A. Cordón-Franco · A. Fernández-Margarit · F.F. Lara-Martín

On the quantifier complexity of Δ_{n+1}(T)–induction

Abstract. In this paper we continue the study of the theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$, initiated in [7]. We focus on the quantifier complexity of these fragments and theirs (non)finite axiomatization. A characterization is obtained for the class of theories such that $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ is Π_{n+2}–axiomatizable. In particular, $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1})$ gives an axiomatization of $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_{n+1})$ and is not finitely axiomatizable. This fact relates the fragment $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1})$ to induction rule for Σ_{n+1}–formulas. Our arguments, involving a construction due to R. Kaye (see [9]), provide proofs of Parsons' conservativeness theorem (see [16]) and (a weak version) of a result of L.D. Beklemishev on unnested applications of induction rules for Π_{n+2} and Σ_{n+1} formulas (see [2]).

1. Introduction

In [7] we introduced classes $\Delta_{n+1}(\mathbf{T})$, Σ_{n+1}–formulas that are equivalent in $\mathbf{T}$ to a Π_{n+1}–formula. Here we continue the study of the theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and the relationship between $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ and $\mathbf{I}\Delta_{n+1}(\mathbf{T})$. Through this paper we will use extensively results in [7] (see also [13]). For notation and preliminaries see that paper and [8], [10] for general references.

This paper is devoted to the study of two main topics on the theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$: its axiomatization properties (quantifier complexity and (non)finite axiomatization) and the relationship of these theories with induction rules. The initial motivation for the work we present here was to prove the following result.

Theorem 1.1. (see 2.4, 5.5) $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_{n+1}) \iff \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1})$. *So, the theory* $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1})$ *is* Π_{n+2}*–axiomatizable.*

In [7], this result is used to separate the fragments of Arithmetic introduced there: $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1})$ and $\mathbf{B}^*\Delta_{n+1}(\mathbf{I}\Sigma_{n+1})$.

A basic result on Σ_{n+1}–induction rule is the following conservativeness theorem of C. Parsons (see [16] and **6.5**): $\mathbf{I}\Sigma_{n+1}$ is a Π_{n+2}–conservative extension of

A. Cordón-Franco, A. Fernández-Margarit, F.F. Lara-Martín: Dpto. Ciencias de la Computación e I.A. Facultad de Matemáfticas, Universidad de Sevilla, C/Tarfia s/n, Sevilla (Spain)
e-mail: `fflara@us.es`

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$\mathbf{I}\Delta_0 + \Sigma_{n+1}$–IR (the closure of $\mathbf{I}\Delta_0$ under the Σ_{n+1}–induction rule). From this fact and theorem **1.1**, it follows that

Theorem 1.2. (Beklemishev) $\mathbf{I}\Delta_0 + \Sigma_{n+1}\text{–}IR \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1})$.

Even more, L.D. Beklemishev has observed (personal communication) that: modulo Parsons' theorem, **1.1** and **1.2** are equivalent; and, from the techniques used in the proof of theorem **1.1** (a generalized construction of Ackermann's function: the sequence of formulas $\mathbb{F}_{n,k}(x) = y$, $k \in \omega$, in section **4**) an alternative proof of Parsons' conservativeness theorem can be obtained.

These facts show the close relation between the topics we deal with here: induction rules and axiomatizations of $\mathbf{I}\Delta_{n+1}(\mathbf{T})$. Now we give another natural connection between the above topics. Theorem **1.1** aims at the following general question on axiomatizations of $\mathbf{I}\Delta_{n+1}(\mathbf{T})$:

(P1) For a theory $\mathbf{T}$, determine
  (a) the quantifier complexity of $\mathbf{I}\Delta_{n+1}(\mathbf{T})$, and
  (b) when $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$.

Informally, **(P1)** asks for an equivalence between recursion and induction: Are there natural classes of recursive functions that can be described in terms of induction principles? A classical problem is the characterization of $\mathcal{R}(\mathbf{T})$, the class of provably total recursive functions of $\mathbf{T}$. For theories axiomatizated by induction schemes the problem is: What functions can be proved to be total using only certain form of (restricted) induction?

Question **(P1)** is related to a kind of reverse problem. Let $\mathcal{C}$ be a class of provably recursive functions of $\mathbf{I}\Sigma_{n+1}$ and $\text{Total}_{\mathcal{C}}$ a class of Π_2 sentences asserting that each function in $\mathcal{C}$ is total.

(P2) Is there a theory $\mathbf{T}$ such that $\mathbf{I}\Sigma_n + \text{Total}_{\mathcal{C}} \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$?

Remark 1.3. Last question suggests that those theories such that $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ are equivalent can be characterized in a functional way . In particular, for these theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ is Π_{n+2}–axiomatizable. In order to describe this functional approach, let us recall some notations and definitions from [7]. We denote by $\mathcal{L}$ the language of Arithmetic and by $\mathcal{N}$ the standard model. If Φ is a class of formulas we write $\psi(x_1, \dots, x_n) \in \Phi^-$ if $\psi(\vec{x}) \in \Phi$ and $x_1, \dots, x_n$ are all the variables that occur free in ψ. Let Γ be a class of formulas of $\mathcal{L}$ with only two free variables, x and y say. For a formula $\varphi(x, y)$, the conjunction of

(–) $\forall x\, \forall y_1\, \forall y_2\, [\varphi(x, y_1) \wedge \varphi(x, y_2) \rightarrow y_1 = y_2]$ and
(–) $\forall x_1\, \forall x_2\, \forall y_1\, \forall y_2\, [x_1 \leq x_2 \wedge \varphi(x_1, y_1) \wedge \varphi(x_2, y_2) \rightarrow y_1 \leq y_2]$,

will be denoted by IPF(φ). Let IPF(Γ) $= \{\text{IPF}(\varphi(x, y)) : \varphi(x, y) \in \Gamma\}$ and $\Gamma^* = \{\forall x\, \exists! y\, \varphi(x, y) : \varphi \in \Gamma\} + \text{IPF}(\Gamma)$.

Let $\Gamma \subseteq \Pi_n$. We say that Γ is a Π_n–functional class if $\mathbf{I}\Sigma_n + \Gamma^*$ is consistent. A theory $\mathbf{T}$ is Π_n–functional if there exists a Π_n–functional class, Γ, such that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma^*)$.

We say that $\varphi(u, x, y) \in \Sigma_{n+1}^-$ is a Π_n–envelope of $\mathbf{T}$ in $\mathbf{T}_0$ if

1. $\mathbf{T} \vdash \Gamma_\varphi^*$, (where $\Gamma_\varphi = \{\varphi(k, x, y) : k \in \omega\}$).

2. For all $k \in \omega$, $\mathbf{T}_0 \vdash \varphi(k+1, x, y) \to \exists z < y\, \varphi(k, x, z)$.
3. For each $\psi(x, y) \in \Pi_n^-$ such that $\mathbf{T} \vdash \forall x\, \exists y\, \psi(x, y)$, there exists $k \in \omega$ such that $\mathbf{T}_0 \vdash \varphi(k, x, y) \to \exists z < y\, \psi(x, z)$.

Definition 1.4. *1. A Π_n–functional class Γ is inductive if for all $\psi \in \Gamma$*
(a) $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*) \vdash$ *IPF*(ψ).
(b) $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*) \vdash \exists y\, \psi(0, y) \wedge \forall x\, [\exists y\, \psi(x, y) \to \exists y\, \psi(x+1, y)]$.
2. *A theory* $\mathbf{T}$ *is inductive Π_n–functional if there is an inductive Π_n–functional class Γ such that* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Sigma_n + \Gamma^*$. *(In this case we say that Γ is an inductive Π_n–functional class for* $\mathbf{T}$*).*
3. *Let $\varphi(u, x, y) \in \Pi_n^-$ be a Π_n–envelope. We say that $\varphi(u, x, y)$ is an inductive Π_n–envelope if $\Gamma_\varphi = \{\varphi(k, x, y) : k \in \omega\}$ is an inductive Π_n–functional class.*

Remark 1.5. Part *(b)* of the definition of inductive Π_n–functional class contains the premises of the induction rule for the formula $\exists y\, \psi(x, y)$. This shows again the relationship between the two topics we are interested in here. By the next proposition, inductive Π_n–functional classes characterize the Π_n–functional theories such that $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ is equivalent to $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$.

Proposition 1.6. *Let* $\mathbf{T}$ *be a Π_n–functional theory. The following properties are equivalent:*

1. *Every Π_n–functional class for* $\mathbf{T}$ *is inductive.*
2. $\mathbf{T}$ *is an inductive Π_n–functional theory.*
3. $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \iff \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$.

Proof. It is trivial that *(1)* $\Longrightarrow$ *(2)*.
(*(2)* $\Longrightarrow$ *(3)*): Let Γ be an inductive Π_n–functional class for $\mathbf{T}$. Then

1.6.1. $\mathbf{I}\Sigma_n + \Gamma^* \iff \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*)$.

Proof. ($\Longrightarrow$): This follows from [7]–**3.4**.
($\Longleftarrow$): By part *(1.a)* of **1.4**, it is enough to prove that for each $\varphi(x, y) \in \Gamma$, $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*) \vdash \forall x\, \exists y\, \varphi(x, y)$. Since $\mathbf{I}\Sigma_n + \Gamma^* \vdash \forall x\, \exists y\, \varphi(x, y)$, then $\exists y\, \varphi(x, y) \in \Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*)$. So, $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*) \vdash \mathbf{I}_{\exists y\, \varphi(x,y)}$. Hence, by part *(1.b)* of **1.4** we get the result. □

By **1.6.1**, we obtain *(3)* as follows

$$\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \iff \mathbf{I}\Sigma_n + \Gamma^* \iff \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*) \iff \mathbf{I}\Delta_{n+1}(\mathbf{T}).$$

(*(3)* $\Longrightarrow$ *(1)*): Let Γ be a Π_n–functional class for $\mathbf{T}$. For every $\varphi(x, y) \in \Gamma$, IPF(φ), $\exists y\, \varphi(0, y)$ and $\forall x\, [\exists y\, \varphi(x, y) \to \exists y\, \varphi(x+1, y)]$ are Π_{n+2}–formulas that are provable in $\mathbf{T}$. So, by *(3)*, they are also provable in $\mathbf{I}\Delta_{n+1}(\mathbf{T})$; hence, also in $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*)$. □

Remark 1.7. Now, in connection with question **(P2)**, we present some inductive Π_0–functional classes and the recursive functions they describe.

Elementary recursive functions. Let us first recall some basic facts on the exponential function. Let **exp** be the sentence $\forall x\, \exists y\, (2^x = y)$, where $2^x = y$ denotes a Δ_0 formula which defines the exponential function in the standard model and such that (see [8]):

(1) $\mathbf{I}\Delta_0 \vdash 2^{x_1} = y_1 \wedge 2^{x_2} = y_2 \wedge x_1 \leq x_2 \rightarrow y_1 \leq y_2$.
(2) $\mathbf{I}\Delta_0 \vdash 2^0 = 1$.
(3) $\mathbf{I}\Delta_0 \vdash 2^x = y \leftrightarrow \exists z\, [2^{x+1} = z \wedge 2 \cdot y = z]$.

From (1)–(3) and **1.6.1** we have that

1.7.1. (i) $\{2^x = y\}$ is an inductive Π_0–functional class.
(ii) $\mathbf{I}\Delta_0 + \mathbf{exp} \iff \mathbf{I}\Delta_1(\mathbf{I}\Delta_0 + \mathbf{exp})$.

By **1.7.1**–*(ii)*, if $\mathbf{I}\Delta_0$ is extended with axioms asserting that every elementary recursive function is total then we obtain induction for every $\Delta_1(\mathbf{I}\Delta_0 + \mathbf{exp})$–formula. It also holds that each elementary recursive set is definable by such a formula. Let us also observe that $\mathbf{I}\Delta_1(\mathbf{I}\Delta_0 + \mathbf{exp})$ is finitely axiomatizable.

Primitive recursive functions. In **4.3** we shall define a sequence of functions: $F_0(x) = (x+1)^2$, $F_{k+1}(x) = F_k^{x+2}(x+1)$. Let $F : \omega^2 \longrightarrow \omega$ be the function defined by: $F(k, m) = F_k(m)$ (F is essentially Ackermann's function). In section **5** (see also [1] or [18]) it will be proved that there exists $\varphi(u, x, y) \in \Delta_0$ such that

1.7.2. (i) $\varphi(u, x, y)$ is an inductive (strong) Π_0–envelope of $\mathbf{I}\Sigma_1$ in $\mathbf{I}\Delta_0$.
(ii) *For each $k \in \omega$, and for all $m, r \in \omega$, $F_k(m) = r \iff \mathcal{N} \models \varphi(k, m, r)$.*

Let $\Gamma_{\text{Ack}} = \{\varphi(k, x, y) : k \in \omega\}$. It holds that:

1.7.3. (i) $\mathbf{Th}_{\Pi_2}(\mathbf{I}\Sigma_1) \iff \mathbf{I}\Delta_0 + \Gamma^*_{\text{Ack}} \iff \mathbf{I}\Delta_1(\mathbf{I}\Sigma_1)$.
(ii) $\mathbf{I}\Delta_1(\mathbf{I}\Sigma_1)$ is Π_2–axiomatizable.
(iii) $\mathbf{I}\Sigma_1$ *and* $\mathbf{I}\Delta_1(\mathbf{I}\Sigma_1)$ *have the same class of recursive functions.*

Proof. (i) follows from **1.6** and **1.7.2**. (ii) and *(iii)* follow from *(i)*. □

By **1.7.3**–*(i)*, if we add to $\mathbf{I}\Delta_0$ axioms expressing that each primitive recursive function is total, then we obtain induction for every $\Delta_1(\mathbf{I}\Sigma_1)$–formula. Moreover, each primitive recursive set is definable by such a formula. But, $\mathbf{I}\Delta_1(\mathbf{I}\Sigma_1)$ is not finitely axiomatizable (see **5.4**).

Grzegorczyk's hierarchy, $\mathcal{E}^k$, $k \geq 3$. For each level of Grzegorczyk's hierarchy, $\mathcal{E}^k$, $k \geq 3$, (see [17]) we have a similar result using the theory $\mathbf{I}\Delta_0 + \forall x\, \exists y\, [\mathbb{F}_{0,k-2}(x) = y]$ (see **4.6**). So, if $\mathbf{I}\Delta_0$ is extended with axioms asserting that each function in $\mathcal{E}^k$ is total, then we obtain induction for every $\Delta_1(\mathbf{I}\Delta_0 + \forall x\, \exists y\, [\mathbb{F}_{0,k-2}(x) = y])$–formula.

As it is well known, $\mathcal{R}(\mathbf{I}\Delta_0) = \mathcal{M}^2$ (see [19]). Let us consider the classes $\mathcal{M}^2$, $\mathcal{E}^k$, $k \geq 3$ and $\mathcal{PR}$ (primitive recursive functions). As we have seen, these classes satisfy problem **(P2)**. In section **2**, we shall see that **(P2)** holds for any class $\mathcal{C}$ of nondecreasing provably recursive functions of $\mathbf{I}\Sigma_{n+1}$.

We conclude this section presenting the main results that will be obtained through this paper. Next theorem sums up the results on axiomatizations properties of $\mathbf{I}\Delta_{n+1}(\mathbf{T})$.

Theorem 1.8. (see 2.4, 5.4)

1. *Let* $\mathbf{T}$ *be a theory.*
   (a) $(n \geq 1)$ $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ *is not* Σ_{n+2}*–axiomatizable.*
   (b) *If* $\mathbf{I}\Sigma_{n+1} \not\Longrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$*, then* $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ *is* Π_{n+3} *axiomatizable but it is not* Σ_{n+3} *axiomatizable.*
   (c) *Assume that* $\mathbf{T}$ *has* Δ_{n+1}*–induction. If* $\mathbf{I}\Sigma_{n+1} \Longrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$*, then* $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ *is* Π_{n+2} *axiomatizable. Even more,*

$$\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T}).$$

2. *If* $\mathbf{T}$ *is a consistent extension of* $\mathbf{I}\Sigma_{n+1}$*, then* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ *and* $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ *are not finitely axiomatizable.*

Part *(1)* of the above theorem is proved in section **2** through a result on (non-existence of) Σ_{n+3}–axiomatizable extensions of $\mathbf{I}\Sigma_{n+1}$.

