On axiom schemes for T-provably Δ_1 formulas

A. Cordón-Franco · A. Fernández-Margarit · F. F. Lara-Martín

Abstract This paper investigates the status of the fragments of Peano Arithmetic obtained by restricting induction, collection and least number axiom schemes to formulas which are Δ_1 provably in an arithmetic theory T. In particular, we determine the provably total computable functions of this kind of theories. As an application, we obtain a reduction of the problem whether $I\Delta_0 + \neg exp$ implies $B\Sigma_1$ to a purely recursion-theoretic question.

Keywords Fragments of Peano Arithmetic · Δ_1 formulas · Provably total computable functions

Mathematics Subject Classification 03F30 · 03D20

1 Introduction

Among the subsystems of first order Peano Arithmetic (PA), fragments for Δ_1-formulas are not completely understood yet. A well-known problem posed by Paris [6] asks whether, over the theory of bounded induction $I\Delta_0$, the induction principle for Δ_n-formulas $I\Delta_n$ and the collection principle for Σ_n-formulas $B\Sigma_n$ are equivalent. By a result of R. Gandy (unpublished, see [12]), $B\Sigma_n$ is equivalent to the least number principle for Δ_n-formulas $L\Delta_n$. Hence, Paris' question can be reformulated as asking whether $I\Delta_n$ and $L\Delta_n$ are equivalent. In 2004 Slaman [21] obtained a partial answer to the problem. He proved $I\Delta_n$ and $B\Sigma_n$ to be equivalent over $I\Delta_0 + exp$, where *exp* is the axiom asserting the totality of the exponential function. Since $I\Delta_2$ proves *exp*, this answered the problem completely for each $n \geq 2$. As to the case $n = 1$, building on Slaman's work Thapen [23] showed that $B\Sigma_1$ is provable from $I\Delta_1$ plus a very weak form of exponentiation: "for all x, x^y exists for some y such that $x < p(y)$", where p can be any primitive recursive function. (An alternative proof of this result was given in [20]). However, the problem of proving or disproving the equivalence over $I\Delta_0$ for $n = 1$ is still pending.

Motivated by this question, we initiated in [11] and [7] the study of fragments of *PA* for formulas that are Δ_1 provably in an *external* theory T. More precisely, let T be an extension of $I\Delta_0$ in the language of arithmetic. The theory $I\Delta_1(T)$ is axiomatized over Robinson's Q by the axiom scheme

$$(I_\varphi) \quad \varphi(0, \mathbf{v}) \wedge \forall x\, (\varphi(x, \mathbf{v}) \rightarrow \varphi(x+1, \mathbf{v})) \;\rightarrow\; \forall x\, \varphi(x, \mathbf{v}),$$

where $\varphi(x, \mathbf{v}) \in \Delta_1(T)$, i.e., $\varphi(x, \mathbf{v}) \in \Sigma_1$ and there is some $\psi(x, \mathbf{v}) \in \Pi_1$ such that $T \vdash \forall x, \mathbf{v}\, (\varphi(x, \mathbf{v}) \leftrightarrow \psi(x, \mathbf{v}))$. The theory $L\Delta_1(T)$ is Q together with

$$(L_\varphi) \quad \exists x\, \varphi(x, \mathbf{v}) \;\rightarrow\; \exists x\, (\varphi(x, \mathbf{v}) \wedge \forall y < x\, \neg\varphi(y, \mathbf{v})),$$

where $\varphi(x, \mathbf{v}) \in \Delta_1(T)$. The theory $B\Delta_1(T)$ consists of $I\Delta_0$ plus

$$(B_\varphi) \quad \forall x\, \exists y\, \varphi(x, y, \mathbf{v}) \;\rightarrow\; \forall z\, \exists u\, \forall x \leq z\, \exists y \leq u\, \varphi(x, y, \mathbf{v}),$$

where $\varphi(x, y, \mathbf{v}) \in \Sigma_1$ and $T \vdash \forall x\, \exists y\, \varphi(x, y, \mathbf{v})$ (so $\exists y\, \varphi(x, y, \mathbf{v}) \in \Delta_1(T)$).

A variant of Paris' problem then arises: *For which theories T does the equivalence $I\Delta_1(T) \equiv L\Delta_1(T)$ hold?*

Besides this original motivation, $\Delta_1(T)$-schemes have turned out to be interesting subsystems of *PA* in their own right. On the one hand, $\Delta_1(T)$ formulas appear naturally in the study of fragments of *PA*, remarkably in connection with the computable functions provably total in T. In fact, as we shall show in this paper, $\Delta_1(T)$-schemes exhibit a nice computational behavior: it is possible to give neat characterizations of their provably total functions by means of some subrecursive operators. On the other hand, $\Delta_1(T)$-schemes are closely related to theories of arithmetic described in terms of *inference rules*. In fact, $T + I\Delta_1(T)$ coincides with the closure of T under unnested applications of the Δ_1-induction rule [T, Δ_1-IR]. Even more, $I\Delta_1(T)$ precisely isolates the amount of induction axioms added to T by unnested applications of Δ_1-IR. Similar remarks apply to $L\Delta_1(T)$ and $B\Delta_1(T)$ considering the Δ_1-minimization rule Δ_1-LR and the Σ_1-collection rule Σ_1-CR, respectively.

In this work we go a step further and show that, as a matter of fact, $\Delta_1(T)$-schemes can be fully characterized as the intersection between a "classic" scheme for Σ_1-formulas and an inference rule theory. More precisely, let $Th_\Gamma(T)$ denote the set of all Γ-consequences of a theory T. Then, for each sentence φ we have

$I\Delta_1(T) \vdash \varphi$ if, and only if, both $I\Sigma_1 \vdash \varphi$ and $[Th_{\Pi_2}(T), \Delta_1\text{-IR}] \vdash \varphi$;
$L\Delta_1(T) \vdash \varphi$ if, and only if, both $I\Sigma_1 \vdash \varphi$ and $[Th_{\Pi_2}(T), \Delta_1\text{-LR}] \vdash \varphi$;

$B\Delta_1(T) \vdash \varphi$ if, and only if, both $B\Sigma_1 \vdash \varphi$ and $[Th_{\Pi_2}(T), \Sigma_1\text{-CR}] \vdash \varphi$.

Thus, the study of $\Delta_1(T)$-schemes can be reduced to investigating how the properties of two theories are transferred to the theory given by the intersection of their theorems. Using this methodology we shall obtain a complete description of the proof-theoretic and computational properties of $\Delta_1(T)$-schemes.
Notably:

- We show that Slaman's theorem transfers to the present context and prove that $I\Delta_1(T)$ and $L\Delta_1(T)$ are equivalent for every T extending $I\Delta_0 + exp$.
- In studying parameter free $\Delta_1(T)$-schemes we introduce parameter free Δ_1-rules Δ_1^--IR and Δ_1^--LR (to our best knowledge, considered here for the first time) and obtain a conservation result, which is of independent interest. Namely, if $T \subseteq \Pi_2$ then $[T, \Delta_1\text{-IR}]$ and $[T, \Delta_1\text{-LR}]$ are conservative over their parameter free counterparts with respect to Σ_2-sentences.
- We determine the provably total computable functions (p.t.c.f.) of $I\Delta_1(T)$ and of $L\Delta_1(T)$ for an arbitrary T extending $I\Delta_0$. We show that the p.t.c.f.'s of $L\Delta_1(T)$ are, precisely, the closure under composition and the bounded minimization operator of the p.t.c.f.'s of T which are primitive recursive. For $I\Delta_1(T)$ we obtain a similar result in terms of the *search operator* introduced in [5]. In addition, in presence of *exp* we give alternative and particularly neat characterizations by means of a suitably modified version of the bounded recursion operator, that we call *C-bounded recursion*.
- We obtain a reduction of the well-known problem whether $I\Delta_0 + \neg exp$ implies $B\Sigma_1$ (for short, the NE Problem) raised by Wilkie and Paris [24] to a purely recursion-theoretic question. Namely, $B\Sigma_1$ is not provable from $I\Delta_0 + \neg exp$ if there is some elementary function f with a Δ_0-definable graph such that the function $x \mapsto max_{i \in [0,x]} f(i)$ cannot be obtained by composition from f and rudimentary functions.

The outline of the paper is as follows. Sections 1 and 2 are introductory. Section 3 contains the proof of the characterization theorem for $\Delta_1(T)$-schemes and several applications. (In particular, we solve a number of questions left over from [11] and [7]). In Sect. 4 we investigate parameter free $\Delta_1(T)$-schemes and parameter free Δ_1-inference rules. Finally, Sect. 5 is devoted to determining the p.t.c.f.'s of $I\Delta_1(T)$ and of $L\Delta_1(T)$ and contains the above-mentioned reduction for the NE Problem.

2 Preliminaries

We assume familiarity with basic notions and results concerning fragments of Peano Arithmetic (all relevant information can be found in [12]). We work in the usual first-order language of arithmetic $\mathcal{L} = \{0, 1, +, \cdot, \leq\}$. We denote by $\mathbb{N}$ the standard model of arithmetic and say that a theory T is sound if all its axioms are true in $\mathbb{N}$. As usual, the formulas of $\mathcal{L}$ are classified in the Σ_n/Π_n hierarchy, Δ_0 denotes the class of bounded formulas, i.e., formulas with bounded quantifiers only, and $\mathcal{B}(\Sigma_n)$ denotes the class of boolean combinations of Σ_n-formulas. For $\Gamma = \Sigma_n$ or Π_n, $I\Gamma$ denotes Q plus the scheme of induction for Γ-formulas, $L\Gamma$ denotes Q plus minimization for

Γ-formulas, and $B\Gamma$ denotes $I\Delta_0$ plus collection for Γ-formulas. Fragments $I\Delta_n$ and $L\Delta_n$ are given by Q together with

$$(\varphi(x,\mathbf{v}) \leftrightarrow \psi(x,\mathbf{v})) \rightarrow I_{\varphi(x,\mathbf{v})}\,; \qquad (\varphi(x,\mathbf{v}) \leftrightarrow \psi(x,\mathbf{v})) \rightarrow L_{\varphi(x,\mathbf{v})},$$

where $\varphi \in \Sigma_n$ and $\psi \in \Pi_n$. Recall from [14] that $E\Gamma^-$ denotes the parameter free version of the theory $E\Gamma$. We also write $\varphi(\mathbf{x}) \in \Gamma^-$ to mean that $\varphi(\mathbf{x})$ is in Γ and contains no other free variables than the ones shown. We will be concerned with theories described in terms of inference rules too. The Γ-induction rule, Γ-IR, and the Γ-collection rule, Γ-CR, are given by

$$\frac{\varphi(0,\mathbf{v}) \wedge \forall x\,(\varphi(x,\mathbf{v}) \rightarrow \varphi(x+1,\mathbf{v}))}{\forall x\,\varphi(x,\mathbf{v})}\,; \qquad \frac{\forall x\,\exists y\,\varphi(x,y,\mathbf{v})}{\forall z\,\exists u\,\forall x \le z\,\exists y \le u\,\varphi(x,y,\mathbf{v})},$$

where $\varphi \in \Gamma$. Similarly, Δ_n-IR and Δ_n-LR are given by

$$\frac{\varphi(x,\mathbf{v}) \leftrightarrow \psi(x,\mathbf{v})}{I_{\varphi(x,\mathbf{v})}}\,; \qquad \frac{\varphi(x,\mathbf{v}) \leftrightarrow \psi(x,\mathbf{v})}{L_{\varphi(x,\mathbf{v})}},$$

with $\varphi \in \Sigma_n$ and $\psi \in \Pi_n$. Following [3], given an inference rule R and a theory T, $T + R$ denotes the closure of T under R and first order logic; while $[T, R]$ denotes the closure of T under *non-nested* applications of R and first order logic. A rule R_1 is *reducible* to R_2 if $[T, R_1] \subseteq [T, R_2]$ for every theory T extending $I\Delta_0$; two rules R_1 and R_2 are *congruent* if they are mutually reducible to each other.

In the present paper by an arbitrary arithmetic theory T we mean any extension of $I\Delta_0$ in the language $\mathcal{L}$. In particular, Cantor's pairing function $\langle x, y\rangle = \frac{(x+y+1)\cdot(x+y)}{2} + x$ and projections $y = (x)_0$ and $y = (x)_1$ will be available in all our theories.

