec{x}, y, \mathbf{f}_{t_1,t_2,g}(\vec{x}, y))$.
  (5) Open Induction: The induction scheme for open formulas of $\mathcal{L}_{pr}^{\mathcal{C}}$.

It is routine to check that $\mathcal{C}$–**BRA** satisfies the following properties stated for **PRA** in [1].

Lemma 6. *It holds that:*

1. *In $\mathcal{C}$–**BRA** the class of open formulas is closed under bounded quantification.*
2. *In $\mathcal{C}$–**BRA** every Δ_0–formula is equivalent to an open formula.*
3. *$\mathcal{C}$–**BRA** supports definition by cases.*
4. *$\mathcal{C}$–**BRA** is universally axiomatizable.*

Lemma 7. $\mathcal{R}(\mathcal{C}\text{–}\mathbf{BRA}) = \mathcal{E}^{\mathcal{C}}$.

Proof. Obviously $\mathcal{E}^{\mathcal{C}} \subseteq \mathcal{R}(\mathcal{C}\text{–}\mathbf{BRA})$. Since $\mathcal{C}$–**BRA** is a universal theory and supports definition by cases, the result follows from Herbrand's theorem. □

Next we investigate the relations between $\mathcal{PR} \cap \mathcal{R}(\mathbf{T})$ and $\mathcal{E}^{\mathcal{C}}$. The key ingredient is Theorem 3 below stating a conservation result between $\mathcal{C}$–**BRA** and $\mathbf{L}\Delta_1(\mathbf{T})$. In the proof of that theorem we use the model–theoretic framework developed by Avigad in [1]. Let us recall some definitions and results from that paper that will be used in what follows.

Definition. We say that a structure $\mathfrak{A}$ is $\exists_2$–closed (or Herbrand saturated, in Avigad's terminology) if for each $\varphi(\vec{x}) \in \exists_2$ and $\vec{a} \in \mathfrak{A}$ such that $\mathfrak{B} \models \varphi(\vec{a})$, for some $\mathfrak{B}$, $\mathfrak{A} \prec_{\forall_1} \mathfrak{B}$, it holds $\mathfrak{A} \models \varphi(\vec{a})$.

As it is proved in [1], every universal theory has a $\exists_2$–closed model. For these models the following results hold (see [1], theorems 3.3 and 3.4):

Proposition 5. *Let $\mathfrak{A}$ be an $\exists_2$-closed model and $\theta(\vec{x}, y, \vec{z})$ an open formula such that $\mathfrak{A} \models \forall \vec{x}\, \exists y\, \theta(\vec{x}, y, \vec{a})$. Then there exist a universal formula $\psi(\vec{z}, \vec{w})$ and terms $t_1, \ldots, t_k$ such that $\mathfrak{A} \models \exists \vec{w}\, \psi(\vec{a}, \vec{w})$ and*

$$\models \psi(\vec{z}, \vec{w}) \rightarrow \theta(\vec{x}, t_1(\vec{x}, \vec{z}, \vec{w}), \vec{z}) \vee \cdots \vee \theta(\vec{x}, t_k(\vec{x}, \vec{z}, \vec{w}), \vec{z}).$$

Proposition 6. *Let $\mathbf{T}_2$ be a universal theory and let $\mathbf{T}_1$ be a theory in the language of $\mathbf{T}_2$. If every $\exists_2$–closed model of $\mathbf{T}_2$ is a model of $\mathbf{T}_1$, then every $\forall_2$–theorem of $\mathbf{T}_1$ is a theorem of $\mathbf{T}_2$.*

Last proposition is used in [1] to obtain new proofs of a number of conservation results. In what follows we use it to prove our main conservation result. First of all, we show that $\exists_2$–closed models of $\mathcal{C}$–**BRA** satisfy Σ_1–collection.