As it was noted in **1.5**, inductive Π_n–functional classes relates quantifier complexity and induction rules. In sections **3** and **4** we develop the basic tools (following a construction due to R. Kaye (see [9])) to obtain explicitly inductive Π_n–functional classes. Given a formula $\varphi(x, y) \in \Pi_n$, defining a total function, by iteration and diagonalization, we define uniformily a family of functions $\mathbb{A}_{\varphi,u}(x) = y$. When the function defined by $\varphi(x, y)$ has a good rate of growth, the above family of functions is a Π_n–envelope of $\mathbf{I}\Sigma_{n+1}^{\varphi,n}$ in $\mathbf{I}\Sigma_n^{\varphi,n}$ (where $\mathbf{I}\Sigma_m^{\varphi,n}$ is a finite extension of $\mathbf{I}\Sigma_m$ asserting that $\varphi(x, y)$ has good properties of growth). If $\mathbf{I}\Sigma_n$ extends $\mathbf{I}\Sigma_n^{\varphi,n}$, then $\mathbb{A}_{\varphi,u}(x) = y$ is an inductive Π_n–envelope.

The theories $\mathbf{I}\Sigma_n^{\varphi}$ give every finite Π_n–functional extension of $\mathbf{I}\Sigma_n$. As an application of these techniques we get part *(2)* of **1.8** and a general version of Parsons' conservativeness theorem. Next theorem sums up the main properties connected with Parsons' theorem.

Theorem 1.9. (see 6.3, 6.4, 6.5)

1. *For all* $k \in \omega$*,*

$$[\mathbf{I}\Sigma_n^{\varphi}, \Pi_{n+2}\text{–}IR]_k \Longleftrightarrow [\mathbf{I}\Sigma_n^{\varphi}, \Sigma_{n+1}\text{–}IR]_k \Longleftrightarrow \mathbf{I}\Sigma_n^{\varphi} + \forall x\, \exists y\, (\mathbb{F}_{\varphi,k}(x) = y)$$

2. $\mathbf{I}\Sigma_n^{\varphi} + \Sigma_{n+1}\text{–}IR \Longleftrightarrow \mathbf{I}\Sigma_n^{\varphi} + \mathbf{ACK}_{\varphi}^{*}$.
3. $\mathbf{I}\Sigma_{n+1}^{\varphi}$ *is a* Π_{n+2}*–conservative extension of* $\mathbf{I}\Sigma_n^{\varphi} + \Sigma_{n+1}\text{–}IR$.
4. *(Parsons)* $\mathbf{I}\Sigma_{n+1}$ *is a* Π_{n+2}*–conservative extension of* $\mathbf{I}\Delta_0 + \Sigma_{n+1}\text{–}IR$.

We conclude by giving a proof, for Π_n–functional theories, of a result of Beklemishev on unnested applications of Σ_{n+1} and Π_{n+2}–induction rules (see [2], corollary **9.1**).

Theorem 1.10. (see 6.7) *Let* $\mathbf{T}$ *be* Π_{n+2}*–axiomatizable extension of* $\mathbf{I}\Sigma_n$*. If* $\mathbf{T}$ *is* Π_n*–functional, then* $[\mathbf{T}, \Sigma_{n+1}\text{–}IR] \Longleftrightarrow [\mathbf{T}, \Pi_{n+2}\text{–}IR]$.

2. Quantifier complexity of Δ_{n+1}(T)–induction

The aim of this section is to prove theorem **1.8**–*(1)* (see also [6]). To this end we first study Σ_{n+3} extensions of $\mathbf{I\Sigma}_{n+1}$. Next lemma is a generalization of a result of D. Leivant (see [14]), and it is used in [7] to prove **3.7.4**.

Lemma 2.1. *Let* $\mathbf{T}$ *be a consistent and* Σ_{n+3} *axiomatizable theory. Then* $\mathbf{T} \not\Longrightarrow \mathbf{I\Sigma}_{n+1}$.

Proof. Assume towards a contradiction that $\mathbf{T} \Longrightarrow \mathbf{I\Sigma}_{n+1}$. Since $\mathbf{I\Sigma}_{n+1}$ is finitely axiomatizable, there exists a sentence $\varphi \in \Sigma_{n+3}$ such that $\mathbf{T} \vdash \varphi$ and $\varphi \Longrightarrow \mathbf{I\Sigma}_{n+1}$. Let $\theta(x) \in \Pi^-_{n+2}$ such that $\varphi \equiv \exists x\, \theta(x)$. Let $\mathfrak{A} \models \mathbf{T}$ nonstandard. Since $\mathbf{T} \vdash \varphi$, there exists $a \in \mathfrak{A}$ such that $\mathfrak{A} \models \theta(a)$. Let $b \in \mathfrak{A}$ nonstandard and $c = \langle a, b\rangle$. We have that

2.1.1. $\mathcal{K}_{n+1}(\mathfrak{A}, c) \models \varphi$.

Proof. Since $\theta(x) \in \Pi_{n+2}$, $a \in \mathcal{K}_{n+1}(\mathfrak{A}, c)$, $\mathcal{K}_{n+1}(\mathfrak{A}, c) \prec_{n+1} \mathfrak{A}$ and $\mathfrak{A} \models \theta(a)$, then $\mathcal{K}_{n+1}(\mathfrak{A}, c) \models \theta(a)$; hence, $\mathcal{K}_{n+1}(\mathfrak{A}, c) \models \varphi$. □

As φ extends $\mathbf{I\Sigma}_{n+1}$, by **2.1.1**, $\mathcal{K}_{n+1}(\mathfrak{A}, c) \models \mathbf{I\Sigma}_{n+1}$. Since $\mathcal{K}_{n+1}(\mathfrak{A}, c)$ is nonstandard, this gives the desired contradiction. □

Theorem 2.2. *If* $\mathfrak{A} \not\models \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ *and* $\mathfrak{A} \models \mathbf{I\Delta}_{n+1}(\mathbf{T})$*, then* $\mathfrak{A} \models \mathbf{I\Sigma}_{n+1}$.

Proof. Let us see that $\mathfrak{A} \models \mathbf{I\Pi}_{n+1}$. Let $\varphi(x, v) \in \Pi_{n+1}$ and $a \in \mathfrak{A}$ such that

(1) $\mathfrak{A} \models \varphi(0, a)$, and $\mathfrak{A} \models \varphi(x, a) \to \varphi(x + 1, a)$.

Let us see that $\mathfrak{A} \models \forall x\, \varphi(x, a)$. Since $\mathfrak{A} \not\models \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$, there exists $\theta(w) \in \Pi^-_{n+1}$ such that $\mathbf{T} \vdash \neg\exists w\, \theta(w)$, and $\mathfrak{A} \models \exists w\, \theta(w)$; so, there exists $b \in \mathfrak{A}$ such that $\mathfrak{A} \models \theta(b)$.

Let $\delta(x, v, w) \in \Pi_{n+1}$ be the following formula $\theta(w) \wedge \varphi(x, v)$. By (1), $\mathfrak{A} \models \delta(0, a, b)$, and $\mathfrak{A} \models \delta(x, a, b) \to \delta(x + 1, a, b)$. Since $\mathbf{T} \vdash \neg\delta(x, v, w)$, then $\delta(x, v, w) \in \Delta^*_{n+1}(\mathbf{T})$. As $\mathfrak{A} \models \mathbf{I\Delta}^*_{n+1}(\mathbf{T})$, it follows that $\mathfrak{A} \models \forall x\, \delta(x, a, b)$; hence, $\mathfrak{A} \models \forall x\, \varphi(x, a)$. □

Theorem 2.3. *Let* $\mathbf{T}$ *be a theory with* Δ_{n+1}*–induction. The following conditions are equivalent.*

1. $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \Longleftrightarrow \mathbf{I\Delta}_{n+1}(\mathbf{T})$.
2. $\mathbf{I\Delta}_{n+1}(\mathbf{T})$ *is* Π_{n+2} *axiomatizable.*
3. $\mathbf{I\Delta}_{n+1}(\mathbf{T})$ *is* Σ_{n+3} *axiomatizable.*
4. $\mathbf{I\Sigma}_{n+1} \Longrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$.

Proof. (*(1)* $\Longrightarrow$ *(2)* $\Longrightarrow$ *(3)*): Trivial.
(*(3)* $\Longrightarrow$ *(1)*): Assume towards a contradiction that *(1)* does not hold. Since $\mathbf{T}$ has Δ_{n+1}–induction, then $\mathbf{I\Delta}_{n+1}(\mathbf{T}) \not\Longleftrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$. Hence, there exists $\theta \in \Pi_{n+2}$ such that $\mathbf{T} \vdash \theta$, and $\mathbf{I\Delta}_{n+1}(\mathbf{T}) \nvdash \theta$. Then, by **2.2**, we get that $\mathbf{I\Delta}_{n+1}(\mathbf{T}) + \neg\theta \Longrightarrow \mathbf{I\Sigma}_{n+1}$. So, $\mathbf{I\Delta}_{n+1}(\mathbf{T}) + \neg\theta$ is a consistent extension of $\mathbf{I\Sigma}_{n+1}$ and, by *(3)*, Σ_{n+3} axiomatizable. Which contradicts **2.1**.

((1) $\Longrightarrow$ (4)): Since $\mathbf{I}\Sigma_{n+1} \Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$, the result follows from (1).
((4) $\Longrightarrow$ (1)): As $\mathbf{T}$ has Δ_{n+1}–induction, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$. For the converse, assume towards a contradiction that $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \not\Longrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$. Then there exists $\mathfrak{A}$ such that $\mathfrak{A} \models \mathbf{I}\Delta_{n+1}(\mathbf{T})$, and $\mathfrak{A} \not\models \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$. Then, by **2.2**, $\mathfrak{A} \models \mathbf{I}\Sigma_{n+1}$. So, by (4), $\mathfrak{A} \models \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$, contradiction. □

Theorem 2.4. *Let* $\mathbf{T}$ *be a theory.*

1. $(n \geq 1)$ $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ *is not* Σ_{n+2}*–axiomatizable.*
2. *If* $\mathbf{I}\Sigma_{n+1} \not\Longrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$*, then* $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ *is* Π_{n+3} *axiomatizable but it is not* Σ_{n+3} *axiomatizable.*
3. *Assume that* $\mathbf{T}$ *has* Δ_{n+1}*–induction. If* $\mathbf{I}\Sigma_{n+1} \Longrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$*, then* $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ *is* Π_{n+2} *axiomatizable; even more,*

$$\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{Th}_{n+2}(\mathbf{T})$$

Proof. ((1)): Since $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{I}\Sigma_n$, the result follows from **2.1**.
((2)): It is obvious that $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ is Π_{n+3}–axiomatizable. Moreover, by the hypothesis, $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \not\Longrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$. Then, as in the proof of (3)$\Longrightarrow$(1) in **2.3**, (which now does not need the asumption that $\mathbf{T}$ has Δ_{n+1}–induction) we get that $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ is not a Σ_{n+3} axiomatizable theory.
((3)): It is a consequence of **2.3**. □

Remark 2.5. **(On Σ_2–axiomatization)**. From **2.4**–(1), $\mathbf{I}\Delta_{n+1}(\mathbf{T})$, $n \geq 1$, is not Σ_{n+2}–axiomatizable. For $n = 0$, there exist theories (for instance, $\mathbf{I}\Delta_0$) such that $\mathbf{I}\Delta_1(\mathbf{T})$ is Σ_2–axiomatizable (indeed Π_1–axiomatizable). Next result gives theories such that $\mathbf{I}\Delta_1(\mathbf{T})$ is not Σ_2–axiomatizable.

2.5.1. Let $\mathbf{T}$ be a Σ_2–axiomatizable extension of $\mathbf{I}\Delta_0$ and $\varphi(x, y) \in \Delta_0$ such that $\mathbf{T} \vdash \forall x\, \exists y\, \varphi(x, y)$. Then there exists a term $t(x)$ such that

$$\mathbf{T} \vdash \exists u\, \forall x\, [u < x \rightarrow \exists y \leq t(x)\, \varphi(x, y)]$$

Proof. Since Σ_2 is closed under conjunction, if $\mathbf{T} \vdash \forall x\, \exists y\, \varphi(x, y)$ then there exists $\psi \in \Sigma_2$ such that $\mathbf{T} \vdash \psi$ and $\mathbf{I}\Delta_0 + \psi \vdash \forall x\, \exists y\, \varphi(x, y)$. Let $\delta(x) \in \Pi_1$ such that ψ is $\exists x\, \delta(x)$. By way of contradiction assume that for each term $t(x)$ of $\mathcal{L}$, $\mathbf{T} \not\vdash \exists u\, \forall x\, [u < x \rightarrow \exists y \leq t(x)\, \varphi(x, y)]$. Let $\mathbf{c}$ and $\mathbf{d}$ be new constants symbols and $\mathbf{T}'$ the theory

$$\mathbf{T} + \delta(\mathbf{c}) + \mathbf{c} < \mathbf{d} + \{\neg\exists y \leq t(\mathbf{d})\, \varphi(\mathbf{d}, y) : \ t(x) \text{ term of } \mathcal{L}\}$$

By compactness, $\mathbf{T}'$ is consistent. Let $\mathfrak{A}$ be a model of $\mathbf{T}'$; a and b, respectively, the interpretations of $\mathbf{c}$ and $\mathbf{d}$ in $\mathfrak{A}$ and $\mathfrak{B}$ the initial segment defined in $\mathfrak{A}$ by $\{t(b) : \ t(x) \text{ term of } \mathcal{L}\}$. Then $\mathfrak{B} \prec_0 \mathfrak{A}$. So, as $a < b$, $\mathfrak{B} \models \mathbf{I}\Delta_0 + \psi$, which contradicts $\mathfrak{B} \not\models \forall x\, \exists y\, \varphi(x, y)$. □

This result generalizes a similar property on $\mathbf{I}\Pi_1^-$ obtained in [5]. The proof we have presented here can be used to obtain the following result for Σ_{n+2}–axiomatizable theories $(n \geq 1)$.

2.5.2. ($n \geq 1$) Let $\varphi(u, x, y)$ be a strong Π_n–envelope of $\mathbf{I}\Sigma_n$ in $\mathbf{I}\Sigma_n$. Let $\mathbf{T}$ be a Σ_{n+2}–axiomatizable theory and $\psi(x, y) \in \Sigma_{n+1}$ such that $\mathbf{B}\Sigma_{n+1} + \mathbf{T} \vdash \forall x \, \exists y \, \psi(x, y)$ then there exists a term $t(x)$ of $\mathcal{L}(\Gamma_\varphi)$ such that

$$(\mathbf{T} + \mathbf{I}\Sigma_n)_{\Gamma_\varphi} \vdash \exists u \, \forall x \, [u < x \rightarrow \exists y \leq t(x) \, \psi(x, y)]$$

By **2.5.1**, if $\mathbf{T}$ is a Σ_2–axiomatizable sound theory, then every function in $\mathcal{R}(\mathbf{T})$ is bounded by a polynomial. From this we get that

2.5.3. Let $\mathbf{T}$ be an extension of $\mathbf{I}\Delta_0$ such that $\mathcal{N} \models \mathbf{T}$. If there exists $f \in \mathcal{R}(\mathbf{T})$ not bounded by a polynomial then $\mathbf{T}$ is not Σ_2–axiomatizable.

2.5.4. If $\mathbf{T} \vdash \mathbf{exp}$ then $\mathbf{I}\Delta_1(\mathbf{T})$ is not Σ_2–axiomatizable.

Proof. Let $2^x = y$ be a Δ_0 formula as in **1.7**. Since $\mathbf{T} \vdash \forall x \, \exists y \, (2^x = y)$, then $\exists y \, (2^x = y)$ is a $\Delta_1(\mathbf{T})$–formula. Hence, $\mathbf{I}\Delta_1(\mathbf{T}) \vdash \forall x \, \exists y \, (2^x = y)$; so, as $\mathbf{I}\Delta_1(\mathbf{T})$ is a sound theory, the result follows from **2.5.3**. □

Remark 2.6. Let us see how we can answer **(P2)** using the above results. Let $\mathcal{C}$ be a class of nondecreasing provably recursive functions of $\mathbf{I}\Sigma_{n+1}$. Assume that for each $f \in \mathcal{C}$ there exists a formula $\varphi_f(x, y) \in \Pi_0$ defining f in $\mathcal{N}$ and such that $\mathbf{I}\Sigma_{n+1} \vdash \forall x \, \exists! y \, \varphi_f(x, y)$.

Let $\Gamma = \{\varphi_f(x, y) : f \in \mathcal{C}\}$. Then, by **2.4**–*(3)*,

$$\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*) \Longleftrightarrow \mathbf{I}\Sigma_n + \Gamma^*$$

3. Ackermann's functions

In this section we give a generalization of Ackermann's function. Similar constructions have been considered by P. D'Aquino (see [1]), R. Kaye (see [9]) and R. Sommer (see [18]). The aim of the definition we develop here is to describe inductive Π_n–functional subtheories of $\mathbf{I}\Sigma_{n+1}$. To this end, the construction proceeds using iteration and diagonalization as in Grzegorczyk's Hierarchy.