Finally, if $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures we write $\mathfrak{A} \prec_\Gamma \mathfrak{B}$ to mean that $\mathfrak{A}$ is a Γ-elementary substructure of $\mathfrak{B}$, i.e., for all $\varphi(\mathbf{x}) \in \Gamma$ and $\mathbf{a} \in \mathfrak{A}$, $\mathfrak{A} \models \varphi(\mathbf{a})$ if, and only if, $\mathfrak{B} \models \varphi(\mathbf{a})$. We denote by $\mathcal{K}_n(\mathfrak{A}, p)$ the submodel of $\mathfrak{A}$ consisting of elements which are Σ_n-definable (possibly with a parameter p). Submodels of Σ_n-definable elements are natural examples of Σ_n-elementary substructures. In addition, since [16] and [17] it has been known that they provide examples of arithmetic structures where Σ_n-collection fails. In [9] we obtained the following strengthening of these old results.

Proposition 1 ([9], Theorem 3.6)

1. *If* $\mathfrak{A} \models I\Delta_0$ *and* $p \in \mathfrak{A}$ *is nonstandard,* $\mathcal{K}_1(\mathfrak{A}, p) \not\models B\Sigma_1 + exp$.
2. *If* $\mathfrak{A} \models I\Delta_0$ *and* $\mathcal{K}_1(\mathfrak{A})$ *is nonstandard,* $\mathcal{K}_1(\mathfrak{A}) \not\models L\Delta_1^- + exp$.
3. *If* $\mathfrak{A} \models B\Sigma_1$ *and* $p \in \mathfrak{A}$ *is nonstandard and* Π_1*-minimal (i.e.,* p *is the least element satisfying some* Π_1*-formula), then* $\mathcal{K}_1(\mathfrak{A}, p) \not\models B\Sigma_1^- + exp$.

3 Models of $\Delta_1(T)$-schemes

Let T be a fixed but arbitrary extension of $I\Delta_0$. In this section we prove our characterization theorem for $\Delta_1(T)$-schemes and obtain their basic proof-theoretic properties.

Although we shall concentrate on the case $n = 1$, our results easily generalize to $\Delta_n(T)$-schemes for an arbitrary $n \geq 1$. First, recall from [11] that

Lemma 1 1. $L\Delta_1(T) \vdash I\Delta_1(T)$.
2. $L\Delta_1(T) \vdash B\Delta_1(T)$.
3. $Th_{\Pi_2}(T) + B\Delta_1(T) \vdash L\Delta_1(T)$.

The proofs are easy adaptations of the proofs that $L\Sigma_1 \vdash I\Sigma_1$ and $L\Delta_1 \equiv B\Sigma_1$ (see e.g., [12]). In particular, it follows that over T, $L\Delta_1(T)$ and $B\Delta_1(T)$ are deductively equivalent, which is a reformulation of the fact that Δ_1-LR and Σ_1-CR are congruent rules.

Turning to the characterization theorem, we will reformulate the theorems of a $\Delta_1(T)$-scheme as the intersection of the theorems of other two theories. Or, equivalently, we will reformulate the class of models of a $\Delta_1(T)$-scheme as the union of the models of other two theories. This motivates the following definition.

Definition 1 Let S and T be $\mathcal{L}$-theories and let $Ax(S)$ and $Ax(T)$ be the sets of their non-logical axioms. Then $S \vee T$ is the theory whose non-logical axioms are the set of sentences $\{\varphi \vee \theta : \varphi \in Ax(S) \text{ and } \theta \in Ax(T)\}$.

Lemma 2 *$\mathfrak{A} \models S \vee T$ if and only if either $\mathfrak{A} \models S$ or $\mathfrak{A} \models T$. Hence, for each φ, $S \vee T \vdash \varphi$ if and only if both $S \vdash \varphi$ and $T \vdash \varphi$.*

We are now ready to state our result.

Theorem 1 (Transfer theorem)

1. $I\Delta_1(T) \vdash Th_{\Pi_2}(T) \vee I\Sigma_1$.
2. $L\Delta_1(T) \vdash Th_{\Pi_2}(T) \vee I\Sigma_1$.
3. $B\Delta_1(T) \vdash Th_{\Pi_2}(T) \vee B\Sigma_1$.

Proof We only write the proof of part 1. The remaining cases are analogous. Suppose $\mathfrak{A} \models I\Delta_1(T)$ and $\mathfrak{A} \not\models Th_{\Pi_2}(T)$. To see that $\mathfrak{A} \models I\Sigma_1$ consider $\varphi(x, v) \in \Sigma_1$. Since $\mathfrak{A} \not\models Th_{\Pi_2}(T)$, there are $\theta(w) \in \Sigma_1$ and $b \in \mathfrak{A}$ such that $T \vdash \forall w\, \theta(w)$ and $\mathfrak{A} \models \neg\theta(b)$. Put $\delta(x, v, w) \equiv \varphi(x, v) \vee \theta(w)$. Clearly, T proves $\forall v, w, x\, \delta(x, v, w)$ and so $\delta(x, v, w) \in \Delta_1(T)$. Hence, for all $a \in \mathfrak{A}$, $\mathfrak{A} \models I_{\delta(x,a,b)}$ by $I\Delta_1(T)$. But $\mathfrak{A} \models \varphi(x, a) \leftrightarrow \delta(x, a, b)$ since $\mathfrak{A} \models \neg\theta(b)$. Thus, $I_{\varphi(x,a)}$ is true in $\mathfrak{A}$. □

From Lemma 2 and Theorem 1 it follows that

Corollary 1 (Characterization theorem)

1. $I\Delta_1(T) \equiv [Th_{\Pi_2}(T), \Delta_1\text{-}IR] \vee I\Sigma_1$.
2. $L\Delta_1(T) \equiv [Th_{\Pi_2}(T), \Sigma_1\text{-}CR] \vee I\Sigma_1$.
3. $B\Delta_1(T) \equiv [Th_{\Pi_2}(T), \Sigma_1\text{-}CR] \vee B\Sigma_1$.

As a first application, we obtain a partial solution to the variant of Paris' problem for $\Delta_1(T)$-schemes. Since Corollary 1 associates $I\Delta_1(T)$ and $L\Delta_1(T)$ to the same classic scheme $I\Sigma_1$, it will suffice to show that Δ_1-IR and Σ_1-CR are congruent rules.

Proposition 2 *Suppose $T \vdash exp$. Then $[T, \Delta_1\text{-}IR] \equiv [T, \Sigma_1\text{-}CR]$.*

Proof Since Σ_1-CR and Δ_1-LR are congruent rules, it is clear that $[T, \Sigma_1\text{-CR}]$ implies $[T, \Delta_1\text{-IR}]$. The converse will follow by adapting Slaman's proof that $I\Delta_1 + exp \vdash B\Sigma_1$ (see Theorem 2.1 of [21]). Suppose $\mathfrak{A} \models T$ and $[T, \Sigma_1\text{-CR}]$ fails in $\mathfrak{A}$. Note that Σ_1-CR is reducible to its parameter free version Σ_1^--CR which in turn is reducible to Π_0^--CR. Hence, there is $\theta(x, y) \in \Pi_0^-$ such that

- $T \vdash \forall x\, \exists y\, \theta(x, y)$;
- B_θ fails in $\mathfrak{A}$ and so $\mathfrak{A} \models \forall u\, \exists x \le a\, \forall y \le u\, \neg\theta(x, y)$ for some $a \in \mathfrak{A}$.

Let $\delta(z)$ denote the Π_1-formula $\forall u\, \exists x \le z\, \forall y \le u\, \neg\theta(x, y)$. Slaman's proof shows us how to produce a failure of $I\Delta_1$ from a failure of $B\Sigma_1$. Inspection of that proof gives us that there are $\varphi(x, z) \in \Sigma_1$ and $\psi(x, z) \in \Pi_1$ such that

- $T \vdash \forall z\, (\delta(z) \to \forall x\, (\varphi(x, z) \leftrightarrow \psi(x, z))$
- $I_{\varphi(x,a)}$ fails in $\mathfrak{A}$.

Still we cannot conclude, as $\varphi(x, z)$ need not be in $\Delta_1(T)$. However, it suffices to modify $\varphi(x, z)$ a bit to produce a failure of $[T, \Delta_1\text{-IR}]$. To that end, write $\delta(z)$ as $\forall y\, \delta'(z, y)$, $\varphi(x, z)$ as $\exists y\, \varphi'(x, y, z)$, and $\psi(x, z)$ as $\forall y\, \psi'(x, y, z)$, with $\delta', \varphi', \psi' \in \Delta_0$. Then, we have

$$T \vdash \forall x, z\, \exists y\, [\neg\delta'(z, y) \vee \neg\psi'(x, y, z) \vee \varphi'(x, y, z)].$$

Write $\theta'(x, y, z)$ for the Δ_0-formula in square brackets above and consider

$$\begin{aligned} \varphi^\star(x, z) &\equiv \exists y\, (y = \mu t.\theta'(x, t, z) \ \wedge\ \varphi'(x, y, z)) \\ \psi^\star(x, z) &\equiv \forall y\, (y = \mu t.\theta'(x, t, z) \ \to\ \varphi'(x, y, z)) \end{aligned}$$

It is clear that $T \vdash \forall x, z\, (\varphi^\star(x, z) \leftrightarrow \psi^\star(x, z))$. In addition, it is easy to see that $T \vdash \delta(z) \to (\varphi^\star(x, z) \leftrightarrow \varphi(x, z))$ and so $I_{\varphi^\star(x,a)}$ fails in $\mathfrak{A}$ since $\mathfrak{A} \models \delta(a)$. Therefore, $\mathfrak{A} \not\models [T, \Delta_1\text{-IR}]$. □

Theorem 2 *Suppose $T \vdash exp$. Then $I\Delta_1(T) \equiv L\Delta_1(T)$.*

Proof Suppose $\mathfrak{A} \models I\Delta_1(T)$. If $\mathfrak{A} \models Th_{\Pi_2}(T)$ then $\mathfrak{A} \models B\Delta_1(T)$ by Proposition 2 and so $\mathfrak{A} \models L\Delta_1(T)$ by Lemma 1. If $\mathfrak{A} \not\models Th_{\Pi_2}(T)$ then $\mathfrak{A}$ satisfies $I\Sigma_1$ by Theorem 1. □

Remark 1
1. In [11] the authors proved the equivalence $I\Delta_1(T) \equiv L\Delta_1(T)$ provided T is an extension of $I\Delta_0$ closed under Σ_1-CR, and asked whether this condition is also necessary for that equivalence (see part 3 of Problem 7.1 in [11]). Theorem 2 answers in the negative that question.
2. It follows from Theorem 1 that $I\Delta_1(T) \vdash Th_{\Pi_2}(T)$ whenever $Th_{\Pi_2}(T) \subseteq I\Sigma_1$. This answers in the negative Problem 7.1 in [7], where the authors asked whether a theory T satisfying that $I\Delta_1(T) \vdash Th_{\Pi_2}(T)$ must be closed under Δ_1-IR.
3. It follows from Lemma 1 and Theorem 2 that $I\Delta_1(T) \vdash B\Delta_1(T)$ if $T \vdash exp$. But, in general, $B\Delta_1(T)$ does not imply $I\Delta_1(T)$ (for example, if $T = I\Delta_0 + exp$ then $I\Delta_1(T) \vdash exp$ whereas $B\Delta_1(T) \subseteq B\Sigma_1$). This differs from the classic case where $B\Delta_1 (\equiv B\Sigma_1) \vdash I\Delta_1$.

A second application of the Transfer Theorem is an unboundedness result for $\Delta_1(T)$-schemes. The so-called Kreisel–Lévy unboundedness theorems [15] are results stating that a certain fragment of arithmetic has no extensions of bounded quantifier complexity of a certain kind. Here we obtain the following variant of this family of results.

Proposition 3 (Unboundedness) *Suppose* $S \subseteq \Sigma_3$.

1. *If* $S \vdash I\Delta_1(T)$ *then* $S \vdash Th_{\Pi_2}(T)$.
2. *If* $S \vdash exp$ *and* $S \vdash B\Delta_1(T)$ *then* $S \vdash Th_{\Pi_2}(T)$.