Lemma 8. *Let $\mathfrak{A} \models \mathcal{C}$–**BRA** be an $\exists_2$–closed model.*

1. *In $\mathfrak{A}$ each Δ_1–formula is equivalent to an open formula.*
2. *$\mathfrak{A} \models \mathbf{B}\Sigma_1$.*

Proof. **(1)** Let $\varphi(x, y, v)$, $\psi(x, y, v) \in \Delta_0$ and $a \in \mathfrak{A}$ such that

$$\mathfrak{A} \models \exists y\, \varphi(x, y, a) \leftrightarrow \forall y\, \psi(x, y, a).$$

Let $\theta(x, y, v)$ be the formula $\varphi(x, y, v) \vee \neg\psi(x, y, v)$. Then $\mathfrak{A} \models \forall x\, \exists y\, \theta(x, y, a)$. By Lemma 6, there exist φ_0 and θ_0 quantifier–free formulas such that

$$\mathcal{C}\text{–}\mathbf{BRA} \vdash (\varphi_0 \leftrightarrow \varphi) \wedge (\theta_0 \leftrightarrow \theta).$$

Then $\mathfrak{A} \models \forall x\, \exists y\, \theta_0(x, y, a)$ and, by Proposition 5 (recall that $\mathcal{C}$–**BRA** supports definition by cases), there exist $b \in \mathfrak{A}$ and a term $t(x, v, w)$ such that $\mathfrak{A} \models \forall x\, \theta_0(x, t(x, a, b), a)$. As a consequence, $\mathfrak{A} \models \exists y\, \varphi(x, y, a) \leftrightarrow \varphi_0(x, t(x, a, b), a)$.

(2) By Lemma 6, the class of open formulas is closed in $\mathcal{C}$–**BRA** under bounded quantification. Since $\mathcal{C}$–**BRA** proves open induction, a standard argument (see lemma I.2.12 in [10]) shows that minimization scheme for open formulas holds in $\mathcal{C}$–**BRA**. So, by **(1)**, $\mathfrak{A} \models \mathbf{L}\Delta_1$. But $\mathbf{L}\Delta_1 \iff \mathbf{B}\Sigma_1$ (see [10] lemmas I.2.16, I.2.17), hence $\mathfrak{A} \models \mathbf{B}\Sigma_1$. □

Theorem 3. *Let* $\mathbf{T}$ *be a (sound)* Π_2*–axiomatized extension of* $\mathbf{I}\Delta_0$ *and* $\mathcal{C} = \mathcal{R}(\mathbf{T})$. *For each* Π_2*–sentence* θ, *if* $\mathbf{L}\Delta_1(\mathbf{T}) \vdash \theta$ *then* $\mathcal{C}$–$\mathbf{BRA} \vdash \theta$.

Proof. Since $\mathcal{C}$–**BRA** is a universal theory, following [1], we prove that every $\exists_2$–closed model of $\mathcal{C}$–**BRA**, $\mathfrak{A}$, is a model of $\mathbf{L}\Delta_1(\mathbf{T})$. Then the result follows by Proposition 6. In a first step we prove $\mathfrak{A} \models \mathbf{I}\Delta_1(\mathbf{T})$.

Let $\varphi(x, y, \vec{v}), \psi(x, y, \vec{v}) \in \Delta_0$ such that $\mathbf{T} \vdash \exists y\, \varphi(x, y, \vec{v}) \leftrightarrow \forall y\, \psi(x, y, \vec{v})$. We may assume that $\mathbf{T} \vdash \varphi(x, y_1, \vec{v}) \wedge \varphi(x, y_2, \vec{v}) \rightarrow y_1 = y_2$ (if not, we take as φ the formula $\varphi(x, y, \vec{v}) \wedge \forall z < y\, \neg\varphi(x, z, \vec{v})$). Let $\theta(x, y, \vec{v}) \in \Delta_0$ the formula $\varphi(x, y, \vec{v}) \vee \neg\psi(x, y, \vec{v})$. Then,

$$\mathbf{T} \vdash \forall x\, \exists y\, (\theta(x, y, \vec{v}) \wedge \forall z < y\, \neg\theta(x, z, \vec{v})).$$

Since $\mathbf{T}$ is a sound theory, the formula $\theta(x, y, \vec{v}) \wedge \forall z < y\, \neg\theta(x, z, \vec{v})$ defines a p.t.c.f. of $\mathbf{T}$, say f. Then $\mathcal{N} \models \forall y\, (\varphi(x, y, \vec{v}) \rightarrow y \le f(x, \vec{v}))$. Now, we continue the proof as in [1], theorem 4.1.