Remark 3.1. **(Set Theory in $\mathbf{I}\Delta_0$)** Here we shall see how set theory can be described in $\mathbf{I}\Delta_0$. We shall informally give a Δ_0 formula, denoted by $x \in u$, such that in each model of $\mathbf{I}\Delta_0$ some of its elements can be considered as finite sets. See [15] for details. Let us consider the following Δ_0–formulas (where $y|x$ is the formula $\exists z \leq x \, (y \cdot z = x)$),

$$\begin{aligned} \text{irred}(x) &\equiv 2 \leq x \wedge \forall y \leq x \, (y|x \rightarrow y = 1 \vee y = x), \\ \text{pot}_2(x) &\equiv 1 \leq x \wedge \forall u \leq x \, (\text{irred}(u) \wedge u|x \rightarrow u = 2), \\ \text{pot}_4(x) &\equiv \text{pot}_2(x) \wedge \exists y \leq x \, (\text{pot}_2(y) \wedge y \cdot y = x). \end{aligned}$$

And $\text{Lp}_2(x) = y$ and $\text{Lp}_4(x) = y$, respectively, are the formulas

$$[x = 0 \wedge y = 1] \vee [x < y \leq 2 \cdot x \wedge \text{pot}_2(y) \wedge \forall z < y \, (\text{pot}_2(z) \rightarrow z \leq x)]$$

$$[x = 0 \wedge y = 1] \vee [x < y \leq 4 \cdot x \wedge \text{pot}_4(y) \wedge \forall z < y \, (\text{pot}_4(z) \rightarrow z \leq x)]$$

3.1.1. (i) $\mathbf{I}\Delta_0 \vdash \forall x \exists! y\, (\mathrm{Lp}_2(x) = y) \wedge \forall x\, \exists! y\, (\mathrm{Lp}_4(x) = y)$.
(ii) $\mathbf{I}\Delta_0 \vdash 1 \leq x \rightarrow \mathrm{Lp}_2(x) \leq 2 \cdot x \wedge \mathrm{Lp}_4(x) \leq 4 \cdot x$.

Formula $x \in u$ is given using the formulas $\mathrm{pot}_2(v)$ and $\mathrm{pot}_4(v)$. We say that $x \in u$ if

(–) x written in base 2, as a sequence of 0, 1, appears in u written in base 4, as a sequence of 0, 1, 2, 3, between two consecutive occurrences of 2.

Now we give without proofs some basic properties of the formula $x \in u$. Let $\mathrm{Conj}(u) \in \Delta_0$ be the formula (we read $\mathrm{Conj}(u)$ as "u is a set")

$$\neg \exists v < u\, \forall x < u\, (x \in u \leftrightarrow x \in v)$$

3.1.2. (i) $\mathbf{I}\Delta_0 \vdash x \in u \rightarrow x < u$.
(ii) $\mathbf{I}\Delta_0 \vdash x \in u \rightarrow \exists v\, [2 \cdot v < u \wedge \forall y\, (y \neq x \wedge y \in u \rightarrow y \in v)]$.
(iii) $\mathbf{I}\Delta_0 \vdash \mathrm{Conj}(0) \wedge \forall x\, (x \notin 0)$.
(iv) $\mathbf{I}\Delta_0 \vdash \mathrm{Conj}(u) \wedge \mathrm{Conj}(v) \wedge \forall x\, (x \in u \leftrightarrow x \in v) \rightarrow u = v$.

3.1.3. (Σ_n–separation). Let $\varphi(x) \in \Sigma_n \cup \Pi_n$. Then

$$\mathbf{I}\Sigma_n \vdash \forall y\, \exists z \leq y\, [\mathrm{Conj}(z) \wedge \forall x\, (x \in z \leftrightarrow x \in y \wedge \varphi(x))]$$

Let $\{x\} = z$ be the Δ_0 formula: $\mathrm{Conj}(z) \wedge \forall y < z\, (y \in z \leftrightarrow y = x)$.
Let $x \cup y = z$ be the Δ_0 formula: $\mathrm{Conj}(z) \wedge \forall u < z\, [u \in z \leftrightarrow u \in x \vee u \in y]$.

3.1.4. (i) $\mathbf{I}\Delta_0 \vdash \forall x\, \exists! y \leq (6 \cdot \mathrm{Lp}_2(x))^2\, [\{x\} = y]$.
(ii) $\mathbf{I}\Delta_0 \vdash \forall x\, \forall y\, \exists! z \leq x + y \cdot \mathrm{Lp}_4(x)\, [x \cup y = z]$.

3.1. Iteration: $IT_\varphi(z, x, y)$

Remark 3.2. In what follows we consider a theory $\mathbf{T}$, extension of $\mathbf{I}\Sigma_n$, and $\varphi(x, y) \in \Pi_n^-$ such that

(1) $\mathbf{T} \vdash \mathrm{IPF}(\varphi(x, y))$, and
(2) $\mathbf{T} \vdash \varphi(x, y) \rightarrow x^2 < y$.

That is, $\varphi(x, y)$ defines in $\mathbf{T}$ a partial increasing function bigger, when defined, than the square. It is easy to see that

3.2.1. $\mathbf{T} \vdash \varphi(x, y) \rightarrow (x + 1)^3 < (y + 1)^2$.

Informally, we denote $\varphi(x, y)$ by $F_\varphi(x) = y$. In the next results we are going to prove in $\mathbf{T}$ some properties by induction, it will be easy to verify in each case that $\mathbf{T}$ proves enough induction to carry on the argument. We will use Cantor's function, $J : \omega^2 \longrightarrow \omega$, defined by

$$J(x, y) = z \quad \equiv \quad (x + y) \cdot (x + y + 1) + 2 \cdot x = 2 \cdot z$$

Definition 3.3. *Let $itcl_\varphi(w, z, x, y) \in \Pi_n$ (in $\mathbf{B}\Sigma_n$ for $n \geq 1$) be*

$$\begin{cases} J(z, y) \in w \wedge J(0, x) \in w \wedge \\ \forall z', y' < w \begin{cases} J(z', y') \in w \rightarrow \begin{cases} (z' = 0 \wedge y' = x) \vee \\ 0 < z' \wedge \exists v < w \begin{cases} \varphi(v, y') \wedge \\ J(z' - 1, v) \in w \end{cases} \end{cases} \end{cases} \end{cases}$$

Remark 3.4. The formula $\mathrm{itcl}_\varphi(w, z, x, y)$ expresses that w is a "computation" of $F^z_\varphi(x) = y$. The following properties are provable in $\mathbf{T}$.

(i) $\mathrm{itcl}_\varphi(w, z, x, y) \to (z = 0 \to x = y) \wedge (z = 1 \to \varphi(x, y))$.
(ii) $\mathrm{itcl}_\varphi(w, z + 1, x, y) \to \exists y' < w\, (\mathrm{itcl}_\varphi(w, z, x, y') \wedge \varphi(y', y))$.
(iii) $\mathrm{itcl}_\varphi(w, z, x, y) \to z,\, x \leq y \leq w \wedge (z \neq 0 \to x^2 < y)$.
(iv) $\mathrm{itcl}_\varphi(w, z, x, y) \to J(z, y) \leq 4 \cdot y^2$.
(v) $\mathrm{itcl}_\varphi(w_1, z, x, y_1) \wedge \mathrm{itcl}_\varphi(w_2, z, x, y_2) \to y_1 = y_2$.

Theorem 3.5. $\mathbf{T} \vdash itcl_\varphi(w, z, x, y) \to \exists w' \leq 9 \cdot 4^3 \cdot (y + 1)^{54}\, itcl_\varphi(w', z, x, y)$.

Proof. Let $\mathfrak{A} \models \mathbf{T}$ and $a, c \in \mathfrak{A}$. By induction we shall see that for all $b \in \mathfrak{A}$

$$\text{(I) } \forall y < c\, [\mathrm{itcl}_\varphi(c, b, a, y) \to \exists w' \leq 9 \cdot 4^3 \cdot (y + 1)^{54}\, \mathrm{itcl}_\varphi(w', b, a, y)]$$

$(b = 0)$: Suppose that $\mathrm{itcl}_\varphi(c, 0, a, d)$. Then $d = a$. Let $c' = \{J(0, a)\}$. Then

$$\begin{aligned} c' &\leq 36 \cdot (\mathrm{Lp}_2(J(0, a)))^2 && [\![\mathbf{3.1.4}\text{–}(i)]\!] \\ &\leq 36 \cdot (\mathrm{Lp}_2((a + 1)^2))^2 && [\![J(0, a) \leq (a + 1)^2]\!] \\ &\leq 36 \cdot (2 \cdot (a + 1)^2)^2 && [\![\mathbf{3.1.1}\text{–}(ii)]\!] \\ &\leq 9 \cdot 4^3 \cdot (d + 1)^{54} && [\![d = a]\!] \end{aligned}$$

We also have that $\mathrm{itcl}_\varphi(c', 0, a, d)$. This proves (I) for $b = 0$.
$(b \to b + 1)$: Suppose that $\mathrm{itcl}_\varphi(c, b + 1, a, d)$. Then there is $d_0 < c$ such that $\varphi(d_0, d)$ and $\mathrm{itcl}_\varphi(c, b, a, d_0)$. By induction hypothesis, there exists $c_0 \leq 9 \cdot 4^3 \cdot (d_0+1)^{54}$ such that $\mathrm{itcl}_\varphi(c_0, b, a, d_0)$. Let $c' = c_0 \cup \{J(b+1, d)\}$. Then $\mathrm{itcl}_\varphi(c', b+1, a, d)$. We also have that

$$\begin{aligned} \{J(b + 1, d)\} &\leq 36 \cdot (\mathrm{Lp}_2(J(b + 1, d)))^2 && [\![\mathbf{3.1.4}\text{–}(i)]\!] \\ &\leq 36 \cdot (2 - \cdot J(b + 1, d))^2) && [\![\mathbf{3.1.1}\text{–}(ii)]\!] \\ &\leq 36 \cdot 4 \cdot (4 \cdot d^2)^2 && [\![\mathbf{3.4}\text{–}(iv),\ (J(b + 1, d) \in c)]\!] \\ &< 36 \cdot 4^3 \cdot (d + 1)^4 \end{aligned}$$

Hence, by **3.1.4**–*(ii)*, $c' \leq 9 \cdot 4^3 \cdot (d_0 + 1)^{54} \cdot 2^{14} \cdot (d + 1)^4$. By **3.2.1**, $(d_0 + 1)^3 < (d + 1)^2$. So, $c' \leq 9 \cdot 4^3 \cdot (d + 1)^{54}$.

This proves (I) for all b. So, the result follows. □

Definition 3.6. *Let us consider the* Π_n *formulas (in* $\mathbf{B}\Sigma_n$ *for* $n \geq 1$*)*

$$\begin{aligned} IT_\varphi(z, x, y) &\equiv \exists w \leq 9 \cdot 4^3 \cdot (y + 1)^{54}\, itcl_\varphi(w, z, x, y), \\ D_\varphi(x, y) &\equiv IT_\varphi(x + 2, x + 1, y) \end{aligned}$$

(The formula $IT_\varphi(z, x, y)$ *expresses that* $F^z_\varphi(x) = y$*).*

By straightforward arguments, using induction, it is proved that

Lemma 3.7. *1.* $\mathbf{T} \vdash \forall x\, \forall y\, [IT_\varphi(0, x, y) \leftrightarrow x = y]$.
2. $\mathbf{T} \vdash \forall x\, \forall y\, [\varphi(x, y) \leftrightarrow IT_\varphi(1, x, y)]$.
3. $\mathbf{T} \vdash IT_\varphi(z + 1, x, y) \leftrightarrow \exists y_0 \leq y\, [IT_\varphi(z, x, y_0) \wedge \varphi(y_0, y)]$.
4. $\mathbf{T} \vdash IT_\varphi(z, x, y_1) \wedge IT_\varphi(z, x, y_2) \to y_1 = y_2$.
5. $\mathbf{T} \vdash IT_\varphi(z, x, y) \to \forall z_0 < z\, \exists y_0 < y\, [IT_\varphi(z_0, x, y_0)]$.
6. $\mathbf{T} + \forall x\, \exists y\, \varphi(x, y) \vdash IT_\varphi(z, x, y) \to \exists y'\, IT_\varphi(z + 1, x, y')$.
7. $\mathbf{T} \vdash D_\varphi(x, y) \to x^2 < y \wedge \exists z < y\, \varphi(x, z)$.
8. $\mathbf{T} \vdash x_1 \leq x_2 \wedge D_\varphi(x_1, y_1) \wedge D_\varphi(x_2, y_2) \to y_1 \leq y_2$.

3.2. Ackermann's finite approximations: $Ack_\varphi(w, u, z, x, y)$

In what follows we shall use the following notation

(–) $\langle u, z, x, y\rangle$ for $J(J(u, x), J(z, y))$.
(–) Let $[w, x] = w'$ be the Δ_0–formula

$$w' = (\mu v)[\forall v' < w\,(v' \in v \leftrightarrow J(J(0, x), v') \in w \vee v' = J(0, x))]$$

That is, $[w, x] = \{J(z, y) : \langle 0, z, x, y\rangle \in w\} \cup \{J(0, x)\}$.
Observe that $\mathbf{I}\Delta_0 \vdash \forall x, w\, \exists! w'\,([w, x] = w')$.

Definition 3.8. *Let $Ack_\varphi(w, u, z, x, y) \in \Pi_n$ (in $\mathbf{B}\Sigma_n$ for $n \geq 1$) be*

$$\langle u, z, x, y\rangle \in w \wedge 1 \leq z \wedge$$
$$\forall u', z', x', y' < w \left\{ \begin{array}{l} \langle u', z', x', y'\rangle \in w \;\rightarrow \\ \rightarrow \left\{ \begin{array}{l} 1 \leq z' \wedge \\ \left\{ \begin{array}{l} [u' = 0 \wedge itcl_\varphi([w, x'], z', x', y')] \vee \\ \left\{ \begin{array}{l} 0 < u' \wedge \\ \left\{ \begin{array}{l} [z' = 1 \wedge \langle u' - 1, x' + 2, x' + 1, y'\rangle \in w] \vee \\ \left\{ \begin{array}{l} 2 \leq z' \wedge \\ \exists v < w \left\{ \begin{array}{l} \langle u', z' - 1, x', v\rangle \in w \\ \langle u', 1, v, y'\rangle \in w \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right. \end{array} \right.$$

Now we give an informal description of the meaning of $\mathrm{Ack}_\varphi(w, u, z, x, y)$. We are going to define (by recursion on u), see **3.17** for the formal definition, a sequence of functions

(–) $F_{\varphi,0}(x) = F_\varphi(x)$.
(–) $F_{\varphi,u+1}(x) = F^{x+2}_{\varphi,u}(x + 1)$.

The intended meaning of $\mathrm{Ack}_\varphi(w, u, z, x, y)$ is that w is a finite approximation of $F^z_u(x) = y$.

Lemma 3.9. *The following formulas are provable in* $\mathbf{T}$.

1. $Ack_\varphi(w, 0, 1, x, y) \rightarrow \varphi(x, y)$.
2. $Ack_\varphi(w, u, z, x, y) \rightarrow \left\{ \begin{array}{l} u = 0 \rightarrow itcl_\varphi([w, x], z, x, y) \\ u \neq 0 \wedge z = 1 \rightarrow Ack_\varphi(w, u - 1, x + 2, x + 1, y) \\ u \neq 0 \wedge 2 \leq z \rightarrow \exists v < w \left\{ \begin{array}{l} Ack_\varphi(w, u, z - 1, x, v) \\ Ack_\varphi(w, u, 1, v, y) \end{array} \right. \end{array} \right.$
3. $Ack_\varphi(w_1, u, z, x, y_1) \wedge Ack_\varphi(w_2, u, z, x, y_2) \rightarrow y_1 = y_2$.

Remark 3.10. In what follows we shall prove some results using the exponential function. Since $\mathbf{T}$ is an extension of $\mathbf{I}\Sigma_n$, for $n \geq 1$ we can use freely this function. But for $n = 0$ let us observe that we do not assume that $\mathbf{T} \vdash \mathbf{exp}$. That means that if in an expression appears an exponential term we must prove first that it exists. Nevertheless, in order to abbreviate expressions that appear below we shall write $x^y < z$ instead of the more accurate $\exists v < z\,(x^y = v)$. Now we give an example, that will be used in **3.11**–*(4)*, of this kind of arguments.