Proof We only prove part 2. The proof of part 1 is similar. Towards a contradiction, assume $S \vdash B\Delta_1(T) + exp$ and S does not imply $Th_{\Pi_2}(T)$. Let θ be a Π_2 sentence such that $T \vdash \theta$ and $S \nvdash \theta$. It follows from Theorem 1 for $B\Delta_1(T)$ that $S + \neg\theta \vdash B\Sigma_1 + exp$. Since $B\Sigma_1 + exp$ is finitely axiomatizable, there is a single Σ_3 sentence φ such that $\varphi + \neg\theta$ is a consistent extension of $B\Sigma_1 + exp$. Let $\mathfrak{A}$ be a nonstandard model of $\varphi + \neg\theta$. Put $\varphi \equiv \exists x\, \varphi'(x)$ and $\neg\theta \equiv \exists x\, \theta'(x)$, with $\varphi'(x) \in \Pi_2$ and $\theta'(x) \in \Pi_1$, and pick $a, b, c \in \mathfrak{A}$ such that a is nonstandard and $\mathfrak{A} \models \varphi'(b) \wedge \theta'(c)$. Finally consider $d = \langle a, b, c\rangle$. Then, the submodel of definable elements $\mathcal{K}_1(\mathfrak{A}, d)$ also satisfies $\varphi'(b) \wedge \theta'(c)$ since $\mathcal{K}_1(\mathfrak{A}, d) \prec_{\Sigma_1} \mathfrak{A}$. So, $\mathcal{K}_1(\mathfrak{A}, d)$ is a model of $B\Sigma_1 + exp$, which contradicts Proposition 1. □

Since the sentence expressing that a Σ_1 formula is equivalent to a Π_1 formula has complexity Π_2, it is clear that $\Delta_1(T)$-schemes only depend on the Π_2-theorems of T. Somewhat surprisingly, it follows from the Unboundedness results that we can also recover the Π_2-theorems of T from the corresponding $\Delta_1(T)$-schemes, no matters how strong T might be.

Proposition 4

1. *Suppose* S *and* T *are closed under* Δ_1*-IR. Then,* $I\Delta_1(S) \equiv I\Delta_1(T)$ *if and only if* $Th_{\Pi_2}(S) = Th_{\Pi_2}(T)$.
2. *Suppose* S *and* T *are closed under* Σ_1*-CR and prove exp. Then,* $B\Delta_1(S) \equiv B\Delta_1(T)$ *if and only if* $Th_{\Pi_2}(S) = Th_{\Pi_2}(T)$.

Proof We only write the proof of part 2. Assume $B\Delta_1(S) \vdash B\Delta_1(T)$. Since S is closed under Σ_1-CR, $Th_{\Pi_2}(S)$ implies $B\Delta_1(S)$. So, $Th_{\Pi_2}(S)$ implies $B\Delta_1(T)$ and then $Th_{\Pi_2}(T) \subseteq Th_{\Pi_2}(S)$ by Proposition 3. The opposite direction follows by symmetry. □

As an immediate consequence, we obtain that

Theorem 3 (Hierarchy theorem)

1. $I\Delta_0 \equiv I\Delta_1(I\Delta_0) \subsetneq I\Delta_1(I\Sigma_1) \subsetneq I\Delta_1(I\Sigma_2) \subsetneq I\Delta_1(I\Sigma_3) \subsetneq \cdots \subseteq I\Sigma_1$
2. $I\Delta_0 \equiv B\Delta_1(I\Delta_0) \subsetneq B\Delta_1(I\Sigma_1) \subsetneq B\Delta_1(I\Sigma_2) \subsetneq B\Delta_1(I\Sigma_3) \subsetneq \cdots \subseteq B\Sigma_1$

Using a modified version of the model-theoretic notion of an *envelope*, Theorem 6.6 in [11] gives another proof that $I\Delta_1(I\Sigma_n)$, $n \geq 0$ form a hierarchy. In contrast, a hierarchy theorem for $B\Delta_1(I\Sigma_n)$, $n \geq 0$, was left over (see Problem 7.5 in [11]).

Theorem 3 answers that question as well as provides a much simpler proof of the hierarchy theorem for the induction case.

We close this section by showing how to use the Unboundedness theorem to determine the usual proof-theoretic properties of $\Delta_1(T)$-schemes. Rather than being systematic, we prefer to illustrate this methodology with a few salient examples.

Proposition 5 (Quantifier complexity)

1. *If* $I\Sigma_1 \vdash Th_{\Pi_2}(T)$*, then* $I\Delta_1(T)$ *is* Π_2*-axiomatizable. If* $I\Sigma_1 \nvdash Th_{\Pi_2}(T)$*, then* $I\Delta_1(T)$ *is* Π_3 *and not* Σ_3*-axiomatizable.*
2. *If* $I\Delta_0 + exp \nvdash Th_{\Pi_2}(T)$*, then* $B\Delta_1(T)$ *is* Π_3 *and not* Σ_3*-axiomatizable.*

Proof Note that the natural axiomatizations of $I\Delta_1(T)$ and $B\Delta_1(T)$ are of quantifier complexity Π_3.

(1) On the one hand, if $I\Sigma_1 \vdash Th_{\Pi_2}(T)$ then it follows from Theorem 1 that $I\Delta_1(T) \vdash Th_{\Pi_2}(T)$. Hence $I\Delta_1(T)$ is equivalent to $[Th_{\Pi_2}(T), \Delta_1\text{-IR}]$ and this last theory is Π_2-axiomatizable. On the other hand, if $I\Delta_1(T)$ were to be Σ_3-axiomatizable then it would follow from Proposition 3 that $I\Delta_1(T) \vdash Th_{\Pi_2}(T)$ and so $I\Sigma_1 \vdash Th_{\Pi_2}(T)$ too.

(2) If $B\Delta_1(T)$ were to be Σ_3-axiomatizable then it would follow from Proposition 3 that $B\Delta_1(T) + exp \vdash Th_{\Pi_2}(T)$ and hence $I\Delta_0 + exp \vdash Th_{\Pi_2}(T)$ too, for $B\Sigma_1 + exp$ is well-known to be Π_2-conservative over $I\Delta_0 + exp$. □

Notice that it follows from Proposition 5 that $I\Delta_1(T)$ is Π_2-axiomatizable if, and only if, $I\Sigma_1 \vdash Th_{\Pi_2}(T)$. This settles the motivating question of [7]: *under which conditions is* $I\Delta_1(T)$ *a* Π_2*-axiomatizable theory*?

Proposition 6 (Finite axiomatizability)

1. $I\Delta_1(T)$ *is finitely axiomatizable if and only if so is* $[Th_{\Pi_2}(T), \Delta_1\text{-}IR]$.
2. *Suppose* $T \vdash exp$*. If* $B\Delta_1(T)$ *is finitely axiomatizable, so is* $[Th_{\Pi_2}(T), \Sigma_1\text{-}CR]$.
3. *So, if* T *is a consistent extension of* $I\Sigma_1$*, neither* $I\Delta_1(T)$ *nor* $B\Delta_1(T)$ *is finitely axiomatizable.*

Proof (1) By Corollary 1 we have $I\Delta_1(T) \equiv [Th_{\Pi_2}(T), \Delta_1\text{-IR}] \vee I\Sigma_1$. So, if $[Th_{\Pi_2}(T), \Delta_1\text{-IR}]$ has a finite axiomatization then the second theory in the previous equivalence provides a finite axiomatization of $I\Delta_1(T)$. For the opposite direction, assume that $I\Delta_1(T)$ is finitely axiomatizable. Then there is a single Π_2-sentence, φ, such that $Th_{\Pi_2}(T) + I\Delta_1(T) \vdash \varphi \vdash I\Delta_1(T)$. But it follows from Proposition 3 that $\varphi \vdash Th_{\Pi_2}(T)$ and hence $\varphi \equiv [Th_{\Pi_2}(T), \Delta_1\text{-IR}]$.

(2) Reason as in the second part of the proof of part 1.

(3) Assume T is consistent and implies $I\Sigma_1$. Then, $Th_{\Pi_2}(T)$ is closed under Δ_1-IR and Σ_1-CR and is known to be not finitely axiomatizable (for a proof see, e.g., Theorem 5.3 of [7]). □

Remark 2 (The theory $B\Delta_1(I\Delta_0 + exp)$ and the NE Problem) In contrast to the induction case, Proposition 3 for $B\Delta_1(T)$ has only been obtained for Σ_3-extensions proving *exp*. As a consequence, this additional assumption has also appeared in the subsequent results on $B\Delta_1(T)$. Eliminating this use of *exp* is apparently quite difficult, for it is related to the well-known open problem whether $I\Delta_0$ plus the negation of *exp* implies $B\Sigma_1$ (for short, the NE Problem) raised by Wilkie and Paris in [24]. Actually, we have

Lemma 3 *The following are equivalent.*

1. $I\Delta_0 + \neg exp \vdash B\Sigma_1$.
2. $B\Delta_1(I\Delta_0 + exp) \equiv I\Delta_0$.

Proof (*1*⇒*2*) By part 1, $I\Delta_0 + \neg exp \vdash B\Delta_1(I\Delta_0 + exp)$. But $I\Delta_0 + exp$ also implies $B\Delta_1(I\Delta_0 + exp)$ since $I\Delta_0 + exp$ is closed under Σ_1-CR and hence part 2 follows. (*2*⇒*1*) Note that $B\Delta_1(I\Delta_0 + exp) + \neg exp \vdash B\Sigma_1$ by Theorem 1. □

Hence, eliminating *exp* in Proposition 3 would give that $I\Delta_0$ is strictly weaker than $B\Delta_1(I\Delta_0 + exp)$, thus settling the NE Problem. (A recent discussion on the difficulty and significance of this problem can be found in [1]).

4 Parameter-free $\Delta_1(T)$-schemes

This section investigates the effect of disallowing parameters in $\Delta_1(T)$-schemes and in Δ_1-inference rules. Recall that $I\Delta_1(T)^-$, $L\Delta_1(T)^-$ and $B\Delta_1(T)^-$ denote the parameter free versions of the corresponding theories. Similarly, we define

$$\Delta_1^-\text{-IR} : \frac{\forall x\,(\varphi(x) \leftrightarrow \psi(x))}{I_{\varphi(x)}}; \qquad \Delta_1^-\text{-LR} : \frac{\forall x\,(\varphi(x) \leftrightarrow \psi(x))}{L_{\varphi(x)}},$$

where $\varphi(x) \in \Sigma_1^-$ and $\psi(x) \in \Pi_1^-$. We have not introduced the inference rule associated to $B\Delta_1(T)^-$, for Σ_1-CR is reducible to its parameter free counterpart. In contrast, Δ_1-IR and Δ_1-LR are no longer reducible to their parameter free versions. To see that, recall from [13] that $UI\Delta_1$ denotes a variant of the Δ_1-induction scheme where parameters are distributed uniformly. Namely, $UI\Delta_1$ is Q together with

$$\forall \mathbf{v}\,\forall x\,(\varphi(x,\mathbf{v}) \leftrightarrow \psi(x,\mathbf{v})) \;\rightarrow\; \forall \mathbf{v}\, I_{\varphi(x,\mathbf{v})},$$

where $\varphi \in \Sigma_1$ and $\psi \in \Pi_1$. Since $I\Delta_1^-$ does not imply $UI\Delta_1$ (see e.g., Theorem 1.2 in [9]), there are $\varphi(x,\mathbf{v}) \in \Sigma_1$ and $\psi(x,\mathbf{v}) \in \Pi_1$ satisfying that $T = I\Delta_1^- + \forall\mathbf{v}\,\forall x\,(\varphi(x,\mathbf{v}) \leftrightarrow \psi(x,\mathbf{v}))$ does not prove $\forall\mathbf{v}\, I_{\varphi(x,\mathbf{v})}$. Thus, such a theory T is closed under Δ_1^--IR and, however, does not imply $[T, \Delta_1\text{-IR}]$. A similar remark applies to Δ_1-LR considering $UL\Delta_1 \equiv B\Sigma_1^-$.