Let $\mathfrak{A}$ be an $\exists_2$–closed model of $\mathcal{C}$–**BRA** and φ_0 an open formula equivalent to φ in $\mathcal{C}$–**BRA**. Let us see that $\mathfrak{A} \models \mathbf{I}_{\exists y\, \varphi_0}$. Assume that, for some $\vec{a} \in \mathfrak{A}$,

$$\mathfrak{A} \models \exists y\, \varphi_0(0, y, \vec{a}) \wedge \forall x\, (\exists y\, \varphi_0(x, y, \vec{a}) \rightarrow \exists y\, \varphi_0(x+1, y, \vec{a})).$$

Then, as in [1], since $\mathcal{C}$–**BRA** supports definition by cases, by Proposition 5, there exist $\vec{b}, c \in \mathfrak{A}$ and a function symbol $\mathbf{g}(x, y, \vec{v}, \vec{w})$ such that

$$\mathfrak{A} \models \varphi_0(0, c, \vec{a}) \wedge \forall x, y\, (\varphi_0(x, y, \vec{a}) \rightarrow \varphi_0(x+1, \mathbf{g}(x, y, \vec{a}, \vec{b}), \vec{a})).$$

Let us denote by g the function defined by $\mathbf{g}$ in $\mathcal{N}$. Let $h_0 : \omega^{n+2} \rightarrow \omega$ be defined in $\mathcal{N}$ by

$$h_0(x, y, z, \vec{v}, \vec{w}) = \begin{cases} g(x, z, \vec{v}, \vec{w}), & \text{if } \varphi_0(x+1, g(x, z, \vec{v}, \vec{w}), \vec{v}); \\ 0, & \text{otherwise.} \end{cases}$$

Then $h_0 \in \mathcal{E}^{\mathcal{C}}$. Let f_0 be the function defined by primitive recursion as follows:

$$f_0(0, y, \vec{v}, \vec{w}) = y, \qquad f_0(x+1, y, \vec{v}, \vec{w}) = h_0(x, y, f_0(x, y, \vec{v}, \vec{w}), \vec{v}, \vec{w}).$$

Then $f_0(x, y, \vec{v}, \vec{w}) \le f'(x, y, \vec{v}, \vec{w}) = f(x+1, \vec{v}) + y$. So, $f_0 \in \mathcal{E}^{\mathcal{C}}$, since it is defined by f'–bounded recursion and $f' \in \mathcal{C}$. Let $\mathbf{f}_0$ be the function symbol corresponding to f_0. Then $\mathfrak{A}$ satisfies that

$$\varphi_0(0, \mathbf{f}_0(0, c, \vec{a}, \vec{b}), \vec{a}) \wedge \forall x\, (\varphi_0(x, \mathbf{f}_0(x, c, \vec{a}, \vec{b}), \vec{a}) \rightarrow \varphi_0(x+1, \mathbf{f}_0(x+1, c, \vec{a}, \vec{b}), \vec{a})).$$

Since $\mathfrak{A}$ satisfies open induction, $\mathfrak{A} \models \forall x\, \varphi_0(x, \mathbf{f}_0(x, c, \vec{a}, \vec{b}), \vec{a})$. Hence, it follows that $\mathfrak{A} \models \forall x\, \exists y\, \varphi(x, y, \vec{a})$. So, $\mathfrak{A} \models \mathbf{I}\Delta_1(\mathbf{T})$.

Let us see that $\mathfrak{A} \models \mathbf{L}\Delta_1(\mathbf{T})$. We distinguish two cases:

1. If $\mathfrak{A} \not\models \mathbf{T}$ then, since $\mathbf{T}$ is Π_2–axiomatized, by Proposition 4, $\mathfrak{A} \models \mathbf{I}\Sigma_1$.
2. If $\mathfrak{A} \models \mathbf{T}$, then, by Lemma 8, $\mathfrak{A} \models \mathbf{T} + \mathbf{B}\Sigma_1$; hence, $\mathfrak{A} \models \mathbf{L}\Delta_1(\mathbf{T})$. □

As a consequence, we get some results relating $\mathcal{E}^{\mathcal{C}}$ and $\mathcal{C} \cap \mathcal{PR}$. The notion of Δ_0–generativeness provides a necessary and sufficient condition for $\mathcal{E}^{\mathcal{C}} = \mathcal{C} \cap \mathcal{PR}$.