3.10.1. Let $\mathfrak{A} \models \mathbf{T}$ and $a, b' \in \mathfrak{A}$. Then for all $b \in \mathfrak{A},\ b \geq 1$,

$$\forall y < b' \,[\mathrm{IT}_\varphi(b, a, y) \rightarrow a^{2^b} < y]$$

Proof. By induction on b.
($b = 1$): Suppose that $\mathrm{IT}_\varphi(1, a, d)$. Then, by **3.4**–*(iii)*, $a^{2^1} = a^2 < d$.
($b \rightarrow b+1$): Suppose that $\mathrm{IT}_\varphi(b+1, a, d)$. By **3.7**–*(3)*, there exists $d_1 < d$ such that $\mathrm{IT}_\varphi(b, a, d_1)$ and $\varphi(d_1, d)$, By induction hypothesis, there exists $a^{2^b} < d_1$. Then $a^{2^{(b+1)}} = (a^{2^b})^2 < (d_1)^2 < d$, where the last inequality follows from $\varphi(d_1, d)$ and **3.2**–(2). □

Lemma 3.11. *1.* $\mathbf{T} \vdash Ack_\varphi(w, u, z, x, y) \rightarrow x^2 < y$.
2. $\mathbf{T} \vdash Ack_\varphi(w, u, z, x, y) \rightarrow u + z + x \leq y$.
3. $\mathbf{T} \vdash Ack_\varphi(w, u, z, x, y) \rightarrow \langle u, z, x, y\rangle \leq 25 \cdot y^4$.
4. $\mathbf{T} \vdash Ack_\varphi(w, u + 1, z, x, y) \rightarrow (x + 1)^{2^{((u+1)+z+x)}} < y$.

Proof. Le us see *(4)*. Let $\mathfrak{A} \models \mathbf{T}$ and $c \in \mathfrak{A}$. We shall prove by induction on $e \in \mathfrak{A}$ that

$$\text{(I)} \quad \forall z, x, y < c \,[\mathrm{Ack}_\varphi(c, e + 1, z, x, y) \rightarrow (x + 1)^{2^{((e+1)+z+x)}} < y]$$

(In the proof we must show that the exponential term that appears in the above expression does exist).
($e = 0$): By induction we shall prove that for all $b \in \mathfrak{A}$, $b \geq 1$,

$$\text{(II)} \quad \forall x, y < c \,[\mathrm{Ack}_\varphi(c, 1, b, x, y) \rightarrow (x + 1)^{2^{(1+b+x)}} < y]$$

($b = 1$): Suppose that $\mathrm{Ack}_\varphi(c, 1, 1, a, d)$, then $\mathrm{Ack}_\varphi(c, 0, a+2, a+1, d)$. So, $\mathrm{itcl}_\varphi([c, a + 1], a + 2, a + 1, d)$. Then, by **3.5**, $\mathrm{IT}_\varphi(a + 2, a + 1, d)$. So, by **3.10.1**, there exists $(a + 1)^{2^{(a+2)}} < d$.
($b \rightarrow b + 1$): Suppose that $\mathrm{Ack}_\varphi(c, 1, b + 1, a, d)$. Since $2 \leq b + 1$, by **3.9**, there exists $d_0 < c$ such that
(i) $\mathrm{Ack}_\varphi(c, 1, b, a, d_0)$, and
(ii) $\mathrm{Ack}_\varphi(c, 1, 1, d_0, d)$.
By *(i)* and induction hypothesis (on b), there exists $(a+1)^{2^{(1+b+a)}} < d_0$. By *(ii)* and *(1)*, $(d_0)^2 < d$; hence,

$$(a + 1)^{2^{(1+(b+1)+a)}} = ((a + 1)^{2^{(1+b+a)}})^2 < (d_0)^2 < d$$

This proves (II) for all $b \geq 1$.

($e \rightarrow e + 1$): By induction on $b \in \mathfrak{A}$, $b \geq 1$, as for $e = 0$, it is proved that

$$\text{(III)} \quad \forall x, y < c \,[\mathrm{Ack}_\varphi(c, e + 2, b, x, y) \rightarrow (x + 1)^{2^{((e+2)+b+x)}} < y]$$

This proves (I) for all e and completes the proof of *(4)*. □

Lemma 3.12. *The following formula is a theorem of* **T**

$$1 \le z \wedge itcl_\varphi(w, z, x, y) \rightarrow$$
$$\rightarrow \exists w' \le (y+1)^{33} \begin{cases} Ack_\varphi(w', 0, z, x, y) \wedge \\ \forall z' \le z \, \forall y' < w \, [J(z', y') \in [w', x] \leftrightarrow J(z', y') \in w] \end{cases}$$

Proposition 3.13. *The following formula is a theorem of* **T**

$$\left.\begin{matrix} Ack_\varphi(w, u, z, x, y) \\ \exists v \, [(y+1)^{72\cdot(u+1)} = v] \end{matrix}\right\} \rightarrow \exists w' \le (y+1)^{72\cdot(u+1)} \, Ack_\varphi(w', u, z, x, y)$$

Proof. Let $\mathfrak{A} \models \mathbf{T}$, $c,\ d' \in \mathfrak{A}$. By induction we shall prove that for all $e \in \mathfrak{A}$

$$(\mathrm{I}) \forall z, x, y < c \begin{bmatrix} \mathrm{Ack}_\varphi(c, e, z, x, y) \wedge \exists v \le d' \, [(y+1)^{72\cdot(e+1)} = v] \rightarrow \\ \rightarrow \exists w' \le (y+1)^{72\cdot(e+1)} \, \mathrm{Ack}_\varphi(w', e, z, x, y) \end{bmatrix}$$

($e = 0$): Suppose that $\mathrm{Ack}_\varphi(c, 0, b, a, d)$ and $\exists v \le d' \, [(d+1)^{27\cdot(0+1)} = v]$. Then $\mathrm{itcl}_\varphi([c, a], b, a, d)$. So, by **3.12**, there is $c' \le (d+1)^{33} \le (d+1)^{72\cdot(0+1)}$ such that $\mathrm{Ack}_\varphi(c', 0, b, a, d)$.
($e \rightarrow e+1$): By induction we shall prove that for all $b \in \mathfrak{A}$, $b \ge 1$,

$$(\mathrm{II}) \forall x, y < c \begin{bmatrix} \mathrm{Ack}_\varphi(c, e+1, b, x, y) \wedge \exists v \le d' \, [(y+1)^{72\cdot(e+2)} = v] \rightarrow \\ \rightarrow \exists w_0 \le (y+1)^{72\cdot(e+1)} \, \mathrm{Ack}_\varphi(w_0, e+1, b, x, y) \end{bmatrix}$$

($b = 1$): Suppose that $\mathrm{Ack}_\varphi(c, e+1, 1, a, d)$ and there is $(d+1)^{72\cdot(e+2)}$. Then $\mathrm{Ack}_\varphi(c, e, a+2, a+1, d)$ and there exists $(d+1)^{72\cdot(e+1)}$. By induction hypothesis (on e), there exists $c_0 \le (d+1)^{72\cdot(e+1)}$ such that $\mathrm{Ack}_\varphi(c_0, e, a+2, a+1, d)$. Let $c' = c_0 \cup \{\langle e+1, 1, a, d\rangle\}$. Then it holds that $\mathrm{Ack}_\varphi(c', e+1, 1, a, d)$. We also have that

$$\begin{aligned} c' &\le c_0 + \mathrm{Lp}_4(c_0) \cdot \{\langle e+1, 1, a, d\rangle\} && [\![\mathbf{3.1.4}\text{–}(ii)]\!] \\ &\le c_0 + 4 \cdot c_0 \cdot 36 \cdot (\mathrm{Lp}_2(\langle e+1, 1, a, d\rangle))^2 && [\![\mathbf{3.1.1}\text{–}(ii),\ \mathbf{3.1.4}\text{–}(i)]\!] \\ &\le c_0 + 4 \cdot c_0 \cdot 36 \cdot (2 \cdot (25 \cdot d^4))^2 && [\![\mathbf{3.1.1}\text{–}(ii),\ \mathbf{3.11}\text{–}(3)]\!] \\ &\le c_0 + 4 \cdot c_0 \cdot 36 \cdot 4 \cdot 25^2 \cdot d^8 && \\ &\le (d+1)^{72\cdot(e+1)} \cdot (1 + 2^6 \cdot 2^{13} \cdot d^8) && [\![c_0 \le (d+1)^{72\cdot(e+1)}]\!] \\ &\le (d+1)^{72\cdot(e+1)} \cdot (d+1)^{27} && [\![2 \le d+1]\!] \\ &\le (d+1)^{72\cdot(e+2)} \end{aligned}$$

($b \rightarrow b+1$): Assume that $\mathrm{Ack}_\varphi(c, e+1, b+1, a, d)$ and there exists $(d+1)^{72\cdot(e+2)}$. Then, by **3.9**, there exists $d_0 < c$ such that
(i) $\mathrm{Ack}_\varphi(c, e+1, b, a, d_0)$,
(ii) $\mathrm{Ack}_\varphi(c, e+1, 1, d_0, d)$. So, $\mathrm{Ack}_\varphi(c, e, d_0+2, d_0+1, d)$.
Since $d_0 < d$, there exists $(d_0+1)^{72\cdot(e+2)}$. Then, by *(i)* and induction hypothesis (on b), we have that there exists $c_0 \le (d_0+1)^{72\cdot(e+2)}$ such that $\mathrm{Ack}_\varphi(c_0, e+1, b, a, d_0)$. By *(ii)* and induction hypothesis (on e), there is $c_1 \le (d+1)^{72\cdot(e+1)}$ such that $\mathrm{Ack}_\varphi(c_1, e, d_0+2, d_0+1, d)$. Let

$$c' = c_0 \cup c_1 \cup \{\langle e+1, b, a, d_0\rangle\} \cup \{\langle e+1, 1, d_0, d\rangle\}$$

Then $\mathrm{Ack}_\varphi(c', e+1, b+1, a, d)$, we also have

$$\begin{aligned}\{\langle e+1, 1, d_0, d\rangle\} &\leq 36 \cdot (\mathrm{Lp}_2(\langle e+1, 1, d_0, d\rangle))^2 && [\![\mathbf{3.1.4}\text{–}(i)]\!]\\ &\leq 36 \cdot 4 \cdot (\langle e+1, 1, d_0, d\rangle)^2 && [\![\mathbf{3.1.1}\text{–}(ii)]\!]\\ &\leq 36 \cdot 4 \cdot 25^2 \cdot d^8 && [\![\mathbf{3.11}\text{–}(3)]\!]\\ &\leq d^{26} && [\![2 \leq d]\!]\end{aligned}$$

Similarly, $\{\langle e+1, b, a, d_0\rangle\} \leq d^{26}$. So, $c' \leq (d+1)^{72\cdot(e+2)}$.

This proves that (I) holds for all e and completes the proof. □

3.3. Ackermann's Functions: $\mathbb{A}_\varphi(u, z, x, y)$

In the above definitions and results we have used $J(z, y) \in w$, for w such that $\mathrm{itcl}_\varphi(w, z, x, y)$, to express that $F^z_\varphi(x) = y$. Now we also use $J(i, j) \in s$ to represent that s is seen as a sequence and j is the i-th element of s. When we use $J(i, j) \in s$ with this meaning we denote that expression by $(s)_i = j$. Let $\mathrm{Func}(s)$ be the conjunction of the following Δ_0 formulas:

(–) $\forall i, j_1, j_2 < s\,[(s)_i = j_1 \wedge (s)_i = j_2 \rightarrow j_1 = j_2]$, and
(–) $\forall i < s\,[\exists j < s\,((s)_i = j) \rightarrow (\forall i')_{1\leq i'\leq i}\,\exists j' < s\,((s)_{i'} = j')]$.

Definition 3.14. *Let* $Cp_\varphi(s, u, x, y) \in \Pi_n$ *(in* $\mathbf{B}\Sigma_n$ *for* $n \geq 1$*) be*

$$\begin{cases} 1 \leq u \wedge Func(s) \wedge (s)_u = x \wedge D_\varphi((s)_1, y) \wedge \\ (\forall u')_{1<u'\leq u}\,\exists w \leq y\, Ack_\varphi(w, u'-1, (s)_{u'}+1, (s)_{u'}+1, (s)_{u'-1}) \end{cases}$$

Suppose that $\mathrm{Cp}_\varphi(s, u, x, y)$. Then $(s)_u = x$, $y = F^{(s)_1+2}_\varphi((s)_1+1)$, and for every u', $1 < u' \leq u$, $F_{\varphi,u'-1}((s)_{u'-1}) = F_{\varphi,u'}((s)_{u'})$.

Lemma 3.15. *The following formulas are provable in* $\mathbf{T}$*:*

1. $Cp_\varphi(s, u, x, y) \rightarrow (\forall u')_{1\leq u'\leq u}\,[u + x \leq (s)_{u'} \wedge Cp_\varphi(s, u', (s)_{u'}, y)]$.
2. $\exists s\, Cp_\varphi(s, u, x, y) \leftrightarrow 1 \leq u \wedge \exists w\, Ack_\varphi(w, u, 1, x, y)$.

Theorem 3.16. $\mathbf{T} \vdash Cp_\varphi(s, u, x, y) \rightarrow \exists s' \leq 36 \cdot 4 \cdot y^6\, Cp_\varphi(s', u, x, y)$.

Proof. Let $\mathfrak{A} \models \mathbf{T}$ and $s, d \in \mathfrak{A}$. By induction we prove that for all $e \in \mathfrak{A}$, $e \geq 1$,

$$(\mathrm{I})\forall x < d \left[\begin{array}{l}\mathrm{Cp}_\varphi(s, e, x, d) \;\rightarrow \\ \qquad\rightarrow \begin{cases}\exists v \leq d^6\,[((s)_1+1)^{12\cdot(e+1)} = v] \wedge \\ \exists s' \leq 36\cdot 4\cdot((s)_1+1)^{12\cdot(e+1)}\,\mathrm{Cp}_\varphi(s', e, x, d)\end{cases}\end{array}\right]$$

$(e = 1)$: If $\mathrm{Cp}_\varphi(s, 1, a, d)$ then $a = (s)_1$ and $\mathrm{D}_\varphi(a, d)$. Let $s' = \{J(1, a)\}$ (that is, $(s')_1 = a$). Then $\mathrm{Cp}_\varphi(s', 1, a, d)$. So, by **3.15**–*(2)*, there exists c' such that $\mathrm{Ack}_\varphi(c', 1, 1, a, d)$; hence, by **3.11**–*(2)*, $1 + 1 + a \leq d$. So, from **3.1.4**–*(i)* and **3.1.1**–*(ii)* we get that

$$s' \leq 36 \cdot (\mathrm{Lp}_2(J(1, a)))^2 \leq 36 \cdot 4 \cdot (a+1)^4 \leq 36 \cdot 4 \cdot (a+1)^{12\cdot(1+1)}$$

Since $\mathrm{Ack}_\varphi(c, 1, 1, a, d)$, by **3.11**–*(4)*, $(a+1)^{2^{(1+1+a)}} < d$; hence,

$$(a+1)^{12\cdot(1+1)} = ((a+1)^4)^6 \le ((a+1)^{2^{(1+1+a)}})^6 \le d^6$$

So, $s' \le 36 \cdot 4 \cdot d^6$. This proves (I) for $e = 1$.
$(e \to e+1)$: Suppose that $\mathrm{Cp}_\varphi(s, e+1, a, d)$. Then

(i) $e + 1 + a \le (s)_1$ (by **3.15**–*(1)*), and
(ii) $\mathrm{Cp}_\varphi(s, e, (s)_e, d)$.

By *(ii)* and induction hypothesis, there exist $((s)_1 + 1)^{12\cdot(e+1)} \le d^6$ and $s_0 \le 36 \cdot 4 \cdot ((s)_1 + 1)^{12\cdot(e+1)}$ such that $\mathrm{Cp}_\varphi(s_0, e, (s)_e, d)$.

Let $s' = s_0 \cup \{J(e+1, a)\}$ (that is, $(s')_{e+1} = a$). Observe that for each u', $1 \le u' \le e$, $(s')_{u'} = (s)_{u'}$. Then $s' \le 36 \cdot 4 \cdot ((s)_1 + 1)^{12\cdot(e+2)}$ and $\mathrm{Cp}_\varphi(s', e+1, a, d)$. Since $\mathrm{D}_\varphi((s)_1, d)$, by **3.12**, there exists c_1 such that $\mathrm{Ack}_\varphi(c_1, 0, (s)_1 + 2, (s)_1 + 1, d)$. So, by **3.11**–*(4)*, $((s)_1 + 2)^{2\cdot((s)_1+1)} < d$. Hence, by *(i)*, it holds that

$$((s)_1 + 1)^{12\cdot(e+2)} = (((s)_1 + 1)^{2\cdot(e+2)})^6 \le (((s)_1 + 1)^{2\cdot((s)_1+1)})^6 \le d^6$$

This proves (I) for all $e \ge 1$, which completes the proof. □

Definition 3.17. *The Ackermann's function of φ, $\mathbb{A}_\varphi(u, z, x, y)$, is the following Π_n formula (in $\mathbf{B}\Sigma_n$ for $n \ge 1$).*

$$\begin{cases} (z = 0 \wedge x = y) \vee \\ \begin{cases} z \ne 0 \wedge \\ \begin{cases} (u = 0 \wedge IT_\varphi(z, x, y)) \vee \\ \begin{cases} u \ne 0 \wedge \\ \exists s, x' < 36 \cdot 4 \cdot y^6 \begin{cases} Cp_\varphi(s, u, x', y) \wedge \\ \begin{cases} (z = 1 \wedge x = x') \vee \\ 2 \le z \wedge \exists w < y\, Ack_\varphi(w, u, z-1, x, x') \end{cases} \end{cases} \end{cases} \end{cases} \end{cases} \end{cases}$$

We shall usually denote the Ackermann's function of φ, $\mathbb{A}_\varphi(u, z, x, y)$, by $\mathbb{A}^z_{\varphi,u}(x) = y$ and $\mathbb{A}^1_{\varphi,u}(x) = y$ by $\mathbb{A}_{\varphi,u}(x) = y$.