Regarding $\Delta_1(T)$-schemes, it follows from our results on quantifier complexity in Sect. 3 that disallowing parameters also makes a difference. Let us see that for the induction case. First, observe that $I\Delta_1(T)^-$ has quantifier complexity $\mathcal{B}(\Sigma_2)$, i.e., boolean combinations of Σ_2-sentences. Second, by Proposition 5, $I\Delta_1(T)$ is not Σ_3-axiomatizable whenever $I\Sigma_1 \nvdash Th_{\Pi_2}(T)$. Thus, $I\Delta_1(T)^-$ is strictly weaker than $I\Delta_1(T)$ if $Th_{\Pi_2}(T) \nsubseteq I\Sigma_1$. Similar remarks apply to the collection and minimization cases.

Our starting point is a Transfer Theorem for these theories.

Theorem 4 (Transfer theorem for parameter free fragments)

1. $I\Delta_1(T)^- \vdash Th_{\Pi_2}(T) \vee I\Sigma_1^-$.
2. $B\Delta_1(T)^- \vdash Th_{\Pi_2}(T) \vee B\Sigma_1^-$.
3. $L\Delta_1(T)^- \vdash Th_{\mathcal{B}(\Sigma_1)}(T) \vee I\Pi_1^-$.

Proof (1) Suppose $\mathfrak{A} \models I\Delta_1(T)^-$ and $\mathfrak{A} \not\models Th_{\Pi_2}(T)$. To see $\mathfrak{A} \models I\Sigma_1^-$, assume $a \in \mathfrak{A}$ and $\mathfrak{A} \models \varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1))$, with $\varphi(x) \in \Sigma_1^-$. We must show $\mathfrak{A} \models \varphi(a)$. Since $\mathfrak{A} \not\models Th_{\Pi_2}(T)$, there are $\theta(w) \in \Sigma_1$ and $b \in \mathfrak{A}$ such that $T \vdash \forall w\,\theta(w)$ and $\mathfrak{A} \models \neg\theta(b)$. We reason as in the proof of Theorem 1, but now we need to codify the induction variable x and the parameter w in a single variable u. To this end, consider

$$\delta(u) \equiv \varphi((u)_0) \vee \theta(((u)_0 + (u)_1)_0)$$

It is clear that $T \vdash \forall u\,\delta(u)$ and so $\delta(u)$ is in $\Delta_1(T)$. Observe that if $u = \langle x, w\rangle$, it follows from the definition of the pairing function that $u+1 = \langle x+1, w-1\rangle$ if $w \neq 0$; and that $u+1 = \langle 0, x+1\rangle$ if $w = 0$. Having this fact in mind, it is easy to check that the assumption of the induction axiom for $\delta(u)$ holds in $\mathfrak{A}$ and so $\mathfrak{A} \models \forall u\,\delta(u)$ by $I\Delta_1(T)^-$. Now consider $c = \langle a, \langle b, a\rangle - a\rangle$ and reason in the model $\mathfrak{A}$. It follows from $\delta(c)$ that $\varphi(a) \vee \theta(b)$ and so $\varphi(a)$ since b was chosen so that $\neg\theta(b)$.

(2) Suppose $\mathfrak{A} \models B\Delta_1(T)^-$ and $\mathfrak{A} \not\models Th_{\Pi_2}(T)$. Then there are $\theta(w) \in \Sigma_1$ and $b \in \mathfrak{A}$ such that $T \vdash \forall w\,\theta(w)$ and $\mathfrak{A} \models \neg\theta(b)$. To see $\mathfrak{A} \models B\Sigma_1^-$, assume $a \in \mathfrak{A}$ and $\mathfrak{A} \models \forall x\,\exists y\,\varphi(x, y)$, with $\varphi(x, y) \in \Sigma_1$. Put

$$\delta(u, y) \equiv \varphi((u)_0, y) \vee (\theta((u)_1) \wedge y = 0)$$

Clearly, $\delta(u, y) \in \Sigma_1$ and $T \vdash \forall u\,\exists y\,\delta(u, y)$. By $B\Delta_1(T)^-$ there is c such that $\mathfrak{A} \models \forall u \leq \langle a, b\rangle\,\exists y \leq c\,(\varphi((u)_0, y) \vee \theta((u)_1))$. So, $\mathfrak{A} \models \forall x \leq a\,\exists y \leq c\,\varphi(x, y)$ since $\mathfrak{A} \models \neg\theta(b)$ and $\mathfrak{A} \models x \leq a \rightarrow \langle x, b\rangle \leq \langle a, b\rangle$.

(3) Suppose $\mathfrak{A} \models L\Delta_1(T)^-$ and there is a $\mathcal{B}(\Sigma_1)$ sentence θ such that $T \vdash \theta$ and $\mathfrak{A} \not\models \theta$. We shall show that $\mathfrak{A}$ satisfies the least number axiom scheme for parameter free Σ_1 formulas $L\Sigma_1^-$ (which is well-known to be equivalent to $I\Pi_1^-$). To this end, assume $\varphi(x) \in \Sigma_1^-$ and $\mathfrak{A} \models \exists x\,\varphi(x)$. By logical operations, $\theta \equiv (\theta_1^0 \vee \theta_2^0) \wedge \ldots \wedge (\theta_1^k \vee \theta_2^k)$, with $\theta_1^i \in \Sigma_1$ and $\theta_2^i \in \Pi_1$. So, there are $\theta_1 \in \Sigma_1$ and $\theta_2 \in \Pi_1$ satisfying that $T \vdash \theta_1 \vee \theta_2$ and $\mathfrak{A} \models \neg\theta_1 \wedge \neg\theta_2$. Put $\theta_2 \equiv \forall z\,\theta_2'(z)$, with $\theta_2' \in \Delta_0$, and define $\delta(u)$ to be the Σ_1 formula

$$\neg\theta_2'((u)_1) \wedge [\theta_1 \vee \varphi((u)_0)]$$

It follows from $T \vdash \theta_1 \vee \forall z\,\theta_2'(z)$ that $\delta(u)$ is equivalent in T to $\neg\,\theta_2'((u)_1)$ and so $\delta(u) \in \Delta_1(T)^-$. It follows from $\mathfrak{A} \models \neg\theta_2 \wedge \exists u\,\varphi(u)$ that $\mathfrak{A} \models \exists u\,\delta(u)$. By applying $L\Delta_1(T)^-$ in $\mathfrak{A}$, we get that there exists c such that $c = \mu u.\,\delta(u)$. Using the monotonicity of the pairing function, it is easy to check that $(c)_0$ is the least element satisfying $\varphi(x)$ in $\mathfrak{A}$. □

Corollary 2 (Characterization theorem)

1. $I\Delta_1(T)^- \equiv [Th_{\Pi_2}(T),\, \Delta_1^-\text{-}IR] \vee I\Sigma_1^-$.
2. $B\Delta_1(T)^- \equiv [Th_{\Pi_2}(T),\, \Sigma_1\text{-}CR] \vee B\Sigma_1^-$.
3. *If T is closed under Σ_1-CR, then* $L\Delta_1(T)^- \equiv Th_{\mathcal{B}(\Sigma_1)}(T) \vee I\Pi_1^-$.

Proof Parts 1 and 2 are immediate consequences of Theorem 4. To get part 3, only the fact that $Th_{\mathcal{B}(\Sigma_1)}(T) \vdash L\Delta_1(T)^-$ needs some explanations. Consider $\varphi(x) \in \Sigma_1^-$ and $\psi(x) \in \Pi_1^-$ such that $T \vdash \forall x\,(\varphi(x) \leftrightarrow \psi(x))$. Write $\varphi(x) \equiv \exists y\, \varphi_0(x, y)$ and $\psi(x) \equiv \forall y\, \psi_0(x, y)$, with $\varphi_0, \psi_0 \in \Delta_0$. Since $T \vdash L\Delta_1(T)^-$ and $T \vdash \forall x\,(\varphi(x) \leftrightarrow \psi(x))$, we have

$$T \vdash \exists x\, \varphi(x) \rightarrow \exists x\,(\varphi(x) \wedge \forall z < x\, \exists y\, \neg\psi_0(z, y))$$

Using again that $T \vdash \forall x\,(\varphi(x) \leftrightarrow \psi(x))$, we get $T \vdash \forall x\, \exists y\,(\neg\psi_0(x, y) \vee \varphi_0(x, y))$ and $T \vdash \forall x\,(\exists y\, \neg\psi_0(x, y) \leftrightarrow \forall y\, \neg\varphi_0(x, y))$. So, since T is closed under Σ_1-CR, we have

$$T \vdash \exists x\, \varphi(x) \rightarrow \exists x\,(\varphi(x) \wedge \exists u\, \forall z < x\, \exists y \leq u\, \neg\psi_0(z, y))$$

But the above sentence has complexity $\mathcal{B}(\Sigma_1)$ and implies $L_{\varphi(x)}$ modulo the theory $Th_{\Pi_1}(T)$, as $Th_{\Pi_1}(T)$ proves $\forall x\,(\exists y\, \neg\psi_0(x, y) \rightarrow \forall y\, \neg\varphi_0(x, y))$. $\square$

Remark 3 It is natural to ask whether the Transfer Theorem for $L\Delta_1(T)^-$ can be improved to $L\Delta_1(T)^- \vdash Th_{\Pi_2}(T) \vee I\Pi_1^-$. The answer to this question is, in general, negative. For instance, consider $T = I\Pi_1^-$. Firstly, since $I\Pi_1^-$ is an extension of $I\Delta_0$ by Σ_2-sentences, it is closed under Σ_1-CR and hence $L\Delta_1(I\Pi_1^-)^- \equiv Th_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ by Corollary 2. Secondly, it is clear that $Th_{\Pi_2}(I\Pi_1^-) \vee I\Pi_1^- \equiv Th_{\Pi_2}(I\Pi_1^-)$. Thirdly, Theorem 4.7 of [10] states that $Th_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ is strictly weaker than $Th_{\Pi_2}(I\Pi_1^-)$.

Equipped with Theorem 4 the next step is to obtain an unboundedness result for parameter free $\Delta_1(T)$-schemes.

Proposition 7 (Unboundedness) *Suppose $S \subseteq \Pi_2$ and either S is recursively enumerable (r.e.) or T is sound.*

1. *If $S \vdash I\Delta_1(T)^-$ then $S \vdash Th_{\Pi_2}(T)$.*
2. *If S is closed under Σ_1-IR and $S \vdash B\Delta_1(T)^-$ then $S \vdash Th_{\Pi_2}(T)$.*
3. *If $S \vdash L\Delta_1(T)^-$ then $S \vdash Th_{\mathcal{B}(\Sigma_1)}(T)$.*

Proof (1) Towards a contradiction, assume that $S \vdash I\Delta_1(T)^-$ and there is a Π_2 sentence θ such that $T \vdash \theta$ and $S \nvdash \theta$. It follows from Theorem 4 for $I\Delta_1(T)^-$ that $S + \neg\theta$ is a consistent extension of $I\Sigma_1^-$. In addition, we have:

Claim $S + \neg\theta$ does not imply $Th_{\Pi_2}(\mathbb{N})$.

If S is r.e. then $S + \neg\theta \nvdash Th_{\Pi_2}(\mathbb{N})$, for a Π_2^0-complete set cannot follow from a r.e. set of sentences. If T is sound, $\theta \in Th_{\Pi_2}(\mathbb{N})$ and so $S + \neg\theta \nvdash Th_{\Pi_2}(\mathbb{N})$.

It follows from the Claim and $S + \neg\theta \vdash L\Pi_1^-$ (recall that $I\Sigma_1^-$ and $L\Pi_1^-$ are deductively equivalent) that there is $\mathfrak{A} \models S + \neg\theta$ with some nonstandard Π_1-minimal element, say a. Also, put $\neg\theta \equiv \exists x\, \theta'(x)$, with $\theta'(x) \in \Pi_1^-$, and pick b satisfying that $b = \mu x.\theta'(x)$. Consider $c = \langle a, b\rangle$. By Proposition 1, $\mathcal{K}_1(\mathfrak{A}, c) \not\models B\Sigma_1^- + exp$. But it follows from $\mathcal{K}_1(\mathfrak{A}, c) \prec_{\Sigma_1} \mathfrak{A}$ that $\mathcal{K}_1(\mathfrak{A}, c)$ satisfies $S + \neg\theta$ and thus also $I\Sigma_1^-$, which gives the desired contradiction.