Theorem 4. *Let $\mathcal{C}$ be a Δ_0–generated F–algebra. Then*

1. *$\mathcal{PR} \cap \mathcal{C}$ is Δ_0–generated.*
2. *$\mathcal{PR} \cap \mathcal{C} \subseteq \mathcal{E}^{\mathcal{C}} = \mathcal{E}^{\mathcal{PR} \cap \mathcal{C}} \subseteq \mathbf{E}(\mathcal{PR} \cap \mathcal{C})$.*
3. *If $\mathcal{C}$ is closed under bounded minimization, then $\mathcal{E}^{\mathcal{C}} = \mathbf{E}(\mathcal{PR} \cap \mathcal{C})$.*

Proof. By Theorem 1, there is a sound extension of $\mathbf{I}\Delta_0$, $\mathbf{T}$, such that $\mathcal{R}(\mathbf{T}) = \mathcal{C}$.
(1) Since $\mathcal{C} \cap \mathcal{PR} = \mathcal{R}(\mathbf{T} \vee \mathbf{I}\Sigma_1)$, by Theorem 1, $\mathcal{C} \cap \mathcal{PR}$ is Δ_0–generated.
(2) By Corollary 1, $\mathcal{PR} \cap \mathcal{C} \subseteq \mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T}))$. Moreover, by Theorem 3 and Lemma 7, $\mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T})) \subseteq \mathcal{E}^{\mathcal{C}}$; so, $\mathcal{PR} \cap \mathcal{C} \subseteq \mathcal{E}^{\mathcal{C}}$. It is trivial that $\mathcal{E}^{\mathcal{C} \cap \mathcal{PR}} \subseteq \mathbf{E}(\mathcal{C} \cap \mathcal{PR})$.

Now, let us see that $\mathcal{E}^{\mathcal{C}} = \mathcal{E}^{\mathcal{C} \cap \mathcal{PR}}$. It is enough to prove that $\mathcal{E}^{\mathcal{C}} \subseteq \mathcal{E}^{\mathcal{C} \cap \mathcal{PR}}$. We proceed by induction on the definition of $f \in \mathcal{E}^{\mathcal{C}}$. The critical step is the definition by $\mathcal{C}$–bounded recursion. But, let us observe that if $f \in \mathcal{E}^{\mathcal{C}}$ is defined by $\mathcal{C}$–bounded recursion, then f is bounded by a function $g_1 \in \mathcal{C}$ and by a function $g_2 \in \mathcal{PR}$ (in fact, $f \in \mathcal{PR}$). We prove that in this case f is bounded by a function $h \in \mathcal{C} \cap \mathcal{PR}$.

Let $\psi_1(\vec{x}, y, z)$, $\psi_2(\vec{x}, y, z) \in \Delta_0$ such that $\exists z\, \psi_1(\vec{x}, y, z)$ and $\exists z\, \psi_2(\vec{x}, y, z)$ define g_1 and g_2 in $\mathbf{T}$ and $\mathbf{I}\Sigma_1$, respectively. Let $\theta_0(\vec{x}, u) \in \Delta_0$ be the formula

$$\begin{cases} (\psi_1(\vec{x}, (u)_0, (u)_1) \vee \psi_2(\vec{x}, (u)_0, (u)_1)) \wedge \\ \forall v < u\, (\neg\psi_1(\vec{x}, (v)_0, (v)_1) \wedge \neg\psi_2(\vec{x}, (v)_0, (v)_1)). \end{cases}$$

Then $\exists z\, \theta_0(\vec{x}, \langle y, z\rangle)$ defines in $\mathbf{T} \vee \mathbf{I}\Sigma_1$ a function h such that for all $\vec{a} \in \omega$, $h(\vec{a}) = g_1(\vec{a})$ or $h(\vec{a}) = g_2(\vec{a})$. So, $h \in \mathcal{R}(\mathbf{T} \vee \mathbf{I}\Sigma_1) = \mathcal{C} \cap \mathcal{PR}$ and, since $f \leq g_1$ and $f \leq g_2$, we have $f \leq h$.
(3) Since $\mathcal{C}$ is closed under $\mu_{\leq}$, each function in $\mathcal{C}$ is bounded by a nondecreasing function also in $\mathcal{C}$. By induction on the construction of $f \in \mathcal{E}^{\mathcal{PR} \cap \mathcal{C}}$, it is proved that each element of $\mathcal{E}^{\mathcal{C}}$ ($= \mathcal{E}^{\mathcal{PR} \cap \mathcal{C}}$) is bounded by an element of $\mathcal{PR} \cap \mathcal{C}$. So, $\mathcal{E}^{\mathcal{C} \cap \mathcal{PR}}$ is closed under bounded recursion. Hence, **(3)** follows from part **(2)**. □