Remark 3.18. Here we shall give an informal description of the formula $\mathbb{A}^z_{\varphi,u}(x) = y$. Consider $u, z \ge 1$. Let s and x' be such that $\mathrm{Cp}_\varphi(s, u, x', y)$. For every j, $1 \le j \le u$, let us denote $(s)_j = y_j$. We have that

$$y_u = \begin{cases} x, & \text{if } z = 1 \\ x' = \mathbb{A}^{z-1}_{\varphi,u}(x), & \text{if } z > 1 \end{cases}$$

$$y_{j-1} = \mathbb{A}^{y_j+1}_{\varphi,j-1}(y_j + 1) \quad \text{for all } j,\ 1 < j \le u.$$

From this we get that $(\forall j)_{1\le j\le u}\, (\mathbb{A}_{\varphi,j}(y_j) = y)$. In particular, we have that $\mathbb{A}_{\varphi,1}(y_1) = y$, that is $\mathrm{D}_\varphi(y_1, y)$. It also holds that (for *(ii)* use **3.13**):

3.18.1. (i) $\mathbf{T} \vdash \mathbb{A}^z_{\varphi,0}(x) = y \leftrightarrow \mathrm{IT}_\varphi(z, x, y)$.
(ii) $\mathbf{T} \vdash \mathbb{A}_{\varphi,u+1}(x) = y \leftrightarrow \mathbb{A}^{x+2}_{\varphi,u}(x+1) = y$.
(iii) $\mathbf{T} \vdash \mathbb{A}^{z+1}_{\varphi,u}(x) = y \leftrightarrow \mathbb{A}_{\varphi,u}(\mathbb{A}^z_{\varphi,u}(x)) = y$.

4. Π_n–envelopes given by iteration

Now, we present the main tool that will be used in the remainder of the paper. We follow a similar construction devised by R. Kaye (see [9]) to analyse parameter free induction schemes.

Definition 4.1. *1. Let* $\mathbb{K}_0(x) = y$ *be* $(x+1)^2 = y$. *For every* $n \geq 1$ *let* $\mathbb{K}_n(x) = y$ *be* $\mathbb{E}_n(x, x, y)$, *where* $\mathbb{E}_n(u, x, y)$ *is the* Π_n*–q–envelope given in [7]–***5.13**. *(Let us observe that* $\mathbf{I\Sigma}_n \vdash x^2 < \mathbb{K}_n(x)$*).*

2. Let $\varphi(x, y) \in \Pi_n^-$. *We will denote by* $KITF_n(\varphi)$ *the formula:*

$$IPF(\varphi) \wedge \forall x\, \forall y\, (\varphi(x, y) \rightarrow \mathbb{K}_n(x) \leq y) \wedge \forall x\, \exists y\, \varphi(x, y)$$

For each theory $\mathbf{T}$, *let* $\mathbf{T}^{\varphi,n}$ *be the theory* $\mathbf{T} + KITF_n(\varphi)$. *In particular, we will denote by* $\mathbf{I\Sigma}_m^{\varphi,n}$ *the theory* $(\mathbf{I\Sigma}_m)^{\varphi,n}$. *When* $n = m$, *we shall omit the superscript* n *and write* $\mathbf{I\Sigma}_n^{\varphi}$.

Remark 4.2. Let us observe that $\mathrm{KITF}_n(\varphi) \in \Pi_{n+2}$ and

$$\mathbf{I\Sigma}_n^{\varphi} = \mathbf{I\Sigma}_n + \Gamma^* + \forall x\, \forall y\, (\varphi(x, y) \rightarrow \mathbb{K}_n(x) \leq y)$$

(where $\Gamma = \{\varphi(x, y)\}$). So, by [7]–**3.7.2**–(ii), if $\mathbf{I\Sigma}_n^{\varphi}$ is consistent, then $\mathbf{I\Sigma}_n^{\varphi}$ is a Π_n–functional theory. In particular, if $\varphi(x, y)$ is the formula $\mathbb{K}_n(x) = y$ then, by [7]–**5.13**, $\mathbf{I\Sigma}_n \vdash \mathrm{KITF}_n(\varphi)$. Hence, $\mathbf{I\Sigma}_n^{\varphi} \Longleftrightarrow \mathbf{I\Sigma}_n$.

Definition 4.3. *Let* $\mathbf{ACK}_{\varphi} = \{\mathbb{F}_{\varphi,k}(x) = y : k \in \omega\}$, *where*

(–) $\mathbb{F}_{\varphi,0}(x) = y$ *is* $\varphi(x, y)$.
(–) $\mathbb{F}_{\varphi,k+1}(x) = y$ *is* $D_{\mathbb{F}_{\varphi,k}}(x, y)$.

If $\varphi(x, y)$ *is the formula* $\mathbb{K}_n(x) = y$, *then* $\mathbb{F}_{n,k}(x) = y$ *will denote the formula* $\mathbb{F}_{\varphi,k}(x) = y$ *and* $\mathbf{ACK}_n$ *will denote the set* $\mathbf{ACK}_{\varphi}$.

Remark 4.4. Let us observe that, as we will see in **4.5**, if $\mathbf{I\Sigma}_n \vdash \mathrm{KITF}_n(\varphi)$ then $\mathbf{ACK}_{\varphi}$ is an inductive Π_n–functional class. Even more, it holds that

4.4.1. If $\mathbf{I\Sigma}_{n+1} \vdash \mathrm{KITF}_n(\varphi)$ and $\mathbf{I\Sigma}_n \vdash \forall x, y\, (\varphi(x, y) \rightarrow \mathbb{K}_n(x) \leq y)$, then $\mathbf{ACK}_{\varphi}$ is an inductive Π_n–functional class.

Proof. By induction on $k \in \omega$ we prove that $\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_n + \mathbf{ACK}_{\varphi}^*)$ proves

(–) $\mathrm{IPF}(\mathbb{F}_{\varphi,k})$, and
(–) $\exists y\, (\mathbb{F}_{\varphi,k}(0) = y) \wedge \forall x\, [\exists y\, (\mathbb{F}_{\varphi,k}(x) = y) \rightarrow \exists y\, (\mathbb{F}_{\varphi,k}(x + 1) = y)]$.

$\underline{k = 0}$: Since $\mathbf{I\Sigma}_{n+1} \vdash \mathrm{KITF}_n(\varphi)$, by **5.5**–*(1)*, $\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_n + \mathbf{ACK}_n^*) \vdash \mathrm{KITF}_n(\varphi)$. As $\mathbf{I\Sigma}_n \vdash \forall x, y\, (\varphi(x, y) \rightarrow \mathbb{K}_n(x) \leq y)$, then by **6.4** and **6.5**,

$$\mathbf{I\Sigma}_n + \mathbf{ACK}_{\varphi}^* \Longleftrightarrow \mathbf{I\Sigma}_n^{\varphi} + \mathbf{ACK}_{\varphi}^* \Longrightarrow \mathbf{I\Sigma}_n + \mathbf{ACK}_n^*$$

So, $\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_n + \mathbf{ACK}_{\varphi}^*) \vdash \mathrm{KITF}_n(\varphi)$, as required.
$\underline{k \rightarrow k + 1}$: It follows from **4.5**. □

Lemma 4.5. *1. For all* $k \in \omega$,

(a) $\mathbf{I\Sigma}_n^\varphi \vdash \mathbb{F}_{\varphi,k}(x) = y \to \mathbb{K}_n(x) \leq y$.
(b) $\mathbf{I\Sigma}_n^\varphi \vdash IPF(\mathbb{F}_{\varphi,k})$.
2. *For all* $k \in \omega$, $\mathbf{I\Sigma}_n^\varphi + \forall x\, \exists y\, [\mathbb{F}_{\varphi,k}(x) = y]$ *proves*
(a) $\exists y\, [\mathbb{F}_{\varphi,k+1}(0) = y]$.
(b) $\forall x\, [\exists y\, (\mathbb{F}_{\varphi,k+1}(x) = y) \to \exists y\, (\mathbb{F}_{\varphi,k+1}(x+1) = y)]$.
3. $\mathbf{I\Sigma}_{n+1}^{\varphi,n} \vdash \mathbf{ACK}_\varphi^*$.

Proof. (*(1)*): We get *(1.a)* and *(1.b)* by induction on $k \in \omega$ using **3.7**.
(*(2)*): We only need to prove *(2.b)*. Let $\mathfrak{A} \models \mathbf{I\Sigma}_n^\varphi + \forall x\, \exists y\, [\mathbb{F}_{\varphi,k}(x) = y]$ and $a \in \mathfrak{A}$ such that $\mathfrak{A} \models \exists y\, [\mathbb{F}_{\varphi,k+1}(a) = y]$. Let $b \in \mathfrak{A}$ such that $\mathfrak{A} \models \mathbb{F}_{\varphi,k+1}(a) = b$. Then, by induction on d, using **3.7**–*(6)* and *(1)*, it is proved that for all $d \leq a$

$$\exists y_1, y_2 \leq b\, [\mathbb{F}_{\varphi,k}^d(a+2) = y_1 \land \mathbb{F}_{\varphi,k}^{d+2}(a+1) = y_2 \land y_1 \leq y_2]$$

From this, for $d = a$, we have that $\exists y\, [\mathbb{F}_{\varphi,k}^a(a+2) = y]$. Then, by **3.7**–*(6)*, $\exists y\, [\mathbb{F}_{\varphi,k}^{a+3}(a+2) = y]$; hence, $\exists y\, [\mathbb{F}_{\varphi,k+1}(a+1) = y]$.
(*(3)*): By induction on k, it follows from *(1)* and *(2)* that, for all $k \in \omega$,

$$\mathbf{I\Sigma}_{n+1}^{\varphi,n} \vdash \mathrm{KITF}_n(\mathbb{F}_{\varphi,k}(x) = y)$$

as required. □

Theorem 4.6. *For all* $k \in \omega$, $IT_{\mathbb{F}_{\varphi,k}}(z, x, y) \in \Pi_n$ *is a* Π_n*–envelope of* $\mathbf{I\Sigma}_n^\varphi + \forall x\, \exists y\, [\mathbb{F}_{\varphi,k}(x) = y]$ *in* $\mathbf{I\Sigma}_n^\varphi$.

Proof. By **4.5**–*(1)* and **3.7**–*(3)*, $\mathrm{IT}_{\mathbb{F}_{\varphi,k}}(z, x, y)$ is a Π_n–q–envelope. So, by [7]–**5.4**, to see that $\mathrm{IT}_{\mathbb{F}_{\varphi,k}}(z, x, y)$ is a Π_n–envelope is enough to prove that this formula satisfies Π_n–IND. Let $\mathfrak{A} \models \mathbf{I\Sigma}_n^\varphi$ and $a, b \in \mathfrak{A}$ such that for all $m \in \omega$, $\mathfrak{A} \models \exists y < b\, \mathrm{IT}_{\mathbb{F}_{\varphi,k}}(m, a, y)$. For all $m \in \omega$ let $b_m < b$ such that $\mathfrak{A} \models \mathrm{IT}_{\mathbb{F}_{\varphi,k}}(m, a, b_m)$. Let $\mathfrak{I} = \{c \in \mathfrak{A} : \exists m \in \omega\, (c < b_m)\}$. Then $a < \mathfrak{I} < b$ and $\mathfrak{I}$ is a initial segment closed under the Π_n–functions defined in $\mathfrak{A}$ by φ and $\mathbb{F}_{\varphi,k}$. For all $c \in \mathfrak{I}$, by **4.5**–*(1)*, there exists $d \in \mathfrak{I}$ such that $\mathfrak{A} \models \mathbb{K}_n(c) = d$. So, by [7]–**5.13**, $\mathfrak{I} \prec_n^e \mathfrak{A}$. Hence, $\mathfrak{I} \models \mathbf{I\Delta}_0 + \mathrm{KITF}_n(\varphi)$ and $\mathfrak{I} \models \forall x\, \exists y\, (\mathbb{K}_n(x) = y)$. So, $\mathfrak{I} \models \mathbf{I\Delta}_0 + \Gamma_n^*$, where $\Gamma_n = \{\mathbb{K}_n(x) = y\}$. Since (see [7]–**5.13**) Γ_n is a strong Π_n–functional class, then, by [7]–**4.6.1**, $\mathbf{I\Delta}_0 + \Gamma_n^* \Longrightarrow \mathbf{I\Sigma}_n$. So, $\mathfrak{I} \models \mathbf{I\Sigma}_n^\varphi + \forall x\, \exists y\, (\mathbb{F}_{\varphi,k}(x) = y)$. □

Theorem 4.7. *1. For all* $n, k \in \omega$, $\mathbf{I\Sigma}_n^\varphi \vdash \mathbb{F}_{\varphi,k}(x) = y \leftrightarrow \mathbb{A}_{\varphi,k}(x) = y$.
2. $\mathbb{A}_{\varphi,u}(x) = y$ *is a* Π_n*–envelope of* $\mathbf{I\Sigma}_{n+1}^{\varphi,n}$ *in* $\mathbf{I\Sigma}_n^\varphi$.
3. *There exists a* Π_n*–envelope of* $\mathbf{I\Sigma}_{n+1}^{\varphi,n}$ *in* $\mathbf{I\Sigma}_n^\varphi$, $\psi(u, x, y) \in \Pi_n$, *such that for all* $k \in \omega$, $\mathbf{I\Sigma}_n^\varphi \vdash \mathbb{F}_{\varphi,k}(x) = y \leftrightarrow \psi(k, x, y)$.

Proof. (*(1)*): Let $\mathfrak{A} \models \mathbf{I\Sigma}_n^\varphi$. By induction on $k \in \omega$, let us see that

$$\text{(I)} \quad \mathfrak{A} \models \mathrm{IT}_{\mathbb{F}_{\varphi,k}}(z, x, y) \leftrightarrow \mathbb{A}_{\varphi,k}^z(x) = y.$$

($k = 0$): This follows from the definitions of $\mathbb{F}_{\varphi,0}$ and $\mathbb{A}_{\varphi,0}$.
($k \to k+1$): Let $d \in \mathfrak{A}$. By induction, using **3.18.1**, it is proved that for all $b \in \mathfrak{A}$, $b \geq 1$,

$$\text{(II)} \quad \forall x, y < d\, [\mathrm{IT}_{\mathbb{F}_{\varphi,k+1}}(b, x, y) \leftrightarrow \mathbb{A}_{\varphi,k+1}^b(x) = y]$$

This proves (I) for all k and completes the proof.

(*(2)*): By **4.5**–*(3)*, $\mathbf{I}\Sigma^{\varphi,n}_{n+1} \vdash \mathbf{ACK}^*_\varphi$ and, by **3.7**–*(5)*,

$$\mathbf{I}\Sigma^{\varphi}_{n} \vdash \mathbb{F}_{\varphi,k+1}(x) = y \rightarrow \exists v < y\, (\mathbb{F}_{\varphi,k}(x) = v)$$

Then, by *(1)*, $\mathbb{A}_{\varphi,u}(x) = y$ is a Π_n–q–envelope of $\mathbf{I}\Sigma^{\varphi,n}_{n+1}$ in $\mathbf{I}\Sigma^{\varphi}_{n}$. So, by [7]–**5.4**, it is enough to prove that for every $\mathfrak{A} \models \mathbf{I}\Sigma^{\varphi}_{n}$ and $a, b \in \mathfrak{A}$, $a < b$,

($\star$) if for all $k \in \omega$, $\mathfrak{A} \models \exists y < b\, (\mathbb{A}_{\varphi,k}(a) = y)$ then there exists $\mathfrak{I} \models \mathbf{I}\Sigma^{\varphi,n}_{n+1}$ such that $\mathfrak{I} \prec^e_n \mathfrak{A}$ and $a < \mathfrak{I} < b$.