(2) Assume that S is closed under Σ_1-IR, $S \vdash B\Delta_1(T)^-$ and there is a Π_2 sentence θ such that $T \vdash \theta$ and $S \nvdash \theta$. It follows from Theorem 4 for $B\Delta_1(T)^-$ that $S + \neg\theta$ is a consistent extension of $B\Sigma_1^-$. By the Claim in part 1, there exists $\mathfrak{A} \models S + \neg\theta$ in which $Th_{\Pi_2}(\mathbb{N})$ fails. But it is a well known result of Parsons [18] that $S + I\Sigma_1$ is Π_2-conservative over $S + \Sigma_1\text{-IR} \equiv S$. Hence, there is $\mathfrak{B} \models S + I\Sigma_1$ with $\mathfrak{A} \prec_{\Sigma_1} \mathfrak{B}$. Clearly, $\mathfrak{B} \models S + \neg\theta + I\Sigma_1$ and $Th_{\Pi_2}(\mathbb{N})$ fails in $\mathfrak{B}$ too. By repeating the argument in part 1, we get the desired contradiction.

(3) Assume that $S \vdash L\Delta_1(T)^-$ and there is a $\mathcal{B}(\Sigma_1)$ sentence θ such that $T \vdash \theta$ and $S \nvdash \theta$. It follows from Theorem 4 for $L\Delta_1(T)^-$ that $S + \neg\theta$ is a consistent extension of $I\Pi_1^-$. Again, $S + \neg\theta$ does not imply $Th_{\Pi_1}(\mathbb{N})$, for either S is r.e. or T is sound. Let $\mathfrak{A}$ be a model of $S + \neg\theta$ with $\mathcal{K}_1(\mathfrak{A})$ nonstandard. Since $\mathfrak{A} \models I\Pi_1^-$, $\mathcal{K}_1(\mathfrak{A}) \models exp$ (see Theorem 2.9 in [14]). Since $\mathcal{K}_1(\mathfrak{A}) \prec_{\Sigma_1} \mathfrak{A}$, $\mathcal{K}_1(\mathfrak{A})$ satisfies $S + \neg\theta$ and then also $I\Pi_1^-$. Hence, $\mathcal{K}_1(\mathfrak{A}) \models I\Pi_1^- + exp$, which contradicts Proposition 1. □

Using Proposition 4 and reasoning as in Sect. 3, one can obtain the basic proof-theoretic information on parameter free $\Delta_1(T)$-schemes (relative strength, hierarchy theorem, quantifier complexity, finite axiomatizability,...). This is more or less routine and we omit it. Instead, we turn our attention to conservation results. We first study conservativity between Δ_1-rules (which is of independent interest) and then transfer the results to $\Delta_1(T)$-schemes.

Proposition 8 *Suppose* $T \subseteq \Pi_2$.

1. $[T, \Delta_1\text{-}IR]$ *is* Σ_2*-conservative over* $[T, \Delta_1^-\text{-}IR]$.
2. $[T, \Delta_1\text{-}LR]$ *is* Σ_2*-conservative over* $[T, \Delta_1^-\text{-}LR]$.

Proof (1) We shall show that if $\mathfrak{A} \models [T, \Delta_1^-\text{-IR}]$ then $\mathcal{K}_1(\mathfrak{A}) \models [T, \Delta_1\text{-IR}]$. This suffices to obtain Σ_2-conservation since $\mathcal{K}_1(\mathfrak{A}) \prec_{\Sigma_1} \mathfrak{A}$. Let $\mathfrak{A}$ be a model of $[T, \Delta_1^-\text{-IR}]$. Since $T \subseteq \Pi_2$, $\mathcal{K}_1(\mathfrak{A}) \models T$. To prove $\mathcal{K}_1(\mathfrak{A}) \models I\Delta_1(T)$, consider $a \in \mathcal{K}_1(\mathfrak{A})$, $\varphi(x, v) \in \Sigma_1$, $\psi(x, v) \in \Pi_1$ such that $T \vdash \forall x, v\, (\varphi(x, v) \leftrightarrow \psi(x, v))$. We must prove that $I_{\varphi(x,a)}$ holds in $\mathcal{K}_1(\mathfrak{A})$. But it is easy to see that $I_{\varphi(x,a)}$ is true in $\mathcal{K}_1(\mathfrak{A})$ if and only if it is true in $\mathfrak{A}$. Note that in models of $I\Delta_0$, every Σ_1-definable element can be obtained as the projection of a Δ_0-minimal one. So, there is $\delta(v) \in \Delta_0$ such that $a = (\mu t.\, \delta(t))_0$ in $\mathfrak{A}$. Consider

$$\varphi'(x) \equiv \exists v\, (v = \mu t.\, \delta(t) \wedge \varphi(x, (v)_0)),$$
$$\psi'(x) \equiv \forall v\, (v = \mu t.\, \delta(t) \rightarrow \psi(x, (v)_0)).$$

Clearly, $\varphi'(x) \in \Sigma_1^-$, $\psi(x) \in \Pi_1^-$ and $\mathfrak{A} \models I_{\varphi'(x)} \leftrightarrow I_{\varphi(x,a)}$. However, we cannot infer $I_{\varphi'(x)}$ from $[T, \Delta_1^-\text{-IR}]$, because $\varphi'(x)$ and $\psi'(x)$ are equivalent in $T + \exists v\, \delta(v)$ and not necessarily in T. To get round this problem, put

$$\varphi^\star(x) \equiv \neg\delta(((x)_0 + (x)_1)_0) \vee \varphi'((x)_0),$$
$$\psi^\star(x) \equiv \neg\delta(((x)_0 + (x)_1)_0) \vee \psi'((x)_0).$$

Then, $T \vdash \forall x\,(\varphi^\star(x) \leftrightarrow \psi^\star(x))$ and hence $I_{\varphi^\star(x)}$ is true in $\mathfrak{A}$ by $[T, \Delta_1^-\text{-IR}]$. But using the properties of the pairing function (see the proof of part 1 of Theorem 4 for details), it is easy to check that $I\Delta_0 + \exists v\,\delta(v) \vdash I_{\varphi^\star(x)} \to I_{\varphi'(x)}$. Therefore $I_{\varphi'(x)}$ holds in $\mathfrak{A}$ and so does $I_{\varphi(x,a)}$, as required.

(2) We shall show that if $\mathfrak{A} \models [T, \Delta_1^-\text{-LR}]$ then $\mathcal{K}_1(\mathfrak{A}) \models [T, \Pi_0^-\text{-CR}]$. This suffices as Δ_1-LR and Π_0^--CR are congruent. Let $\mathfrak{A}$ be a model of $[T, \Delta_1^-\text{-LR}]$. Consider $a \in \mathcal{K}_1(\mathfrak{A})$ and $\theta(x, y) \in \Pi_0^-$ such that $T \vdash \forall x\,\exists y\,\theta(x, v)$. We must show that $\exists u\,\forall x \le a\,\exists y \le u\,\theta(x, y)$ is true in $\mathcal{K}_1(\mathfrak{A})$ or, equivalently, in $\mathfrak{A}$. To this end, we reason as in Gandy's proof that $L\Delta_1 \vdash B\Sigma_1$ (see Lemma 2.17, chapter I in [12] for details). Define $\varphi'(x, v)$ and $\psi'(x, v)$ to be, respectively

$$x \le v \wedge \exists u\,(u = \mu t.\,\theta(x,t) \wedge \forall z \in [x, v]\,\exists y \le u\,\theta(z, y))$$
$$x \le v \wedge \forall u\,(u = \mu t.\,\theta(x,t) \to \forall z \in [x, v]\,\exists y \le u\,\theta(z, y))$$

Clearly, $\varphi' \in \Sigma_1$, $\psi' \in \Pi_1$, $T \vdash \forall x, v\,(\varphi'(x, v) \leftrightarrow \psi'(x, v))$ and $\mathfrak{A} \models \varphi'(a, a)$. In addition, Gandy's proof shows that if $c = \mu t.\,\varphi'(t, a)$ and $\mathfrak{A} \models \theta(c, b)$, then $\mathfrak{A} \models \forall x \le a\,\exists y \le b\,\theta(x, y)$. So, it suffices to prove that $L_{\varphi'(x,a)}$ is true in $\mathfrak{A}$. Let $\delta(v) \in \Delta_0$ be such that $a = (\mu t.\,\delta(t))_0$ in $\mathfrak{A}$ and put

$$\varphi^\star(x) \equiv \delta((x)_1) \wedge \exists v\,(v = \mu t.\,\delta(t) \wedge \varphi'((x)_0, (v)_0)),$$
$$\psi^\star(x) \equiv \delta((x)_1) \wedge \forall v\,(v = \mu t.\,\delta(t) \to \psi'((x)_0, (v)_0)),$$

Then, $T \vdash \forall x\,(\varphi^\star(x) \leftrightarrow \psi^\star(x))$ and $\mathfrak{A} \models \exists x\,\varphi^\star(x)$. By $[T, \Delta_1^-\text{-LR}]$ there exists the least element d satisfying $\varphi^\star(x)$ in $\mathfrak{A}$. It follows that $(d)_0$ is the least element satisfying $\varphi'(x, a)$ in $\mathfrak{A}$. □

Let us observe that the assumption $T \subseteq \Pi_2$ in Proposition 8 cannot be eliminated: Π_1 sentences need not be conserved if the quantifier complexity of T exceeds Π_2. To see that, let $Con(PA)$ denote the consistency statement for PA. Then, $I\Delta_1^- + \neg Con(PA)$ does not imply $UI\Delta_1$. (Indeed, it follows from part 3 of Proposition 1 and Lemma 3.7 of [9] that $UI\Delta_1$ is not contained in any recursive set of Σ_2-sentences consistent with $I\Sigma_1$). So, there are $\varphi(x, \mathbf{v}) \in \Sigma_1$ and $\psi(x, \mathbf{v}) \in \Pi_1$ satisfying that $I\Delta_1^- + \neg Con(PA) + \forall \mathbf{v}, x\,(\varphi(x, \mathbf{v}) \leftrightarrow \psi(x, \mathbf{v}))$ does not prove $\forall \mathbf{v}\, I_{\varphi(x,\mathbf{v})}$. If we consider T to be the theory given by $I\Delta_1^- + \forall \mathbf{v}, x\,(\varphi(x, \mathbf{v}) \leftrightarrow \psi(x, \mathbf{v})) + (\forall \mathbf{v}\, I_{\varphi(x,\mathbf{v})} \to Con(PA))$, then $[T, \Delta_1\text{-IR}]$ proves $Con(PA)$ whereas $T + \Delta_1^-\text{-IR}$ (which is equivalent to T) does not.

Theorem 5 1. *$I\Delta_1(T)$ is Σ_2-conservative over $I\Delta_1(T)^-$. Moreover, if T is closed under Δ_1-IR then Σ_3-sentences are also conserved.*
2. *$B\Delta_1(T)$ is Σ_3-conservative over $B\Delta_1(T)^-$.*

Proof (1) Assume that $\mathfrak{A} \models I\Delta_1(T)^-$ and φ is a Σ_2-theorem of $I\Delta_1(T)$. If $\mathfrak{A}$ satisfies $Th_{\Pi_2}(T)$ then $\mathfrak{A} \models [Th_{\Pi_2}(T), \Delta_1^-\text{-IR}]$ and so $\mathfrak{A} \models \varphi$ by Proposition 8. If $\mathfrak{A}$ does not satisfy $Th_{\Pi_2}(T)$ then $\mathfrak{A} \models I\Sigma_1^-$ by Theorem 4 and so $\mathfrak{A} \models \varphi$, for $I\Delta_1(T) \subseteq I\Sigma_1$ and $I\Sigma_1$ is well-known to be Σ_3-conservative over $I\Sigma_1^-$ (see Theorem 2.1 of [14]).