Theorem 5. *Let $\mathcal{C}$ be a Δ_0–generated F–algebra. If $\mathcal{C}$ is closed under bounded minimization, then the following conditions are equivalent:*

1. *$\mathcal{E}^{\mathcal{C}}$ is Δ_0–generated. Or, by Theorem 4, $\mathcal{E}^{\mathcal{C} \cap \mathcal{PR}}$ is Δ_0–generated.*
2. *$\mathcal{E}^{\mathcal{C}} = \mathcal{C} \cap \mathcal{PR}$.*

Proof. $(\mathbf{2}) \Longrightarrow (\mathbf{1})$: By Theorem 4, $\mathcal{C} \cap \mathcal{PR}$ is Δ_0–generated; so, **(1)** holds.

$(\mathbf{1}) \Longrightarrow (\mathbf{2})$: Let $\mathcal{F}_0$ be the class of all functions in $\mathcal{E}^{\mathcal{C}}$ having a Δ_0–definable graph. Then $\mathcal{E}^{\mathcal{C}} = \mathbf{C}(\mathcal{F}_0)$. By the proof of part (3) of Theorem 4, each function in $\mathcal{E}^{\mathcal{C}}$ is bounded by a function in $\mathcal{C}$. Since $\mathcal{E}^{\mathcal{C}}$ is a rudimentary F–algebra, by Lemma 1, $\mathcal{F}_0 \subseteq \mathcal{C}$ and, as a consequence, $\mathcal{E}^{\mathcal{C}} = \mathbf{C}(\mathcal{F}_0) \subseteq \mathcal{C} \cap \mathcal{PR}$. On the other hand, by Theorem 4, $\mathcal{C} \cap \mathcal{PR} \subseteq \mathcal{E}^{\mathcal{C}}$. This proves **(2)**. □

Corollary 2. *The following statements are equivalent:*

1. *$\mathcal{E}^2$ is a Δ_0–generated F–algebra.*
2. *$\mathcal{M}^2 = \mathcal{E}^2$.*
3. *There exists an extension of $\mathbf{I}\Delta_0$, $\mathbf{T}$, such that $\mathcal{E}^2 = \mathcal{R}(\mathbf{T})$.*

Now we give a characterization of $\mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T}))$ in terms of $\mathcal{C}$–bounded recursion.

Theorem 6. *Let $\mathbf{T}$ be a sound extension of $\mathbf{I}\Delta_0 + \mathbf{exp}$ and $\mathcal{C} = \mathcal{R}(\mathbf{T})$. If $\mathcal{C}$ is closed under bounded minimization, then*

$$\mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T})) = \mathcal{R}(\mathbf{L}\Delta_1(\mathbf{T})) = \mathcal{E}^{\mathcal{C}} = \mathcal{C} \cap \mathcal{PR}.$$

Proof. Without loss of generality we can assume that $\mathbf{T}$ is Π_2 axiomatized.