Through the proof we shall write $\mathbb{A}_u(x) = y$ and $\mathbb{F}_k(x) = y$ instead of $\mathbb{A}_{\varphi,u}(x) = y$ and $\mathbb{F}_{\varphi,k}(x) = y$, respectively.

We follow the proof of lemma **4.6** in [18] (which, in turn, follows a construction of Paris and Kirby (see [12])). First of all, let us observe that we can assume that a is nonstandard and $\mathfrak{A} \models$ **exp**:

(–) We can assume that $\omega < a$:
Let $I = \{c \in \mathfrak{A} :\ \exists k \in \omega,\ c < \mathbb{A}_k(a)\}$. Then for each $c < \mathbb{A}_k(a)$,

$$\mathfrak{A} \models \mathbb{K}_n(c) < \mathbb{F}_k(\mathbb{A}_k(a)) = \mathbb{F}^2_k(a) < \mathbb{F}^{a+2}_k(a+1) = \mathbb{A}_{k+1}(a)$$

Hence, $I \prec^e_n \mathfrak{A}$, $a < I < b$ and I is closed under the Π_n–functions defined by $\mathbb{F}_k$. If $I = \omega$, then by overspill there exists $a^* > I$ such that for all $k \in \omega$, $\mathfrak{A} \models \exists y < b\, (\mathbb{A}_k(a^*) = y)$. If I is a nonstandard segment then there exists $a^* \in I$ such, $a^* > \omega$. So, for all $k \in \omega$, $\mathfrak{A} \models \exists y < b\, (\mathbb{A}_k(a^*) = y)$.

(–) We can assume that $\mathfrak{A} \models$ **exp**:
We will use the trick of lemma **3** in [1]. For all $k \in \omega$ it holds that

$$(\bullet)\quad \mathfrak{A} \models \exists y < b\, (\mathbb{A}_k(a) = y \wedge \forall x \leq y\, \exists z < b\, (\mathbb{F}^k_1(x) = z))$$

Let $\psi(k, a, b)$ be the Π_n formula:

$$\exists y < b \begin{cases} \mathbb{A}_k(a) = y\ \wedge\ \forall x \leq y\, \exists z < b\, (\mathbb{F}^k_1(x) = z) \wedge \\ \forall u \leq k\, \exists v < y\, (\mathbb{A}_u(a) = v) \end{cases}$$

By ($\bullet$), for all $k \in \omega$, $\mathfrak{A} \models \psi(k, a, b)$. So, by overspill, there exists $c > \omega$ such that $\mathfrak{A} \models \psi(c, a, b)$. Let b^* such that

$$\begin{cases} b^* < b\ \wedge\ \mathbb{A}_c(a) = b^*\ \wedge\ \forall x \leq b^*\, \exists z < b\, (\mathbb{F}^c_1(x) = z) \wedge \\ \forall u \leq c\, \exists v < b^*\, (\mathbb{A}_u(a) = v) \end{cases}$$

Let $I^* = \{d \in \mathfrak{A} : \exists k \in \omega,\ d < \mathbb{F}^k_1(b^*)\}$. Then $a < I^* < b$ and I^* is closed under the Π_n–function defined by $\mathbb{F}_1$; hence, $I^* \prec^e_n \mathfrak{A}$. So,

$$I^* \models \mathbf{I}\Sigma^{\varphi}_{n} + \forall x\, \exists y\, (\mathbb{F}_1(x) = y).$$

But $\mathbf{I}\Sigma^{\varphi}_{n} + \forall x\, \exists y\, (\mathbb{F}_1(x) = y) \vdash$ **exp**, and $I^* \models \exists y < b^*\, (\mathbb{A}_k(a) = y)$.

So, taking a^* and b^* instead of a and b, and I^* instead of $\mathfrak{A}$, if needed, we can assume in ($\star$) that $\mathfrak{A} \models \mathbf{I\Sigma}_n^{\varphi} + \mathbf{exp}$ and $\omega < a$.

Suppose that for all $k \in \omega, \mathfrak{A} \models \exists y < b\, (\mathbb{A}_k(a) = y)$. By overspill, there exists $c > \omega, c < a$, such that $\mathfrak{A} \models \exists y < b\, (\mathbb{A}_{c+1}(a) = y)$. Let $d = \mathbb{A}_{c+1}(c)$. Then for all $k \in \omega, \mathfrak{A} \models \exists y < d\, (\mathbb{A}_k(c) = y)$. We define an initial segment of $\mathfrak{A}$, $\mathfrak{I}$, as follows. Let $\{\psi_k(w, v) : k \in \omega\}$ be an enumeration of the class of Π_n formulas such that each Π_n formula appears infinitely often. We define two sequences $\{a_k : k \in \omega\}$ and $\{b_k : k \in \omega\}$ of elements of $\mathfrak{A}$ such that

$(1)_k$ $k \neq 0 \Longrightarrow (a_{k-1})^2 < a_k$,
$(2)_k$ $a_0 < a_1 < \cdots < a_k \leq b_k \leq b_{k-1} \leq \cdots \leq b_0$,
$(3)_k$ $k \neq 0 \wedge \langle d_1, \ldots, d_r\rangle \leq a_k \Longrightarrow (\mu w)[\psi_{k-1}(w, d_1, \ldots, d_r)] \notin (a_k, b_k]$,
$(4)_k$ $b_k = \mathbb{A}_{c-k+1}(a_k)$.

We proceed by recursion on k (at the same time we prove that they satisfy $(1)_k$–$(4)_k$).
($k = 0$): Let $a_0 = c$ and $b_0 = d$.
($k \to k+1$): Suppose that we have a_i and b_i, $0 \leq i \leq k$, and they satisfy $(1)_i$–$(4)_i$. Then $b_k = \mathbb{A}_{c-k+1}(a_k)$. Since $\mathbb{A}_{c-k+1}(a_k) = \mathbb{A}_{c-k}^{a_k+2}(a_k + 1)$, then

$$(a_k, b_k] = \bigcup_{0 \leq j \leq a_k+1} (\mathbb{A}_{c-k}^{j}(a_k + 1), \mathbb{A}_{c-k}^{j+1}(a_k + 1)]$$

Now the class $M = \{(\mu w)[\psi_k(w, d_1, \ldots, d_r)] : \langle d_1, \ldots, d_r\rangle \leq a_k\}$ has at most $a_k + 1$ elements; hence, by the Pigeon-Hole Principle, there exists $j \leq a_k + 1$ such that $M \cap (\mathbb{A}_{c-k}^{j}(a_k + 1), \mathbb{A}_{c-k}^{j+1}(a_k + 1)] = \emptyset$. Let

$$a_{k+1} = \mathbb{A}_{c-k}^{j}(a_k + 1), \textit{and } \; b_{k+1} = \mathbb{A}_{c-k}^{j+1}(a_k + 1)$$

By definition of a_{k+1} and b_{k+1}, properties $(2)_{k+1}$–$(4)_{k+1}$ are trivial and $(1)_{k+1}$ follows from the definition of Ackermann's function, $\mathbb{A}$.
Let $\mathfrak{I} = \{d \in \mathfrak{A} : \exists k \in \omega\, (d < a_k)\}$. Then, by $(1)_k$, $\mathfrak{I}$ is an initial substructure of $\mathfrak{A}$; hence, $\mathfrak{I} \models \mathbf{I}\Delta_0$. We also have that $\mathfrak{I}$ is closed under $\mathbb{K}_n(x) = y$; hence, $\mathfrak{I} \prec_n \mathfrak{A}$ and $\mathfrak{I} \models \mathbf{I}\Delta_0^{\varphi,n}$. By $(3)_k$ (since each Π_n– formula appears in $\{\psi_k : k \in \omega\}$ infinitely often) it holds that $\mathfrak{I} \models \mathbf{L}\Sigma_{n+1}$. This proves that $\mathfrak{I} \models \mathbf{I\Sigma}_{n+1}^{\varphi,n}$.
(*(3)*): It follows from *(1)* and *(2)*. □

5. Non–finite axiomatization of $\mathrm{I}\boldsymbol{\Delta}_{n+1}$(T)

In this section, using Ackermann's functions, we shall prove **1.8**–*(2)* and present an alternative proof of theorem **1.1**.

Theorem 5.1. *Assume that* $\mathbf{I\Sigma}_{n+1}^{\varphi,n}$ *is consistent. Then*

$$\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_{n+1}^{\varphi,n}) \Longleftrightarrow \mathbf{I\Sigma}_n^{\varphi} + \mathbf{ACK}_{\varphi}^{*} \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{I\Sigma}_{n+1}^{\varphi,n})^{\varphi,n}.$$

Proof. By **4.7**–*(3)*, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_{n+1}^{\varphi,n}) \Longleftrightarrow \mathbf{I\Sigma}_n^{\varphi} + \mathbf{ACK}_{\varphi}^{*}$. Let us see, by induction on $k \in \omega$, that

$$\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_n^{\varphi} + \mathbf{ACK}_{\varphi}^{*})^{\varphi,n} \vdash \forall x\, \exists y\, [\mathbb{F}_{\varphi,k}(x) = y]$$

$(k = 0)$: It follows from the definition of $\mathbf{T}^{\varphi,n}$ and $\mathbb{F}_{\varphi,0}(x) = y$.
$(k \to k+1)$: Suppose that $\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_n^{\varphi} + \mathbf{ACK}_{\varphi}^{*})^{\varphi,n} \vdash \forall x\, \exists y\, [\mathbb{F}_{\varphi,k}(x) = y]$. Then, by **4.5**–*(2)*, $\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_n^{\varphi} + \mathbf{ACK}_{\varphi}^{*})^{\varphi,n}$ proves that

$$\exists y\, [\mathbb{F}_{\varphi,k+1}(0) = y] \wedge \forall x\, [\exists y\, (\mathbb{F}_{\varphi,k+1}(x) = y) \to \exists y\, (\mathbb{F}_{\varphi,k+1}(x+1) = y)]$$

Since $\exists y\, (\mathbb{F}_{\varphi,k+1}(x) = y) \in \Delta_{n+1}(\mathbf{I\Sigma}_n^{\varphi} + \mathbf{ACK}_{\varphi}^{*})$, then

$$\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_n^{\varphi} + \mathbf{ACK}_{\varphi}^{*})^{\varphi,n} \vdash \forall x\, \exists y\, [\mathbb{F}_{\varphi,k+1}(x, y)],$$

as required. □

Lemma 5.2. *If* $\mathbf{I\Sigma}_{n+1}^{\varphi,n}$ *is consistent, then* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_{n+1}^{\varphi,n})$ *is not finitely axiomatizable.*

Proof. By way of contradiction suppose that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_{n+1}^{\varphi,n})$ is finitely axiomatizable. Then by **5.1** and **3.7**-*(1,5)* there exists $k \in \omega$ such that

$$\mathbf{I\Sigma}_n^{\varphi} + \forall x\, \exists y\, (\mathbb{F}_{\varphi,k}(x) = y) \Longleftrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_{n+1}^{\varphi,n})$$

So, by **5.1**, $\mathbf{I\Sigma}_n^{\varphi} + \forall x\, \exists y\, (\mathbb{F}_{\varphi,k}(x) = y) \vdash \forall x\, \exists y\, (\mathbb{F}_{\varphi,k+1}(x) = y)$. By **4.6**, there exists $m \in \omega$ such that

$$\mathbf{I\Sigma}_n^{\varphi} \vdash \mathrm{IT}_{\mathbb{F}_{\varphi,k}}(m, x, y) \to \exists z < y\, (\mathbb{F}_{\varphi,k+1}(x) = z)$$

Since $\mathrm{IT}_{\mathbb{F}_{\varphi,k}}(z+2, x, y) \to \exists y' < y\, \mathrm{IT}_{\mathbb{F}_{\varphi,k}}(z, x, y')$, then it holds that the theory $\mathbf{I\Sigma}_n^{\varphi} + \forall x\, \exists y\, (\mathbb{F}_{\varphi,k}(x) = y)$ proves that

$$\mathrm{IT}_{\mathbb{F}_{\varphi,k}}(m+2, m, y) \to \exists z < y\, (\mathbb{F}_{\varphi,k+1}(m) = z);$$

$$\mathrm{IT}_{\mathbb{F}_{\varphi,k}}(m+2, m, y) \to \exists z < y\, (\mathrm{IT}_{\mathbb{F}_{\varphi,k}}(m+2, m+1, z));$$

which contradicts **4.5**–*(1.b)*. □

Lemma 5.3. *Let* $\mathbf{T}$ *be a* Π_n*–functional finite* Π_{n+2}*–extension of* $\mathbf{I\Sigma}_n$*. Then there exists* $\varphi(x, y) \in \Pi_n^{-}$ *such that* $\mathbf{T} \Longleftrightarrow \mathbf{I\Sigma}_n^{\varphi}$*.*

Proof. By hypothesis, $\mathbf{T} \Longleftrightarrow \mathbf{I\Sigma}_n + \forall x\, \exists y\, \theta(x, y)$, where $\theta(x, y) \in \Pi_n^{-}$. Let $\varphi(x, y) \in \Pi_n^{-}$ the formula

$$\exists y_1, y_2 \le y\, (\mathbb{K}_n(x) = y_1 \wedge \mathcal{C}_{\theta}(x, y_2) \wedge y = y_1 + y_2)$$

Where the formula $\mathcal{C}_{\theta}(x, y)$ is as in the proof of theorem **3.5** in [7]. Then $\mathbf{I\Sigma}_n \vdash \forall x\, \exists y\, \varphi(x, y) \to \forall x\, \exists y\, \theta(x, y)$ and $\mathbf{T} \vdash \forall x\, \exists y\, \varphi(x, y) \leftrightarrow \forall x\, \exists y\, \theta(x, y)$. Hence, $\mathbf{T} \Longleftrightarrow \mathbf{I\Sigma}_n^{\varphi}$, as required. □

Part *(1)* of next theorem can be also obtained from corollary **3.3** in [2].

Theorem 5.4. *Let* $\mathbf{T}$ *be a consistent extension of* $\mathbf{I\Sigma}_{n+1}$. *Then*

1. $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ *is not finitely axiomatizable.*
2. $\mathbf{I\Delta}_{n+1}(\mathbf{T})$ *is not finitely axiomatizable.*

Proof. (*(1)*): Let us assume that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ is finitely axiomatizable. Then, by **5.3** there exists $\varphi(x, y) \in \Pi_n^-$ such that

$$\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \Longleftrightarrow \mathbf{I\Sigma}_n^{\varphi}$$

Hence, $\mathbf{I\Sigma}_{n+1}^{\varphi,n}$ is consistent and $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_{n+1}^{\varphi,n})$, which contradicts **5.2**.
(*(2)*): Assume that $\mathbf{I\Delta}_{n+1}(\mathbf{T})$ is finitely axiomatizable. Then as in the proof of **5.3**, there exists $\varphi(x, y) \in \Pi_n^-$ such that $\mathbf{I\Sigma}_{n+1}^{\varphi,n}$ is consistent and

$$\mathbf{T} \Longrightarrow \mathbf{I\Sigma}_n^{\varphi} \Longrightarrow \mathbf{I\Delta}_{n+1}(\mathbf{T})$$

Since $\mathbf{T}$ is an extension of $\mathbf{I\Sigma}_{n+1}$, then $\mathbf{I\Delta}_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_{n+1}^{\varphi,n})$.
As $\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_{n+1}^{\varphi,n})^{\varphi,n} \Longrightarrow \mathbf{I\Sigma}_n^{\varphi}$, then (second equivalence follows from **5.1**)

$$\mathbf{I\Delta}_{n+1}(\mathbf{T})^{\varphi,n} \Longleftrightarrow \mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_{n+1}^{\varphi,n})^{\varphi,n} \Longleftrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_{n+1}^{\varphi,n})$$

Hence, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_{n+1}^{\varphi,n})$ is finitely axiomatizable, which contradicts **5.2**. □

Theorem 5.5. *1.* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_{n+1}) \Longleftrightarrow \mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_{n+1}) \Longleftrightarrow \mathbf{I\Sigma}_n + \mathbf{ACK}_n^*$.
2. $\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_{n+1})$ *is* Π_{n+2} *axiomatizable.*
3. $\mathbf{I\Sigma}_{n+1}$ *is a* Π_{n+2}*–conservative extension of* $\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_{n+1})$.
4. $\mathbf{I\Sigma}_{n+1}$ *and* $\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_{n+1})$ *have the same class of recursive functions.*

Proof. Let $\varphi(x, y) \in \Pi_n^-$ be the formula $\mathbb{K}_n(x) = y$. Then, $\mathbf{I\Sigma}_n \vdash \mathrm{KITF}_n(\varphi)$, and $\mathbf{I\Sigma}_n^{\varphi} \Longleftrightarrow \mathbf{I\Sigma}_n$. Hence, *(1)* follows from **5.1**. Parts *(2)*, *(3)* and *(4)* are consequences of *(1)*. □

Proposition 5.6. *1. (*$k > 0$*) There does not exist a class of sentences* $\Phi \subseteq \Sigma_{n+2}$ *such that* $\mathbf{I\Sigma}_n + \Phi$ *is consistent and*

$$\mathbf{I\Sigma}_n + \Phi \Longrightarrow \mathbf{I\Sigma}_n + \forall x\, \exists y\, [\mathbb{F}_{n,k}(x) = y]$$

2. *There does not exist a class of sentences* $\Phi \subseteq \Sigma_{n+2}$ *such that* $\mathbf{I\Sigma}_n + \Phi$ *is consistent and* $\mathbf{I\Sigma}_n + \Phi \Longrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_{n+1})$.