Now assume that T is closed under Δ_1-IR. By Corollary 1 we have $I\Delta_1(T) \equiv Th_{\Pi_2}(T) \vee I\Sigma_1$ and by Corollary 2 we have $I\Delta_1(T)^- \equiv Th_{\Pi_2}(T) \vee I\Sigma_1^-$. So, that $I\Delta_1(T)$ is Σ_3-conservative over $I\Delta_1(T)^-$ follows from Σ_3-conservation of $I\Sigma_1$ over $I\Sigma_1^-$.

(2) By Theorem 2.4 of [14] $B\Sigma_1$ is Σ_3-conservative over $B\Sigma_1^-$. Hence, the result follows since $B\Delta_1(T) \equiv [Th_{\Pi_2}(T), \Sigma_1\text{-CR}] \vee B\Sigma_1$ by Corollary 1 and $B\Delta_1(T)^- \equiv [Th_{\Pi_2}(T), \Sigma_1\text{-CR}] \vee B\Sigma_1^-$ by Corollary 2. □

Note that the situation for $L\Delta_1(T)$ and $L\Delta_1(T)^-$ is completely different: even Π_1-sentences are not necessarily conserved. For example, put $T = I\Sigma_1$. Then $L\Delta_1(I\Sigma_1)$ is equivalent to $Th_{\Pi_2}(I\Sigma_1)$ by Corollary 1 and so proves $Con(I\Pi_1^-)$, whereas $L\Delta_1(I\Sigma_1)^- \subseteq I\Pi_1^-$.

We close this section with some other applications of Proposition 8.

Corollary 3 *Suppose that T is an extension of $I\Delta_0 + exp$.*

1. $[T, \Delta_1^-\textit{-IR}] \equiv [T, \Delta_1^-\textit{-LR}]$.
2. $I\Delta_1(T)^- \vdash L\Delta_1(T)^-$.

Proof (1) One direction is clear. For the other, assume $T \vdash \forall x\,(\varphi(x) \leftrightarrow \psi(x))$. By Proposition 2, $[Th_{\Pi_2}(T), \Delta_1\text{-IR}]$ proves L_φ. But since L_φ is of quantifier complexity Σ_2, $[Th_{\Pi_2}(T), \Delta_1^-\text{-IR}]$ also proves it by Proposition 8.

(2) Assume $\mathfrak{A} \models I\Delta_1(T)^-$. If $\mathfrak{A}$ satisfies $Th_{\Pi_2}(T)$ then $\mathfrak{A} \models L\Delta_1(T)^-$ by part 1. If $\mathfrak{A}$ does not satisfy $Th_{\Pi_2}(T)$, it follows from Theorem 4 that $\mathfrak{A} \models I\Sigma_1^-$. □

Corollary 4 *Suppose $T \subseteq \Pi_2$.*

1. *$T + B\Sigma_1$ is $\mathcal{B}(\Sigma_1)$-conservative over* $[T, \Delta_1^-\textit{-LR}]$.
2. *If $T + \Delta_1$-IR collapses to* $[T, \Delta_1\textit{-IR}]$ *then $T + I\Delta_1$ is $\mathcal{B}(\Sigma_1)$-conservative over* $[T, \Delta_1^-\textit{-IR}]$.

Proof Assume $T \subseteq \Pi_2$. By Theorem 3.2 in [4] $T + B\Sigma_1$ is Π_2-conservative over $T + \Sigma_1$-CR, by Theorem 4.2 in [4] the latter theory collapses to $[T, \Sigma_1\text{-CR}]$; and by Theorem 2 in [5] $T + I\Delta_1$ is Π_2-conservative over $T + \Delta_1$-IR (all these results are proved for theories extending *exp* but this is unessential). Corollary 4 follows combining these results and Proposition 8. □

5 Provably total computable functions

In this section we address the question of what computable functions are provably total in $I\Delta_1(T)$ and in $L\Delta_1(T)$. Recall that a number-theoretic function $f : \mathbb{N}^k \to \mathbb{N}$ is said to be a provably total computable function (p.t.c.f.) of a theory T, written $f \in \mathcal{R}(T)$, if there is a Σ_1 formula of the language of T, $\varphi(\mathbf{x}, y)$, such that:

1. φ defines the graph of f in the standard model of Arithmetic $\mathbb{N}$; and
2. $T \vdash \forall \mathbf{x}\, \exists y\, \varphi(\mathbf{x}, y)$.

We call a formula $\varphi(\mathbf{x}, y)$ satisfying conditions 1 and 2 *a definition of f in T*. As long as T extends $I\Delta_0$, replacing the condition 2 with 2'. $T \vdash \forall \mathbf{x}\, \exists! y\, \varphi(\mathbf{x}, y)$ does not change $\mathcal{R}(T)$. In fact, if $\exists z\, \varphi'(\mathbf{x}, y, z)$ is a definition of f in T, with $\varphi' \in \Delta_0$, it suffices to put $\varphi(\mathbf{x}, y) \equiv \exists u\, (y = (u)_0 \wedge u = \mu t.\, \varphi'(\mathbf{x}, (t)_0, (t)_1))$ to obtain a Σ_1-definition of f satisfying 2'.

We shall characterize the classes of p.t.c.f.'s of $\Delta_1(T)$-schemes as function algebras generated by means of some recursive operators. Let us fix some notation. We write $\mathbf{C}(\mathcal{F})$ for the closure of the set of functions $\mathcal{F}$ under composition and, in general, $\mathbf{O}(\mathcal{F})$ denotes the closure of $\mathcal{F}$ under composition and the recursive operator $\mathbf{O}$. In addition, we write $[\mathcal{F}, \mathbf{O}]$ for the closure of $\mathcal{F}$ under composition and *unnested* application of the operator $\mathbf{O}$. Our base function algebra will be Grzegorczyk's $\mathcal{M}^2$, which is the closure of a set of initial functions (zero, successor, projections, sum and product) under composition and the bounded minimization operator, see [19]. By a result of Takeuti [22] the p.t.c.f.'s of $I\Delta_0$ coincide with $\mathcal{M}^2$ and thus $\mathcal{M}^2 \subseteq \mathcal{R}(T)$ for every extension of $I\Delta_0$.

It is a more or less direct consequence of Herbrand's theorem that there is a correspondence between computable functions with a Δ_0-definable graph and finite, sound extensions of $I\Delta_0$ (see e.g., Proposition 4.2 in [3]).

Lemma 4 *Suppose f is a computable function with a Δ_0-definable graph and let $\theta(x, y) \in \Delta_0$ such that $\mathbb{N} \models \forall x\, \theta(x, f(x))$. Then*

$$\mathcal{R}(I\Delta_0 + \forall x\, \exists y\, \theta(x, y)) = \mathbf{C}\left(\mathcal{M}^2 \cup \{f\}\right).$$

This correspondence is not exact: some information is being lost when going from a theory to the corresponding class of provably total functions. For example, $\mathcal{R}(I\Delta_0) = \mathcal{R}(I\Pi_1^-) = \mathcal{M}^2$ whereas $Th_{\Pi_2}(I\Pi_1^-)$ is strictly stronger than $I\Delta_0$, for $I\Delta_0 + exp$ is Σ_2-conservative over $I\Pi_1^-$ by Theorem 2.9 of [14]. However, it turns out to be true that $\mathcal{R}(T)$ determines the Π_2-consequences of a sound theory T modulo the set of all Π_1-true sentences $Th_{\Pi_1}(\mathbb{N})$.

Lemma 5 *Let T, S be sound extensions of $I\Delta_0$. The following are equivalent:*

1. $\mathcal{R}(T) = \mathcal{R}(S)$.
2. *Over* $Th_{\Pi_1}(\mathbb{N})$, $Th_{\Pi_2}(T) \equiv Th_{\Pi_2}(S)$.

Proof (*1*$\Rightarrow$*2*): By symmetry we only prove $Th_{\Pi_1}(\mathbb{N}) + Th_{\Pi_2}(T) \vdash Th_{\Pi_2}(S)$. Assume $S \vdash \forall x\, \exists y\, \varphi(x, y)$, with $\varphi \in \Delta_0$, and put $\varphi'(x, y) \equiv y = \mu t.\, \varphi(x, t)$. Since S is sound, φ' defines in $\mathbb{N}$ the graph of a computable function f and $f \in \mathcal{R}(S) = \mathcal{R}(T)$. Hence, there is a Σ_1-definition of f in T too, say $\psi(x, y)$. Then $\mathbb{N} \models \forall x, y\, (\psi(x, y) \to \varphi'(x, y))$ and so $T + Th_{\Pi_1}(\mathbb{N}) \vdash \forall x\, \exists y\, \varphi(x, y)$.
(*2*$\Rightarrow$*1*): This part follows because adding true Π_1-sentences to a sound theory does not increase the corresponding class of provably total functions. In fact, if $\varphi(\mathbf{x}, y) \in \Sigma_1$ defines f in $T + \theta$, where θ is a Π_1-sentence true in $\mathbb{N}$, then $(\neg\theta \wedge y = 0) \vee \varphi(\mathbf{x}, y)$ is a Σ_1-definition of f in T. □

Although quite simple, the key observation for determining the p.t.f.c.'s of $I\Delta_1(T)$ and $L\Delta_1(T)$ is the following

Lemma 6 $\mathcal{R}(T \vee S) = \mathcal{R}(T) \cap \mathcal{R}(S)$.

Proof It follows from Lemma 2 that if $\varphi(\mathbf{x}, y)$ and $\psi(\mathbf{x}, y)$ are Σ_1-definitions of f in T and in S, respectively, then $\varphi(\mathbf{x}, y) \vee \psi(\mathbf{x}, y)$ is a Σ_1-definition of f in $T \vee S$. □

By a well-known result due independently to G. Mints, C. Parsons and G. Takeuti, $\mathcal{R}(I\Sigma_1)$ equals to the class of primitive recursive functions PR. In view of Corollary 1, it only remains to determine the p.t.c.f.'s of $[T, \Sigma_1\text{-CR}]$ and $[T, \Delta_1\text{-IR}]$ for T a sound Π_2-extension of $I\Delta_0$. In both cases our results will be, more or less, direct consequences of previous work by Beklemishev. In fact, in Corollary 5.6 of [3] it is shown that if T extends $I\Delta_0 + exp$, then $\mathcal{R}([T, \Sigma_1\text{-CR}])$ coincides with the closure of $\mathcal{R}(T)$ under the bounded recursion operator **BR** or, equivalently, under the bounded minimization operator **M**. Here we give a variant of that result in terms of the maximum operator **Max**. The proof is similar to that of Corollary 5.6 of [3] and we omit it.

Definition 2 (*Bounded Min and Max operators*) Assume $f : \mathbb{N}^{k+1} \to \mathbb{N}$. Then $M(f)$ denotes the function given by $M(f)(x, \mathbf{z}) = \mu i \leq x.\, [f(i, \mathbf{z}) = 0]$ if such an i exists, or $x+1$ otherwise; and $Max(f)$ denotes the function given by $Max(f)(x, \mathbf{z}) = max(\{f(i, \mathbf{z}) : 0 \leq i \leq x\})$.