First we prove the result for $\mathbf{T}$ satisfying $\mathbf{I}\Sigma_1 \Longrightarrow \mathbf{T}$. Then $\mathcal{C} = \mathcal{C} \cap \mathcal{PR}$. By Proposition 4, $\mathbf{L}\Delta_1(\mathbf{T}) \Longrightarrow \mathbf{I}\Delta_1(\mathbf{T}) \Longrightarrow \mathbf{T}$; hence, $\mathbf{T}+\mathbf{L}\Delta_1(\mathbf{T}) \Longleftrightarrow \mathbf{L}\Delta_1(\mathbf{T})$. By Lemma 3, $\mathbf{T}+\mathbf{L}\Delta_1(\mathbf{T}) \Longleftrightarrow \mathbf{T}+\mathbf{B}^*\Delta_1(\mathbf{T})$, so $\mathcal{R}(\mathbf{L}\Delta_1(\mathbf{T})) = \mathcal{R}(\mathbf{T}+\mathbf{B}^*\Delta_1(\mathbf{T}))$. In [9] (see remark 2.8), it is proved that $\mathbf{T}+\mathbf{B}^*\Delta_1(\mathbf{T}) \Longleftrightarrow [\mathbf{T}, \Sigma_1\text{–CR}]$ (the closure of $\mathbf{T}$ under unnested applications of Σ_1–collection rule). Therefore, by corollary 5.6 in [2], since $\mathbf{T} \vdash \mathbf{exp}$ we get

$$\mathcal{R}(\mathbf{L}\Delta_1(\mathbf{T})) = \mathcal{R}([\mathbf{T}, \Sigma_1\text{–CR}]) = \mathbf{E}(\mathcal{C}).$$

By Theorem 4–(3), $\mathcal{E}^{\mathcal{C}} = \mathbf{E}(\mathcal{C} \cap \mathcal{PR})$; so, $\mathcal{R}(\mathbf{L}\Delta_1(\mathbf{T})) = \mathcal{E}^{\mathcal{C}}$, since $\mathcal{C} = \mathcal{C} \cap \mathcal{PR}$. As a consequence, $\mathcal{E}^{\mathcal{C}}$ is Δ_0–generated and, by Theorem 5, $\mathcal{E}^{\mathcal{C}} = \mathcal{C}$. Now the result follows from the chain of inclusions below:

$$\mathcal{R}(\mathbf{L}\Delta_1(\mathbf{T})) = \mathcal{E}^{\mathcal{C}} = \mathcal{C} \subseteq \mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T})) \subseteq \mathcal{R}(\mathbf{L}\Delta_1(\mathbf{T})).$$

Let us prove the general case. Let $\mathbf{T}^I$ be the theory $\mathbf{I}\Sigma_1 \vee \mathbf{T}$. By Lemma 5, $\mathcal{R}(\mathbf{T}^I) = \mathcal{R}(\mathbf{T}) \cap \mathcal{R}(\mathbf{I}\Sigma_1)$. Since $\mathbf{I}\Sigma_1 \Longrightarrow \mathbf{T}^I \Longrightarrow \mathbf{I}\Delta_0 + \mathbf{exp}$, by previous case $\mathcal{R}(\mathbf{L}\Delta_1(\mathbf{T}^I)) = \mathcal{E}^{\mathcal{R}(\mathbf{T}^I)} = \mathcal{R}(\mathbf{T}^I)$, and by Theorem 4, $\mathcal{E}^{\mathcal{R}(\mathbf{T}^I)} = \mathcal{E}^{\mathcal{C} \cap \mathcal{PR}} = \mathcal{E}^{\mathcal{C}}$. So, $\mathcal{E}^{\mathcal{C}}$ is Δ_0–generated and, by Theorem 5, $\mathcal{E}^{\mathcal{C}} = \mathcal{C} \cap \mathcal{PR}$. By Theorem 3,

$$\mathcal{R}(\mathbf{L}\Delta_1(\mathbf{T})) \subseteq \mathcal{E}^{\mathcal{C}} = \mathcal{C} \cap \mathcal{PR} \subseteq \mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T})) \subseteq \mathcal{R}(\mathbf{L}\Delta_1(\mathbf{T})).$$

This concludes the proof of the theorem. □

The hypothesis on $\mathcal{C}$ in Theorem 6, namely, $\mathcal{C}$ is closed under $\mu_{\leq}$, is equivalent to the existence of a theory $\mathbf{T}'$ such that $\mathcal{C} = \mathcal{R}(\mathbf{T}')$ and $\mathbf{T}'$ extends $\mathbf{L}\Delta_1(\mathbf{T}')$.

Below we discuss if this hypothesis can be weakened. This is related to the problem on the equivalence between $\mathbf{UI}\Delta_1$ and $\mathbf{B}\Sigma_1^-$. Here, $\mathbf{B}\Sigma_1^-$ denotes parameter free Σ_1–collection (see [9] for a deeper background on these theories).