Proof. (*(1)*): By way of contradiction suppose that there is a class Φ such that $\mathbf{I\Sigma}_n + \Phi \Longrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_n + \forall x\, \exists y\, [\mathbb{F}_{n,k}(x) = y])$. Let $\varphi(u, x, y)$ be $\mathrm{IT}_{\mathbb{F}_{n,0}}(u, x, y)$. Then, $\varphi(u, x, y)$ is a strong Π_n–envelope of $\mathbf{I\Sigma}_n$ in $\mathbf{I\Sigma}_n$ such that $\mathbf{I\Sigma}_n + \forall x\, \exists y\, [\mathbb{F}_{n,k}(x) = y]$ proves

(–) $\forall u\, \forall x\, \exists y\, \varphi(u, x, y)$, and
(–) $\forall u, x, y_1, y_2\, [\varphi(u, x, y_1) \wedge \varphi(u + 1, x, y_2) \rightarrow y_1 < y_2]$.

Since $\mathbf{I\Sigma}_n + \forall x\, \exists y\, [\mathbb{F}_{n,k}(x) = y]$ is finitely axiomatizable (for $n = 0$, as $k \geq 1$, $\mathbf{I\Sigma}_n + \forall x\, \exists y\, [\mathbb{F}_{0,k}(x) = y] \vdash \mathbf{exp}$), then there exists $\psi \in \Phi$ such that

$$\mathbf{I\Sigma}_n + \psi \Longrightarrow \mathbf{I\Sigma}_n + \forall x\, \exists y\, [\mathbb{F}_{n,k}(x) = y].$$

Let $\mathfrak{A} \models (\mathbf{I\Sigma}_n + \psi)_{\Gamma_\varphi}$, $a \in \mathfrak{A}$ nonstandard such that $\mathfrak{A} \models \psi_0(a)$ (where ψ is $\exists x\, \psi_0(x)$, with $\psi_0(x) \in \Pi_{n+1}$); and let $\mathfrak{B} = \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A}, a)$ as in [7]–**6.5**. Then, by [7]–**6.6**, $\mathfrak{B} \models \mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_n)$. So, $\mathfrak{B} \models \mathbf{I\Sigma}_n$. By [7]–**6.5**–*(2)*, it holds that $\mathfrak{B} \prec_n \mathfrak{A}$ as $\mathcal{L}$–structures, so, $\mathfrak{B} \models \psi_0(a)$. Hence, $\mathfrak{B} \models \exists x\, \psi_0(x)$. So, $\mathfrak{B} \models \mathbf{I\Sigma}_n + \psi$. But, by [7]–**6.6**, $\mathfrak{B} \not\models \mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_n + \forall x\, \exists y\, [\mathbb{F}_{n,k}(x) = y])$. Hence, $\mathfrak{B} \not\models \mathbf{I\Sigma}_n + \forall x\, \exists y\, [\mathbb{F}_{n,k}(x) = y]$. Contradiction.
(*(2)*): It follows from *(1)*. □

6. Induction rules

In this section we shall apply the techniques developed in the above sections to obtain a new proof of Parsons' conservativeness theorem (see [16]) and a weak version of a result of Beklemishev on induction rules (see [2], corollary **9.1**). We are mainly interested in the analysis of the induction rule:

$$\text{IR}: \qquad \frac{\varphi(0), \quad \forall x\, (\varphi(x) \to \varphi(x+1))}{\forall x\, \varphi(x)}$$

and the collection rule:

$$\text{CR}: \qquad \frac{\forall x\, \exists y\, \varphi(x, y)}{\forall z\, \exists u\, \forall x \leq z\, \exists y \leq u\, \varphi(x, y)}$$

Let $\mathbf{T}$ be a theory and Γ a class of formulas. We shall denote by $\mathbf{T} + \Gamma$–IR the closure of $\mathbf{T}$ under first–order logic and applications of IR restricted to formulas $\varphi \in \Gamma$. Following the notation introduced in [2], $[\mathbf{T}, \Gamma$–IR$]$ is the closure of $\mathbf{T}$ under first–order logic and *unnested applications* of Γ–IR: that is, we can only apply Γ–IR if the premises are theorems of $\mathbf{T}$. Finally we define (the theories $\mathbf{T} + \Gamma$–CR and $[\mathbf{T}, \Gamma$–CR$]$ are defined in a similar way)

$$[\mathbf{T},\ \Gamma\text{–IR}]_0 = \mathbf{T}$$

$$[\mathbf{T},\ \Gamma\text{–IR}]_{k+1} = [[\mathbf{T},\ \Gamma\text{–IR}]_k, \Gamma\text{–IR}].$$

Proposition 6.1.

$$[\mathbf{I\Sigma}_n^{\varphi}, \Pi_{n+2}\text{–}IR] \Longleftrightarrow \mathbf{I\Sigma}_n^{\varphi} + \forall x\, \exists y\, D_\varphi(x, y) \Longleftrightarrow [\mathbf{I\Sigma}_n^{\varphi}, \Sigma_{n+1}\text{–}IR]$$

Proof. We recall that $\mathbb{F}_{\varphi,0}(x) = y$ is the formula $\varphi(x, y)$ and $D_\varphi(x, y)$ is $\mathbb{F}_{\varphi,1}(x) = y$. So, by **4.5**–*(2)*, it holds

$$[\mathbf{I\Sigma}_n^{\varphi}, \Pi_{n+2}\text{–IR}] \Longrightarrow [\mathbf{I\Sigma}_n^{\varphi}, \Sigma_{n+1}\text{–IR}] \Longrightarrow \mathbf{I\Sigma}_n^{\varphi} + \forall x\, \exists y\, D_\varphi(x, y),$$

Then, it is enough to prove that

$$\mathbf{I\Sigma}_n^{\varphi} + \forall x\, \exists y\, (D_\varphi(x, y)) \Longrightarrow [\mathbf{I\Sigma}_n^{\varphi}, \Pi_{n+2}\text{–IR}]$$

We must prove that, for each $\psi(u) \in \Pi_{n+2}$, if

$$\mathbf{I}\Sigma_n^{\varphi} \vdash \psi(0) \wedge \forall u\,(\psi(u) \to \psi(u+1))$$

then $\mathbf{I}\Sigma_n^{\varphi} + \forall x\,\exists y\, D_{\varphi}(x, y) \vdash \forall u\,\psi(u)$.

We can assume that $\psi(u)$ is $\forall x\,\exists y\,\theta(u, x, y)$, where $\theta(u, x, y) \in \Pi_n^-$. Indeed, Π_{n+2}–IR is reductible to its parameter free version. This is easily seen in this case, but it holds also for Σ_{n+1}–IR (see lemma **2.1** in [4]). So we have to prove that

$$(\bullet) \qquad \mathbf{I}\Sigma_n^{\varphi} + \forall x\,\exists y\, D_{\varphi}(x, y) \vdash \forall u\,\forall x\,\exists y\,\theta(u, x, y)$$

Next claims will provide bounds which allow us to reduce the quantifier complexity of the formulas considered.

6.1.1. There exists $k \in \omega$ such that $\mathbf{I}\Sigma_n^{\varphi} \vdash \forall x\,\exists y < \mathbb{F}^k_{\varphi,0}(x)\,\theta(0, x, y)$.

Proof. By hypothesis $\mathbf{I}\Sigma_n^{\varphi} \vdash \forall x\,\exists y\,\theta(0, x, y)$; so, the result follows from **4.6** ($\mathbb{F}^u_{\varphi,0}(x) = y$ is a Π_n–envelope of $\mathbf{I}\Sigma_n^{\varphi}$ en $\mathbf{I}\Sigma_n^{\varphi}$). □

6.1.2. Let f and F_{φ} be new function symbols of arity 1. Let $\mathbf{T}^f$ be the theory of language $\mathbf{L} = \mathcal{L} + f + F_{\varphi}$,

$$\mathbf{I}\Sigma_n^{\varphi} + \begin{cases} \forall x_1, x_2\,(x_1 \le x_2 \to f(x_1) \le f(x_2)) + \\ \forall x\,\forall y\,(\varphi(x, y) \leftrightarrow F_{\varphi}(x) = y) \end{cases}$$

Then there exist $t(x)$ and $s(x)$ terms of $\mathbf{L}$ such that:

(i) $\mathbf{T}^f \vdash \forall u\,[\forall x\,\exists y < f(x+u)\,\theta(u, x, y) \to \forall x\,\exists y < t(x+u)\,\theta(u+1, x, y)]$

(ii) $\mathbf{T}^f \vdash \forall u\,\forall x_2 \begin{cases} \forall x_1 < s(x_2+u)\,\exists y < f(x_1+u)\,\theta(u, x_1, y) \to \\ \qquad \exists y < t(x_2+u)\,\theta(u+1, x_2, y) \end{cases}$

Proof. (*(i)*): Let $\mathbf{c}$ be a new constant symbol. We prove that there exists a term $t(x)$ of $\mathbf{L}$ such that

$$\mathbf{T}^f \vdash \forall x\,\exists y < f(x+\mathbf{c})\,\theta(\mathbf{c}, x, y) \to \forall x\,\exists y < t(x+\mathbf{c})\,\theta(\mathbf{c}+1, x, y)$$

For the sake of a contradiction, assume that for each $t(x) \in \mathbf{Term}(\mathbf{L})$, there exist $\mathfrak{A}_t \models \mathbf{T}^f + \forall x\,\exists y < f(x+\mathbf{c})\,\theta(\mathbf{c}, x, y)$ and $a \in \mathfrak{A}_t$ such that

$$\mathfrak{A}_t \models \neg\exists y < t(x+\mathbf{c})\,\theta(\mathbf{c}+1, a, y)$$

Let $\mathbf{d}$ be a new constant symbol and $\mathbf{T}'$ the theory

$$\begin{aligned} &\mathbf{T}^f + \forall x\,\exists y < f(x+\mathbf{c})\,\theta(\mathbf{c}, x, y) \\ &\quad + \{\neg\exists y < t(\mathbf{c}+\mathbf{d})\,\theta(\mathbf{c}+1, \mathbf{d}, y) :\ t(x) \in \mathbf{Term}(\mathbf{L})\}. \end{aligned}$$

By compactness, $\mathbf{T}'$ is consistent. Indeed, if $t_1, \ldots, t_n$ are terms corresponding to a finite part, $\mathbf{T}''$, of $\mathbf{T}'$, and t is $t_1 + \cdots + t_n$, then $\mathfrak{A}_t \models \mathbf{T}''$ (interpreting $\mathbf{d}$ in $\mathfrak{A}_t$ as a).

Let $\mathfrak{A} \models \mathbf{T}'$ and $a = \mathfrak{A}(\mathbf{d})$. Let $\mathfrak{I}$ be the initial segment

$$\mathfrak{I} = \{e \in \mathfrak{A} :\ \text{There exists } t(x) \in \mathbf{Term}(\mathbf{L}),\ \mathfrak{A} \models e < t(a+\mathbf{c})\}.$$

Then $\mathfrak{I}$ is closed under the function defined in $\mathfrak{A}$ by F_φ and, as a consequence, under the function defined by $\mathbb{K}_n$. Hence, $\mathfrak{I} \prec_n \mathfrak{A}$ as $\mathcal{L}$–structures, and $\mathfrak{I}$ is closed under f. From this we get that $\mathfrak{I} \models \mathbf{I}\Sigma_n^\varphi$ and, as $\theta \in \Pi_n$,

$$\mathfrak{I} \models \forall x\, \exists y < f(x + \mathbf{c})\, \theta(\mathbf{c}, x, y).$$

So, $\mathfrak{I} \models \mathbf{T}^f + \forall x\, \exists y < f(x + \mathbf{c})\, \theta(\mathbf{c}, x, y)$. On the other hand,

$$\mathbf{I}\Sigma_n^\varphi \vdash \forall u\, (\forall x\, \exists y\, \theta(u, x, y) \to \forall x\, \exists y\, \theta(u + 1, x, y));$$

hence, $\mathfrak{I} \models \forall x\, \exists y\, \theta(\mathbf{c} + 1, x, y)$. Since $\mathfrak{I} \models \neg\exists y\, \theta(\mathbf{c} + 1, a, y)$, this provides the required contradiction.

(*(ii)*): From *(i)* it follows that $\mathbf{T}^f$ proves that

$$\forall u\, \forall x_2\, \exists x_1\, [\exists y < f(x_1 + u)\, \theta(u, x_1, y) \to \exists y < t(x_2 + u)\, \theta(u + 1, x_2, y)]$$

As in *(i)* it is proved that there exists a term $s(x)$ of $\mathbf{L}$ such that $\mathbf{T}^f$ proves

$$\exists x_1 < s(x_2 + u)\, [\exists y < f(x_1 + u)\, \theta(u, x_1, y) \to \exists y < t(x_2 + u)\, \theta(u + 1, x_2, y)].$$

From this it follows *(ii)*. □

6.1.3. There exist $m, q \in \omega$ such that if $\mathfrak{A} \models \mathbf{I}\Sigma_n^\varphi + \forall x\, \exists y\, D_\varphi(x, y)$ and $a, b \in \mathfrak{A}$, then the following formulas are true in $\mathfrak{A}$:

$$\forall x\, \exists y < \mathbb{F}_{\varphi,0}^{b}(x + a)\, \theta(a, x, y) \to \forall x\, \exists y < \mathbb{F}_{\varphi,0}^{b\cdot m}(x + a)\, \theta(a + 1, x, y),$$

$$\forall x_1 < \mathbb{F}_{\varphi,0}^{q\cdot b}(x_2 + a)\, \exists y < \mathbb{F}_{\varphi,0}^{b}(x_1 + a)\, \theta(a, x_1, y) \to$$
$$\to \exists y < \mathbb{F}_{\varphi,0}^{b\cdot m}(x_2 + a)\, \theta(a + 1, x_2, y)$$

Proof. Let $\mathfrak{B}$ be the expansion of $\mathfrak{A}$ to $\mathbf{L}$ given by $f(x) = \mathbb{F}_{\varphi,0}^{b}(x)$ and $F_\varphi(x) = \mathbb{F}_{\varphi,0}(x)$. Let $t(x)$ and $s(x)$ be two terms as in **6.1.2**. Then, by induction on terms of $\mathbf{L}$, we obtain that there exist $m, q \in \omega$ such that

$$\mathfrak{B} \models t(x) < \mathbb{F}_{\varphi,0}^{m\cdot b}(x) \wedge s(x) < \mathbb{F}_{\varphi,0}^{q\cdot b}(x).$$

This concludes the proof of the claim. □

Now we prove (•). Let k, m y q be as in **6.1.1** and **6.1.3**, and $r = \max(k, m, q, 2)$. Let $\mathfrak{A} \models \mathbf{I}\Sigma_n^\varphi + \forall x\, \exists y\, D_\varphi(x, y)$ and $a, d \in \mathfrak{A}$. Let us see that

$$\mathfrak{A} \models \exists y < \mathbb{F}_{\varphi,0}^{r^{a+1}}(d + a)\, \theta(a, d, y)$$

For each $j \le a$, let $e_j = \frac{(a+1)(a+2)}{2} - \frac{(j+1)(j+2)}{2}$. By induction on $j \le a$ we prove that

$$(\star) \quad \mathfrak{A} \models \forall j \le a\, \forall x \le \mathbb{F}_{\varphi,0}^{r^{e_j}}(a + d)\, \exists y < \mathbb{F}_{\varphi,0}^{r^{j+1}}(x + j)\, \theta(j, x, y).$$

$\underline{j = 0}$: Since $\mathfrak{A} \models \forall x\, \exists y\, \theta(0, x, y)$, the result follows from **6.1.1**.