Proposition 9 *Suppose that T is a sound Π_2-theory extending $I\Delta_0$. Then, $\mathcal{R}([T, \Sigma_1\text{-}CR]) = [\mathcal{R}(T), \mathbf{Max}] = \mathbf{M}(\mathcal{R}(T))$.*

As for the Δ_1-IR case, Beklemishev introduced in [5] a new recursive operator called *search operator* and showed that it corresponds to Δ_1-IR. Given $f : \mathbb{N} \to \mathbb{N}$, the function defined by the search operator (**S**) from f is $S(f)(a, b) = \mu z.\, J(f, a, b, z)$, where $J(f, a, b, z)$ stands for

$$\begin{aligned} \exists x, u, v \leq z \quad & z = \langle x, u, v\rangle \\ & \wedge [\, (a \leq x < b \wedge (u)_0 = 0 \wedge (v)_0 \neq 0 \wedge f(x) = u \wedge f(x+1) = v) \\ & \vee (x = a \wedge (u)_0 \neq 0 \wedge v = 0 \wedge f(a) = u) \\ & \vee (x = b \wedge (v)_0 = 0 \wedge u = 0 \wedge f(b) = v)\,] \end{aligned}$$

In words, either one finds $x \in [a, b)$ such that $(f(x))_0 = 0$ and $(f(x+1))_0 \neq 0$, or one establishes that $(f(a))_0 \neq 0$ or $(f(b))_0 = 0$. Then one outputs such an x as the first coordinate of a witness z that x is as required (see [5] for details). It is important to note that in [5] it is assumed that, by definition, the search operator can only be applied to unary functions with Δ_0-definable graph. Restricting the operator to unary functions is unessential but the restriction to functions with bounded graph is crucial. Here, to make this restriction explicit, we prefer to keep the search operator applicable to any unary function and then introduce the following notations. Let $\mathcal{R}_0(T)$ denote the class of those p.t.c.f.'s of T with a Δ_0-definition in T. Note that, in general, $\mathcal{R}_0(T)$ is not closed under composition and that $\mathcal{R}(T) = \mathbf{C}(\mathcal{R}_0(T))$. In addition,

Lemma 7 *$\mathcal{R}_0(T)$ coincides with the class of the functions in $\mathcal{R}(T)$ whose graph is Δ_0-definable in the standard model.*

Proof One inclusion is obvious. For the other, let $\varphi(\mathbf{x}, y) \in \Sigma_1$ be a definition of a function f in T and let $\theta(\mathbf{x}, y) \in \Delta_0$ defining the graph of f in $\mathbb{N}$. Then $\mathbb{N} \models$

$\forall \mathbf{x}\, \forall y\, (\varphi(\mathbf{x}, y) \to \theta(\mathbf{x}, y))$ and so there is a true Π_1-sentence, say $\forall z\, \delta(z)$ with $\delta \in \Delta_0$, satisfying that $T + \forall z\, \delta(z) \vdash \forall \mathbf{x}\, \exists y\, \theta(\mathbf{x}, y)$. It is easy to see that the Δ_0-formula $\neg\delta(y) \vee \theta(\mathbf{x}, y)$ is a definition of f in T. $\square$

Let $[\mathcal{F}, \mathbf{S}]^w$ denote the smallest set of functions containing $\mathcal{F}$, closed under composition, and satisfying that $S(f)$ belongs to the set whenever $f \in \mathcal{F}$. Note that $[\mathcal{F}, \mathbf{S}]^w$ is contained in, but could be weaker than, $[\mathcal{F}, \mathbf{S}]$ if $\mathcal{F}$ is not closed under composition. Using this terminology, Theorem 3 in [5] can be restated as follows (that result is proved in [5] over $I\Delta_0 + exp$ but this is unessential).

Proposition 10 *Suppose that T is a sound Π_2-theory extending $I\Delta_0$. Then, $\mathcal{R}([T, \Delta_1\text{-}IR]) = [\mathcal{R}_0(T), \mathbf{S}]^w$.*

The above characterization is not as neat as the one obtained for Σ_1-CR. It would be nicer to show $\mathcal{R}([T, \Delta_1\text{-IR}]) = [\mathcal{R}(T), \mathbf{S}]$, i.e., with the search operator being applied to any computable function rather than only to functions with a bounded graph. However, one should take into account the following fact.

Lemma 8 1. $[\mathcal{C}, \mathbf{M}] \subseteq [\mathcal{C}, \mathbf{S}]$ *for each function algebra $\mathcal{C}$ containing $\mathcal{M}^2$.*
2. *Assume that $\mathcal{R}([T, \Delta_1\text{-}IR]) = [\mathcal{R}(T), \mathbf{S}]$ for every sound Π_2-theory T extending $I\Delta_0$. Then $B\Sigma_1^-$ is provable from $Th_{\Pi_1}(\mathbb{N}) + I\Delta_1$. (Whether such a proof exists is still open).*

Proof (1) Pick $f : \mathbb{N}^{k+1} \to \mathbb{N}$ in $\mathbf{C}(\mathcal{C})$. Roughly speaking, in order to obtain the least $i \le x$ such that $f(i, \mathbf{z}) = 0$, it suffices to apply the search operator on the interval $[0, x+2]$ to the function that takes the values

$$\langle 0, 0\rangle, \langle \overline{sg}(f(0, \mathbf{z})), 0\rangle, \ \cdots, \ \langle \overline{sg}(f(x, \mathbf{z})), 0\rangle, \langle 1, 0\rangle,$$

where $\overline{sg}$ denotes Kleene's signum function, which satisfies $\overline{sg}(0) = 1$ and $\overline{sg}(x) = 0$ if $x \neq 0$. More formally, define $f' : \mathbb{N}^{k+2} \to \mathbb{N}$ to be

$$f'(x, \mathbf{z}, w) = \begin{cases} \langle 0, 0\rangle & x = 0 \\ \langle \overline{sg}(f(x-1, \mathbf{z})), 0\rangle & 1 \le x \le w \\ \langle 1, 0\rangle & x > w \end{cases}$$

Then, $f' \in \mathbf{C}(\mathcal{C})$ as $\mathcal{M}^2 \subseteq \mathcal{C}$, and it follows from the definition of the search operator that $M(f)(x, \mathbf{z}) = (S(f')(0, x+2, \mathbf{z}, x+1))_0$, where by abuse of notation we also write $S(f')$ to denote the search operator applied to a function with parameters. It only remains to eliminate the use of parameters $\mathbf{z}, w$ in f'. This can be achieved by putting together pieces of f' as follows. (This idea has been taken from the proof of Lemma 14 in [5] but we need to modify the coding method because we work over $\mathcal{M}^2$ rather than over the class of elementary functions). For simplicity, we first encode $\mathbf{z}, w$ into a single parameter v by putting $f''(x, v) = f'(x, (v)_0, \ldots, (v)_k)$. Now consider

$$g(x) = f''((x)_0, (((x)_0 + (x)_1)_0).$$

It follows from the definition of the pairing function that g on the interval $[\langle 0, \langle v, x\rangle\rangle, \langle 0, \langle v, x\rangle\rangle + x]$ takes the values $f''(0, v), \ldots, f''(x, v)$. If we put

$h(x, v) = \langle 0, \langle v, x\rangle\rangle$ and write $S(g)(h(x, v), h(x, v) + x)$ as $\langle a, b, c\rangle$, then we have $S(f'')(0, x, v) = \langle a - h(x, v), b, c\rangle$ and $S(f'')(0, x, \langle \mathbf{z}, w\rangle) = S(f')(0, x, \mathbf{z}, w)$. So, the latter function is in $[\mathcal{C}, \mathbf{S}]$, as required.

(2) Suppose $\mathfrak{A} \models Th_{\Pi_1}(\mathbb{N}) + I\Delta_1$ and consider T to be the set of all Π_2-sentences true in $\mathfrak{A}$. Then T is a sound Π_2-extension of $I\Delta_0$ closed under Δ_1-IR and hence $\mathcal{R}(T)$ is closed under the search operator by the assumption. It follows from part 1 that $\mathcal{R}(T)$ is also closed under bounded minimization. So $\mathcal{R}(T) = \mathcal{R}([T, \Sigma_1\text{-CR}])$ by Proposition 9 and T extends $[T, \Sigma_1\text{-CR}]$ by Lemma 5. Thus, $\mathfrak{A} \models B\Sigma_1^-$ as required. □

Having justified the introduction of the function algebra $[\mathcal{F}, \mathbf{S}]^w$, we are now in a position to obtain the main theorem of this section.

Theorem 6 *Let T be a sound extension of $I\Delta_0$.*

1. $\mathcal{R}(I\Delta_1(T)^-) = PR \cap \mathcal{R}(T)$.
2. $\mathcal{R}(I\Delta_1(T)) = PR \cap [\mathcal{R}_0(T), \mathbf{S}]^w = [PR \cap \mathcal{R}_0(T), \mathbf{S}]^w$.
3. $\mathcal{R}(L\Delta_1(T)) = PR \cap [\mathcal{R}(T), \mathbf{Max}] = [PR \cap \mathcal{R}(T), \mathbf{Max}]$.

Proof Write T' for $Th_{\Pi_2}(T)$.

(1) It follows from Corollary 2 and Lemma 6 that $\mathcal{R}(I\Delta_1(T)^-)$ equals to $PR \cap \mathcal{R}([T', \Delta_1^-\text{-IR}])$. But $T' + Th_{\Pi_1}(\mathbb{N})$ implies $[T', \Delta_1^-\text{-IR}]$ and $\mathcal{R}(T) = \mathcal{R}([T', \Delta_1^-\text{-IR}])$, for adding true Π_1-sentences to a sound theory does not increase the corresponding class of provably total functions.

(2) First, it follows from Corollary 1, Lemma 6 and Proposition 10 that $\mathcal{R}(I\Delta_1(T)) = PR \cap [\mathcal{R}_0(T), \mathbf{S}]^w$. Second, notice that

Claim $I\Delta_1(T)$ is Π_2-conservative over $I\Delta_1(I\Sigma_1 \vee T)$.

Suppose that $I\Delta_1(T)$ proves θ, with $\theta \in \Pi_2$. Then both $I\Sigma_1$ and $[T', \Delta_1\text{-IR}]$ prove θ too. Towards a contradiction, assume $I\Delta_1(I\Sigma_1 \vee T) \nvdash \theta$. Since $I\Sigma_1 \vdash \theta$, it follows from Theorem 1 that $[Th_{\Pi_2}(I\Sigma_1) \vee T', \Delta_1\text{-IR}] + \neg\theta$ is consistent. Put $S = Th_{\Pi_2}(I\Sigma_1) \vee T'$ and $\neg\theta \equiv \exists z\, \delta(z)$, with $\delta \in \Pi_1$, and suppose that $S + \neg\theta \vdash \forall x\, (\varphi(x, v) \leftrightarrow \psi(x, v))$, with $\varphi \in \Sigma_1$, $\psi \in \Pi_1$. Then S proves $\forall z\, (\delta(z) \to \forall x\, (\varphi(x, v) \leftrightarrow \psi(x, v)))$ and reasoning as in the proof of Proposition 2, we get that $[S, \Delta_1\text{-IR}] + \neg\theta \vdash \forall v\, I_{\varphi(x,v)}$. As a result, $[S, \Delta_1\text{-IR}] + \neg\theta$ implies $[S + \neg\theta, \Delta_1\text{-IR}]$ and hence the latter theory is consistent as well. But we have

$$S + \neg\theta \equiv (Th_{\Pi_2}(I\Sigma_1) + \neg\theta) \vee (T' + \neg\theta) \equiv (T' + \neg\theta)$$

since θ is a Π_2-theorem of $I\Sigma_1$. We have thus obtained that $[T', \Delta_1\text{-IR}] + \neg\theta$ is consistent, which is a contradiction. This completes the proof of the Claim.

It then follows that

$$\begin{aligned}\mathcal{R}(I\Delta_1(T)) = \mathcal{R}(I\Delta_1(I\Sigma_1 \vee T)) &= PR \cap \mathcal{R}([Th_{\Pi_2}(I\Sigma_1) \vee T', \Delta_1\text{-IR}]) \\ &= [\mathcal{R}_0(Th_{\Pi_2}(I\Sigma_1) \vee T'), \mathbf{S}]^w \\ &= [PR \cap \mathcal{R}_0(T), \mathbf{S}]^w\end{aligned}$$

(For the last equality note that each primitive recursive function whose graph is Δ_0-definable in $\mathbb{N}$ has a Δ_0-definition in $I\Sigma_1$ by Lemma 7).

(3) The proof is similar to that of part 2. □

In what follows we show that in presence of *exp*, $\mathcal{R}(I\Delta_1(T))$ and $\mathcal{R}(L\Delta_1(T))$ can also be described in purely recursion-theoretic terms. We introduce a suitably modified version of the bounded recursion operator, called *C-bounded recursion*, and prove that if T is a sound extension of $I\Delta_0 + exp$ then $\mathcal{R}(L\Delta_1(T))$ coincides with the closure of the basic functions under composition and $\mathcal{R}(T)$*-bounded recursion*. (A preliminary version of this result appeared in [8]).