First of all, let us observe that the hypothesis cannot be omitted. In [3] Beklemishev obtains $f \in \mathcal{E}^4$ ($\subseteq \mathcal{PR}$) such that $\mathcal{C} = \mathbf{C}(\mathcal{E}^3 \cup \{f\})$ is not closed under bounded recursion. Let $\mathbf{T}$ be the theory given by $\mathbf{I}\Delta_0 + \mathbf{exp} +$ "f is total". Then $\mathcal{R}(\mathbf{T}) = \mathcal{C}$ and, since $\mathbf{I}\Sigma_1 \Longrightarrow \mathbf{T}$, as in the first part of the proof of Theorem 6, $\mathcal{R}(\mathbf{L}\Delta_1(\mathbf{T})) = \mathbf{E}(\mathcal{C})$. So, $\mathcal{C} = \mathcal{C} \cap \mathcal{PR} \neq \mathcal{R}(\mathbf{L}\Delta_1(\mathbf{T}))$.

A suitable hypothesis on $\mathcal{C}$ to be used in Theorems 5 and 6 instead of the closure under bounded minimization is the following one:

(IC) There exists a theory $\mathbf{T}$ such that $\mathcal{C} = \mathcal{R}(\mathbf{T})$ and $\mathbf{T}$ has Δ_1–induction.

Observe that if Theorem 5 holds under hypothesis **(IC)**, so does Theorem 6. Next lemma allows us to avoid using Theorem 4–(3) in the proof of Theorem 6.

Lemma 9. *Let $\boldsymbol{T}$ be a sound Π_2–axiomatized extension of $\mathbf{I}\Delta_0 + \mathbf{exp}$. Let $\mathcal{C} = \mathcal{R}(\mathbf{T})$. Then $\mathbf{E}(\mathcal{C}) = \mathbf{C}(\mathcal{C} \cup \mathcal{E}^{\mathcal{C}})$.*

Proof. By the first part of the proof of Theorem 6, $\mathbf{E}(\mathcal{C}) = \mathcal{R}(\mathbf{T} + \mathbf{L}\Delta_1(\mathbf{T}))$. Let $\mathbf{T}_{\mathcal{C}}$ be the theory obtained by extending $\mathcal{C}$–**BRA** as follows:

For each formula $\varphi(\vec{x}, y) \in \Delta_0$ such that $\mathbf{T} \vdash \forall \vec{x}\, \exists y\, \varphi(\vec{x}, y)$, we add a new symbol function $\mathbf{f}_\varphi$ and take as an axiom the formula

$$\mathbf{f}_\varphi(\vec{x}) = y \leftrightarrow \varphi(\vec{x}, y) \wedge \forall z < y\, \neg\varphi(\vec{x}, z).$$

Since each Δ_0–formula is equivalent in $\mathcal{C}$–**BRA** to an open formula, $\mathbf{T}_{\mathcal{C}}$ is a universal theory and supports definition by cases. Then, by a standard Herbrand analysis we get that $\mathcal{R}(\mathbf{T}_{\mathcal{C}}) = \mathbf{C}(\mathcal{C} \cup \mathcal{E}^{\mathcal{C}})$.

Moreover, every $\exists_2$–closed model of $\mathbf{T}_{\mathcal{C}}$ is a model of $\mathbf{T} + \mathbf{B}\Sigma_1$, since it is a $\exists_2$–closed model of $\mathcal{C}$–**BRA** (Lemma 8). So, as in Theorem 3, we get that for each formula $\theta \in \Pi_2$,

$$\mathbf{T} + \mathbf{B}\Sigma_1 \vdash \theta \Longrightarrow \mathbf{T}_{\mathcal{C}} \vdash \theta.$$

Since $\mathbf{T} \vdash \mathbf{exp}$, then $\mathbf{E}(\mathcal{C}) = \mathcal{R}(\mathbf{T} + \mathbf{B}\Sigma_1) \subseteq \mathcal{R}(\mathbf{T}_{\mathcal{C}}) = \mathbf{C}(\mathcal{C} \cup \mathcal{E}^{\mathcal{C}}) \subseteq \mathbf{E}(\mathcal{C})$. □

We conclude studying the equivalence between $\mathbf{UI}\Delta_1$ and $\mathbf{B}\Sigma_1^-$.