$\underline{j \to j + 1}$: Assume that $(\star)$ holds for $j < a$. Let $a_2 \le \mathbb{F}_{\varphi,0}^{r^{e_{j+1}}}(a + d)$ and $a' = \max(a_2, a)$. Then $a' \le \mathbb{F}_{\varphi,0}^{r^{e_{j+1}}}(a + d)$ and

$$\mathfrak{A} \models x_1 < \mathbb{F}_{\varphi,0}^{q\cdot r^{j+1}}(a_2 + a) \to x_1 < \mathbb{F}_{\varphi,0}^{q\cdot r^{j+1}}(2a') \le \mathbb{F}_{\varphi,0}^{r^{j+2}}(a') \le \mathbb{F}_{\varphi,0}^{r^{e_j}}(a + d).$$

Hence, by hypothesis, we get that

$$\mathfrak{A} \models \forall x_1 < \mathbb{F}_{\varphi,0}^{q \cdot r^{j+1}}(a_2 + a)\, \exists y < \mathbb{F}_{\varphi,0}^{r^{j+1}}(x_1 + j)\, \theta(j, x_1, y).$$

So, by **6.1.3**, $\mathfrak{A} \models \exists y < \mathbb{F}_{\varphi,0}^{r^{j+1}m}(a_2 + j)\, \theta(j+1, a_2, y)$. Hence,

$$\mathfrak{A} \models \forall x_2 < \mathbb{F}_{\varphi,0}^{r^{e^{j+1}}}(a + d)\, \exists y < \mathbb{F}_{\varphi,0}^{r^{j+2}}(x_2 + j + 1)\, \theta(j+1, x_1, y),$$

and this proves $(\star)$. Taking $j = a$ in $(\star)$, we obtain that

$$\mathfrak{A} \models \forall x \leq \mathbb{F}_{\varphi,0}^{0}(a + d)\, \exists y < \mathbb{F}_{\varphi,0}^{r^{a+1}}(x + a)\, \theta(a, x, y)$$

Since $d < \mathbb{F}_{\varphi,0}^{0}(a+d) = a+d$, we have $\mathfrak{A} \models \exists y < \mathbb{F}_{\varphi,0}^{r^{a+1}}(d+a)\, \theta(a, d, y)$. This concludes the proof of the proposition. □

Lemma 6.2. *Let Φ be a class of Σ_{n+2}–sentences. Then the following theories, $[\mathbf{I\Sigma}_n^{\varphi} + \Phi,\ \Pi_{n+2}\text{–}IR]$ and $[\mathbf{I\Sigma}_n^{\varphi} + \Phi,\ \Sigma_{n+1}\text{–}IR]$, are Π_{n+2}–conservative extensions of $\mathbf{I\Sigma}_n^{\varphi} + \Phi + \forall x\, \exists y\, D_{\varphi}(x, y)$.*

Proof. We only prove the result for Π_{n+2}–IR, the other case being similar.

By **6.1**, $[\mathbf{I\Sigma}_n^{\varphi} + \Phi,\ \Pi_{n+2}\text{–IR}]$ is an extension of $\mathbf{I\Sigma}_n^{\varphi} + \Phi + \forall x\, \exists y\, (D_{\varphi}(x, y))$. Let us see that it is a Π_{n+2}–conservative one. Let $\theta(x) \in \Pi_{n+2}$ such that

(–) $\mathbf{I\Sigma}_n^{\varphi} + \Phi \vdash \theta(0)$, and
(–) $\mathbf{I\Sigma}_n^{\varphi} + \Phi \vdash \forall x\, (\theta(x) \to \theta(x+1))$.

Then there exists $\psi \in \Phi$ such that:

(–) $\mathbf{I\Sigma}_n^{\varphi} \vdash \psi \to \theta(0)$, and
(–) $\mathbf{I\Sigma}_n^{\varphi} \vdash \forall x\, [(\psi \to \theta(x)) \to (\psi \to \theta(x+1))]$.

Since $\psi \to \theta(x)$ is Π_{n+2}, then $[\mathbf{I\Sigma}_n^{\varphi},\ \Pi_{n+2}\text{–IR}] \vdash \forall x\, (\psi \to \theta(x))$. So, by **6.1**,

$$\mathbf{I\Sigma}_n^{\varphi} + \forall x\, \exists y\, D_{\varphi}(x, y) \vdash \forall x\, (\psi \to \theta(x))$$

So, $\mathbf{I\Sigma}_n^{\varphi} + \Phi + \forall x\, \exists y\, D_{\varphi}(x, y) \vdash \forall x\, \theta(x)$. □

Theorem 6.3. *For all $k \in \omega$,*

$$[\mathbf{I\Sigma}_n^{\varphi},\ \Pi_{n+2}\text{–}IR]_k \Longleftrightarrow [\mathbf{I\Sigma}_n^{\varphi},\ \Sigma_{n+1}\text{–}IR]_k \Longleftrightarrow \mathbf{I\Sigma}_n^{\varphi} + \forall x\, \exists y\, (\mathbb{F}_{\varphi,k}(x) = y)$$

Proof. It is enough to prove, by induction on k, that

$$[\mathbf{I\Sigma}_n^{\varphi},\ \Pi_{n+2}\text{–IR}]_k \Longleftrightarrow \mathbf{I\Sigma}_n^{\varphi} + \forall x\, \exists y\, (\mathbb{F}_{\varphi,k}(x) = y).$$

$\underline{k = 0}$: Since $\mathbb{F}_{\varphi,0}(x) = y$ is $\varphi(x, y)$; it holds that

$$[\mathbf{I\Sigma}_n^{\varphi},\ \Pi_{n+2}\text{–IR}]_0 \Longleftrightarrow \mathbf{I\Sigma}_n^{\varphi} \Longleftrightarrow \mathbf{I\Sigma}_n^{\varphi} + \forall x\, \exists y\, (\mathbb{F}_{\varphi,0}(x) = y).$$

$\underline{k \to k+1}$: Suppose that $[\mathbf{I\Sigma}_n^{\varphi},\ \Pi_{n+2}\text{–IR}]_k \Longleftrightarrow \mathbf{I\Sigma}_n^{\varphi} + \forall x\, \exists y\, (\mathbb{F}_{\varphi,k}(x) = y)$. By definition $[\mathbf{I\Sigma}_n^{\varphi},\ \Pi_{n+2}\text{–IR}]_{k+1} = [[\mathbf{I\Sigma}_n^{\varphi},\ \Pi_{n+2}\text{–IR}]_k,\ \Pi_{n+2}\text{–IR}]$, so

$$[\mathbf{I\Sigma}_n^{\varphi},\ \Pi_{n+2}\text{–IR}]_{k+1} \Longleftrightarrow [\mathbf{I\Sigma}_n^{\varphi} + \forall x\, \exists y\, (\mathbb{F}_{\varphi,k}(x) = y),\ \Pi_{n+2}\text{–IR}].$$

Let $\theta(x, y) \in \Pi_n^-$ be the formula $\mathbb{F}_{\varphi,k}(x) = y \wedge \exists z \leq y\, \varphi(x, z)$ and $\mathrm{KIPF}_n(\varphi)$ the Π_{n+1}–formula

$$\mathrm{IPF}(\varphi) \wedge \forall x, y_1, y_2\, (\mathbb{K}_n(x) = y_1 \wedge \varphi(x, y_2) \rightarrow y_1 \leq y_2).$$

Then, by **4.5** and **3.7**, $\mathbf{I\Sigma}_n^\theta + \mathrm{IPF}(\varphi) \Longleftrightarrow \mathbf{I\Sigma}_n^\varphi + \forall x\, \exists y\, (\mathbb{F}_{\varphi,k}(x) = y)$ and

$$\mathbf{I\Sigma}_n^\theta + \mathrm{KIPF}_n(\varphi) + \forall x\, \exists y\, D_\theta(x, y) \Longleftrightarrow \mathbf{I\Sigma}_n^\varphi + \forall x\, \exists y\, (\mathbb{F}_{\varphi,k+1}(x) = y).$$

Now, since $\mathrm{KIPF}_n(\varphi) \in \Pi_{n+1}$, by **6.2**, we get that

$$[\mathbf{I\Sigma}_n^\theta + \mathrm{KIPF}_n(\varphi),\ \Pi_{n+2}\text{–IR}] \Longleftrightarrow \mathbf{I\Sigma}_n^\varphi + \forall x\, \exists y\, (\mathbb{F}_{\varphi,k+1}(x) = y)$$

as required. □

Theorem 6.4. (Generalized Parsons' Theorem)

1. $\mathbf{I\Sigma}_n^\varphi + \Sigma_{n+1}$*–IR* $\Longleftrightarrow \mathbf{I\Sigma}_n^\varphi + \mathbf{ACK}_\varphi^*$

2. $\mathbf{I\Sigma}_{n+1}^{\varphi,n}$ *is a* Π_{n+2}*–conservative extension of* $\mathbf{I\Sigma}_n^\varphi + \Sigma_{n+1}$*–IR.*

Proof. *(1)* follows from **6.3**, and *(2)* from *(1)* and **5.1**. □

Theorem 6.5. *(Parsons)*
$\mathbf{I\Sigma}_{n+1}$ *is a* Π_{n+2}*–conservative extension of* $\mathbf{I\Delta}_0 + \Sigma_{n+1}$*–IR.*

Proof. Let $\varphi(x, y) \in \Pi_n$ be $\mathbb{K}_n(x) = y$. By [7]–**5.13**, $\mathbf{I\Sigma}_n^\varphi \Longleftrightarrow \mathbf{I\Sigma}_n$. Moreover, by lemmas **5.1** and **2.3** in [3], $\mathbf{I\Delta}_0 + \Sigma_{n+1}$–IR is closed under Σ_{n+1}–CR and, $[\mathbf{I\Delta}_0, \Sigma_{n+1}\text{–CR}] \Longrightarrow \mathbf{I\Sigma}_n$. So,

$$\mathbf{I\Delta}_0 + \Sigma_{n+1}\text{–IR} \Longleftrightarrow \mathbf{I\Sigma}_n + \Sigma_{n+1}\text{–IR};$$

hence, the result is a consequence of **6.4**–*(2),* □

From **6.4** and **5.6**, we obtain (see also [4]) the following result.

Corollary 6.6. $\mathbf{I\Sigma}_n + \mathbf{I\Pi}_{n+1}^- \not\Longleftrightarrow \mathbf{I\Sigma}_n + \Sigma_{n+1}$*–IR.*

Proof. Let $\varphi(x, y) \in \Pi_n^-$ be $\mathbb{K}_n(x) = y$. Then, $\mathbf{I\Sigma}_n^\varphi \Longleftrightarrow \mathbf{I\Sigma}_n$. By **6.4**–*(1),* $\mathbf{I\Sigma}_n + \Sigma_{n+1}\text{–IR} \Longleftrightarrow \mathbf{I\Sigma}_n + \mathbf{ACK}_\varphi^*$. Since $\mathbf{I\Pi}_{n+1}^-$ is Σ_{n+2}–axiomatizable, the result follows from **5.6**. □

We conclude with a proof of a (weak) version of corollary **9.1** in [2].

Theorem 6.7. *Let* $\mathbf{T}$ *be a* Π_{n+2}*–axiomatizable extension of* $\mathbf{I\Sigma}_n$*. If* $\mathbf{T}$ *is* Π_n*–functional then* $[\mathbf{T}, \Sigma_{n+1}\text{–IR}] \Longleftrightarrow [\mathbf{T}, \Pi_{n+2}\text{–IR}]$.
(By **6.2***, the result also holds if* $\mathbf{T}$ *is* $\Pi_{n+2} \cup \Sigma_{n+2}$*–axiomatizable).*

Proof. Since both theories are Π_{n+2}–axiomatizable and $[\mathbf{T}, \Pi_{n+2}\text{–IR}]$ is, obviously, an extension of $[\mathbf{T}, \Sigma_{n+1}\text{–IR}]$, it suffices to prove that this extension is Π_{n+2}–conservative. Let $\theta \in \Pi_{n+2}$ a sentence such that $[\mathbf{T}, \Pi_{n+2}\text{–IR}] \vdash \theta$. Then there exists $\psi(x, y) \in \Pi_n^-$ such that

$$[\mathbf{I\Sigma}_n + \forall x\, \exists y\, \psi(x, y), \Pi_{n+2}\text{–IR}] \vdash \theta.$$

As in **5.3**, let $\varphi(x, y) \in \Pi_n^-$ be the formula

$$\exists y_1, y_2 \leq y\, (\mathbb{K}_n(x) = y_1 \wedge \mathcal{C}_\psi(x, y_2) \wedge y = y_1 + y_2).$$

Then $\mathbf{I\Sigma}_n^\varphi \vdash \forall x \exists y\, \psi(x, y)$ and $[\mathbf{I\Sigma}_n^\varphi, \Pi_{n+2}\text{–IR}] \vdash \theta$. So, $[\mathbf{I\Sigma}_n^\varphi, \Sigma_{n+1}\text{–IR}] \vdash \theta$, by **6.1**. Since $\mathbf{T}$ extends $\mathbf{I\Sigma}_n^\varphi$, it follows that $[\mathbf{T}, \Sigma_{n+1}\text{–IR}] \vdash \theta$. □

7. Open questions and concluding remarks

The main problem we have studied in this paper is

(P) Under which conditions is $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ a Π_{n+2}–axiomatizable theory?

In **2.4** we have obtained that if $\mathbf{T}$ has Δ_{n+1}–induction, then $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ is a Π_{n+2}–axiomatizable theory if and only if $\mathbf{I}\Sigma_{n+1}$ extends $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$. Let us add the following property to the ones included in **2.4**:

0. $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$.

Then as in the proof of **2.4**, without assuming that $\mathbf{T}$ has Δ_{n+1}–induction, we get that:

$$(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (0) \Longleftrightarrow (4).$$

This raises the following problem:

Problem 7.1. Let $\mathbf{T}$ be an extension of $\mathbf{I}\Sigma_n$ such that $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ extends $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$. Does $\mathbf{T}$ have Δ_{n+1}–induction?

As we have proved in **2.5**, there exist theories $\mathbf{T}$ such that $\mathbf{I}\Delta_1(\mathbf{T})$ is Σ_2–axiomatizable, e.g. $\mathbf{I}\Delta_0$. Nevertheless, $\mathbf{I}\Delta_1(\mathbf{I}\Delta_0)$ is Π_1–axiomatizable (it is equivalent to $\mathbf{I}\Delta_0$). In **2.5.4** we obtained a condition under which $\mathbf{I}\Delta_1(\mathbf{T})$ is not Σ_2–axiomatizable. The proof of this result rested on **2.5.1**. This raises the following question:

Problem 7.2. (On Σ_2–axiomatization) Let $\mathbf{T}$ be a Π_0–functional theory. Are the following conditions equivalent?

1. $\mathbf{I}\Delta_1(\mathbf{T})$ is Π_1–axiomatizable.
2. For every $\varphi(x, y) \in \Delta_0$ such that $\mathbf{I}\Delta_1(\mathbf{T}) \vdash \forall x\, \exists y\, \varphi(x, y)$ there exists a term $t(x)$ such that $\mathbf{I}\Delta_1(\mathbf{T}) \vdash \forall x\, \exists y \leq t(x)\, \varphi(x, y)$.
3. $\mathbf{I}\Delta_1(\mathbf{T})$ is Σ_2–axiomatizable.
4. For every $\varphi(x, y) \in \Delta_0$ such that $\mathbf{I}\Delta_1(\mathbf{T}) \vdash \forall x\, \exists y\, \varphi(x, y)$ there exists a term $t(x)$ such that $\mathbf{I}\Delta_1(\mathbf{T}) \vdash \exists u\, \forall x\, [u < x \rightarrow \exists y \leq t(x)\, \varphi(x, y)]$.

Let $\mathbf{T}$ be a theory such that $\mathbf{I}\Delta_1(\mathbf{T}) \Longleftrightarrow \mathbf{Th}_{\Pi_2}(\mathbf{T})$. Then

$$(1) \Longleftrightarrow (2) \text{ and } (3) \Longleftrightarrow (4)$$

Indeed, (1) $\Longleftrightarrow$ (2) follows from Parikh's theorem for sound Π_1 axiomatizable theories. Similarly, we get (3) $\Longleftrightarrow$ (4) from **2.5.1**.

By **2.4** we know that $\mathbf{I}\Delta_1(\mathbf{I}\Pi_1^-) \Longleftrightarrow \mathbf{Th}_{\Pi_2}(\mathbf{I}\Pi_1^-)$. From the above remark, $\mathbf{I}\Delta_1(\mathbf{I}\Pi_1^-)$ is Σ_2–axiomatizable. So, if problem **7.2** has an affirmative answer, then $\mathbf{I}\Delta_1(\mathbf{I}\Pi_1^-)$ is Π_1–axiomatizable and each $\Delta_1(\mathbf{I}\Pi_1^-)$ formula is equivalent, in $\mathbf{I}\Pi_1^-$, to a Δ_0 formula. As it is proved in [9], $\mathbf{I}\Pi_1^-$ proves that there exist infinitely many primes. Hence, the above remarks suggest relationships between problem **7.2** and Wilkie's problem on the provability in $\mathbf{I}\Delta_0$ of the existence of infinitely many primes.

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