Proposition 7. *Assume that Theorem 5 holds under hypothesis* **(IC)**. *Let $\boldsymbol{T}$ be a Π_2–axiomatized extension of $\mathbf{Th}_{\Pi_1}(\mathcal{N}) + \mathbf{exp}$. If $\boldsymbol{T}$ has Δ_1–induction then $\boldsymbol{T}$ extends $\mathbf{L}\Delta_1(\mathbf{T})$.*

Proof. As noticed in the proof of Lemma 9, $\mathcal{R}(\mathbf{T} + \mathbf{L}\Delta_1(\mathbf{T})) = \mathbf{E}(\mathcal{C})$. Moreover, by Lemma 9, $\mathbf{E}(\mathcal{C}) = \mathbf{C}(\mathcal{C} \cup \mathcal{E}^{\mathcal{C}})$ and by Theorem 6, $\mathcal{E}^{\mathcal{C}} = \mathcal{R}(\mathbf{I}\Delta_1(\mathbf{T}))$. So,

$$\mathbf{E}(\mathcal{C}) = \mathbf{C}(\mathcal{C} \cup \mathcal{E}^{\mathcal{C}}) \subseteq \mathcal{R}(\mathbf{T} + \mathbf{I}\Delta_1(\mathbf{T})) \subseteq \mathcal{R}(\mathbf{T} + \mathbf{L}\Delta_1(\mathbf{T})) = \mathbf{E}(\mathcal{C}).$$

Therefore, $\mathcal{R}(\mathbf{T}) = \mathcal{R}(\mathbf{T} + \mathbf{L}\Delta_1(\mathbf{T}))$, since $\mathbf{T}$ has Δ_1–induction. By Lemma 4, $\mathbf{T} + \mathbf{L}\Delta_1(\mathbf{T})$ is Π_2 axiomatizable. Hence, by Proposition 1, $\mathbf{T} \Longleftrightarrow \mathbf{T} + \mathbf{L}\Delta_1(\mathbf{T})$ (recall that $\mathbf{T}$ extends $\mathbf{Th}_{\Pi_1}(\mathcal{N})$). In particular, $\mathbf{T} \Longrightarrow \mathbf{L}\Delta_1(\mathbf{T})$. □

Theorem 7. *Assume that Theorem 5 holds under hypothesis* **(IC)***. Then*

$$\mathbf{B}\Sigma_1^- + \mathbf{Th}_{\Pi_1}(\mathcal{N}) + \mathbf{exp} \quad \Longleftrightarrow \quad \mathbf{UI}\Delta_1 + \mathbf{Th}_{\Pi_1}(\mathcal{N}) + \mathbf{exp}.$$

Proof. It is known that $\mathbf{B}\Sigma_1^- \Longrightarrow \mathbf{UI}\Delta_1$; so, we only prove that

$$\mathbf{UI}\Delta_1 + \mathbf{Th}_{\Pi_1}(\mathcal{N}) + \mathbf{exp} \Longrightarrow \mathbf{B}\Sigma_1^- + \mathbf{Th}_{\Pi_1}(\mathcal{N}) + \mathbf{exp}.$$

Let $\mathfrak{A} \models \mathbf{UI}\Delta_1 + \mathbf{Th}_{\Pi_1}(\mathcal{N}) + \mathbf{exp}$ and $\mathbf{T} = \mathbf{Th}_{\Pi_2}(\mathfrak{A})$. Then it is easy to check that $\mathbf{T}$ has Δ_1–induction and extends $\mathbf{Th}_{\Pi_1}(\mathcal{N}) + \mathbf{exp}$. By Proposition 7, $\mathbf{T}$ extends $\mathbf{L}\Delta_1(\mathbf{T})$. As a consequence, $\mathbf{T}$ extends $\mathbf{B}^*\Delta_1(\mathbf{T})$, since $\mathbf{L}\Delta_1(\mathbf{T}) \Longrightarrow \mathbf{B}^*\Delta_1(\mathbf{T})$. From this it follows that $\mathfrak{A} \models \mathbf{B}\Sigma_1^-$ (see remark 2.5.3 in [9]). □

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