Definition 3 (*C-bounded recursion*) A function $f : \mathbb{N}^{k+1} \to \mathbb{N}$ is defined from $g : \mathbb{N}^k \to \mathbb{N}$, $h : \mathbb{N}^{k+2} \to \mathbb{N}$ and $C : \mathbb{N}^{k+1} \to \mathbb{N}$ by *C-bounded recursion*, written $f = \mathbf{BR}_C(g, h)$, if $f \leq C$ and

$$f(\mathbf{x}, 0) = g(\mathbf{x}); \quad f(\mathbf{x}, y + 1) = h(\mathbf{x}, y, f(\mathbf{x}, y)),$$

i.e., f is defined from g and h by primitive recursion and f is bounded by C. Given a function class $\mathcal{C}$, $\mathcal{E}^{\mathcal{C}}$ is the smallest set of functions containing the basic functions (the constant zero, projections, and the successor function) and closed under composition and $\mathcal{C}$-bounded recursion, that is, C-bounded recursion for every $C \in \mathcal{C}$.

We use the notation $\mathcal{E}^{\mathcal{C}}$ in analogy with the well-known Grzegorczyk hierarchy $\mathcal{E}^i$, $i \geq 0$, defined in terms of usual bounded recursion (see e.g., [19]). One can attach to $\mathcal{E}^{\mathcal{C}}$ a first-order theory in an extended language, denoted $\mathcal{C}$-BRA, so that $\mathcal{E}^{\mathcal{C}} = \mathcal{R}(\mathcal{C}\text{-}BRA)$. The definition of $\mathcal{C}$-BRA is inspired by the well-known system PRA for the primitive recursive functions.

Definition 4 Suppose that $\mathcal{C}$ contains $\mathcal{M}^2$ and is closed under composition. The theory $\mathcal{C}$-BRA, *$\mathcal{C}$-Bounded Recursive Arithmetic*, is given by:

Language: $\mathcal{L}^{\mathcal{C}} = \bigcup_{i\in\omega} L_i$, where

- $L_0 = \mathcal{L}$ plus a function symbol B_f for each basic function.
- $L_{j+1} = L_j$ plus a function symbol f_t for each term t of L_j, and a function symbol f_{t_1,t_2} for each pair of terms $t_1(\mathbf{x})$, $t_2(\mathbf{x}, y, z)$ of L_j such that the function defined in the standard model from t_1 and t_2 by primitive recursion is bounded by some function $C \in \mathcal{C}$.

Axioms: (the universal closure of)

(1) Robinson's Q.
(2) $B_S(x) = x + 1, \quad B_{\Pi_i^n}(x_1, \ldots, x_n) = x_i, \quad B_O(x) = 0.$
(3) $\mathbf{f}_t(\mathbf{x}) = t(\mathbf{x})$.
(4) $\mathbf{f}_{t_1,t_2}(\mathbf{x}, 0) = t_1(\mathbf{x}), \quad \mathbf{f}_{t_1,t_2}(\mathbf{x}, z, y + 1) = t_2(\mathbf{x}, y, \mathbf{f}_{t_1,t_2}(\mathbf{x}, y))$.
(5) Open Induction: The induction scheme for open formulas of $\mathcal{L}^{\mathcal{C}}$.

Observe that $\mathcal{C}$-*BRA* is a theory only in an abstract model-theoretic sense (i.e., a set of sentences in a first order language) but, in general, it is not even effectively axiomatized. We shall use this theory as a technical tool in order to prove that $\mathcal{C} \cap PR \subseteq \mathcal{E}^{\mathcal{C}}$ in Proposition 11. Let us also note that bounds (i.e., the functions from $\mathcal{C}$) are not included in the axiomatizations of the recursive schemes (part (4) of the definition) and, so, $\mathcal{C}$-*BRA* cannot prove anything about them. This is natural because, in general,

$\mathcal{C}$ is not contained in $\mathcal{E}^{\mathcal{C}}$: for instance, consider the case when $\mathcal{C}$ contains some non primitive recursive functions, or, alternatively, see Remark 4 below.

It is routine to check that $\mathcal{C}$-*BRA* satisfies the following properties, which are well-known for *PRA*:

- in $\mathcal{C}$-*BRA* every bounded formula is equivalent to an open one;
- $\mathcal{C}$-*BRA* supports definition by cases;
- $\mathcal{C}$-*BRA* admits a purely universal axiomatization.

As a consequence, a standard application of Herbrand's theorem gives us that $\mathcal{R}(\mathcal{C}-BRA) = \mathcal{E}^{\mathcal{C}}$. Equipped with this result, we are able to show that

Proposition 11 *Suppose that $\mathcal{C} = \mathcal{R}(T)$ for T some sound extension of $I\Delta_0$. Then $\mathcal{C} \cap PR \subseteq \mathcal{E}^{\mathcal{C}}$.*

Proof Since $\mathcal{R}(\mathcal{C}\text{-}BRA) = \mathcal{E}^{\mathcal{C}}$ and $\mathcal{C} \cap PR \subseteq \mathcal{R}(I\Delta_1(T))$ by Theorem 6, it is sufficient to prove that

Claim $I\Delta_1(T)$ is Π_2-conservative over $\mathcal{C}$-BRA.

To this end, we follow J. Avigad's proof that $I\Sigma_1$ is Π_2-conservative over PRA given in [2]. The key ingredient is that of an $\exists_2$-*closed* model (or *Herbrand saturated* model in Avigad's terminology). We say that $\mathfrak{A}$ is an $\exists_2$-closed if, for every structure $\mathfrak{B}$, $\mathfrak{A} \prec_{\forall_1} \mathfrak{B}$ implies $\mathfrak{A} \prec_{\exists_2} \mathfrak{B}$. By a union of chain argument every model of a universal theory U can be $\forall_1$-elementary extended to a new model of U which is $\exists_2$-closed. Thus, if every $\exists_2$-closed model of a universal theory U is a model of a theory W, then W is $\forall_2$-conservative over U (this is Theorem 3.4 of [2]).

Turning back to the proof of the Claim, it suffices to show that every $\exists_2$-closed model of $\mathcal{C}-BRA$ satisfies $I\Delta_1(T)$, for each Π_2-formula is equivalent in $\mathcal{C}-BRA$ to a $\forall_2$-formula. Suppose that $\mathfrak{A}$ is an $\exists_2$-closed model of $\mathcal{C}-BRA$. Let $\varphi(x, y, v)$, $\psi(x, y, v) \in \Delta_0$ with $T \vdash \exists y\, \varphi(x, y, v) \leftrightarrow \forall y\, \psi(x, y, v)$. We may assume $I\Delta_0 \vdash \varphi(x, y_1, v) \wedge \varphi(x, y_2, v) \rightarrow y_1 = y_2$, otherwise consider $\varphi(x, y, v) \wedge \forall y' < y\, \neg\varphi(x, y', v)$ instead. Since $T \vdash \forall x, v\, \exists y\, (\varphi(x, y, v) \vee \neg\psi(x, y, v))$ and T is sound, $y = \mu t.\, (\varphi(x, t, v) \vee \neg\psi(x, t, v))$ defines a p.t.f.c. of T, say C. Then,

$$(\dagger) \quad \mathbb{N} \models \varphi(x, y, v) \rightarrow y = C(x, v).$$

Let $a \in \mathfrak{A}$ and let $\varphi_0(x, y, v)$ be an open formula equivalent in $\mathcal{C}$-BRA to φ. We must show that the induction axiom for $\exists y\, \varphi_0(x, y, a)$ is true in $\mathfrak{A}$. To that end, assume $\mathfrak{A} \models \exists y\, \varphi_0(0, y, a) \wedge \forall x\, (\exists y\, \varphi_0(x, y, a) \rightarrow \exists y\, \varphi_0(x + 1, y, a))$. In particular, $\mathfrak{A} \models \forall x, y\, \exists y'\, (\varphi_0(x, y, a) \rightarrow \varphi_0(x+1, y', a))$. Since this last formula has quantifier complexity $\forall_2$, it is provable from the universal diagram of $\mathfrak{A}$ by the closedness condition for $\mathfrak{A}$. Thus, applying Herbrand's theorem and using that $\mathcal{C}-BRA$ supports definition by cases, we obtain that there are $b, c \in \mathfrak{A}$ and a term of $\mathcal{L}^{\mathcal{C}}$, $t(x, y, v, w)$, satisfying that

$$\mathfrak{A} \models \varphi_0(0, c, a) \wedge \forall x, y\, (\varphi_0(x, y, a) \rightarrow \varphi_0(x + 1, t(x, y, a, b), a)).$$

Let h denote the function defined in the standard model by

$$h(x, y, z, v, w) = \begin{cases} t(x, z, v, w) & \text{if } \varphi_0(x+1, t(x, z, v, w), v); \\ 0 & \text{otherwise} \end{cases}$$

Clearly $h \in \mathcal{E}^{\mathcal{C}}$. Let f be the function defined by primitive recursion as follows:

$$f(0, y, v, w) = y, \qquad f(x+1, y, v, w) = h(x, y, f(x, y, v, w), v, w).$$

By (†) $f(x, y, v, w) \leq C'(x, y, v, w) = y + C(x, v)$. So $f \in \mathcal{E}^{\mathcal{C}}$, since it is defined by C'-bounded recursion and $C' \in \mathcal{C}$. Let $\mathbf{f}$ be the function symbol of $\mathcal{L}^{\mathcal{C}}$ corresponding to f. Then $\mathfrak{A}$ satisfies that

$$\varphi_0(0, \mathbf{f}(0, c, a, b), a) \wedge \forall x\, (\varphi_0(x, \mathbf{f}(x, c, a, b), a) \rightarrow \varphi_0(x+1, \mathbf{f}(x+1, c, a, b), a)).$$

Since $\mathfrak{A}$ is a model of open induction, $\mathfrak{A} \models \forall x\, \varphi_0(x, \mathbf{f}(x, c, a, b), a)$ and hence $\mathfrak{A} \models \forall x\, \exists y\, \varphi_0(x, y, a)$, as required. □

Remark 4 It is worth noting that the assumption that $\mathcal{C}$ is the class of p.t.c.f.'s of a theory T cannot be dropped in Proposition 11. For example, put $\mathcal{C} = \mathbf{C}(\mathcal{M}^2 \cup \{Ch_A\})$, where Ch_A is the characteristic function of a primitive recursive set A which is not in the second level of the Grzegorczyk hierarchy $\mathcal{E}^2$. First, $\mathcal{C}$ cannot be written as $\mathcal{R}(T)$ for any theory T in the language of arithmetic, for we have $\mathcal{R}(T) = \mathbf{C}(\mathcal{R}_0(T))$ whereas closing under composition the functions in $\mathcal{C}$ with a Δ_0-definable graph only gives us $\mathcal{M}^2$. Second, $\mathcal{C} \cap PR = \mathcal{C} \nsubseteq \mathcal{E}^{\mathcal{C}} = \mathcal{E}^2$.

Theorem 7 *Let T be a sound extension of $I\Delta_0 + exp$ and let $\mathcal{C} = \mathcal{R}(T)$. Then, $\mathcal{R}(I\Delta_1(T)) = \mathcal{R}(L\Delta_1(T)) = \mathcal{E}^{\mathcal{C}}$.*

Proof It follows from Proposition 11 that $\mathcal{C} \cap PR \subseteq \mathcal{E}^{\mathcal{C}}$ and it follows from Theorem 6 that $\mathcal{R}(L\Delta_1(T)) = [\mathcal{C} \cap PR, \mathbf{Max}] = \mathbf{M}(\mathcal{C} \cap PR)$. But it is easy to see that $\mathcal{E}^{\mathcal{C}}$ is closed under bounded minimization. Thus, $\mathcal{R}(L\Delta_1(T)) \subseteq \mathcal{E}^{\mathcal{C}}$. For the opposite inclusion, note that

Claim $\mathcal{E}^{\mathcal{C}} = \mathcal{E}^{\mathcal{C} \cap PR}$.

We reason by induction on the definition of $f \in \mathcal{E}^{\mathcal{C}}$. The critical step is the definition by $\mathcal{C}$-bounded recursion. Suppose $f = \mathbf{BR}_C(g, h)$ with $C \in \mathcal{C}$. Since f itself is primitive recursive, there are $C_1 \in PR$ and $C_2 \in \mathcal{C}$ with Δ_0-definable graphs such that $f \leq C_1, C_2$. Let $\theta_1(\mathbf{x}, y) \in \Delta_0$ be a definition of C_1 in $I\Sigma_1$ and let $\theta_2(\mathbf{x}, y) \in \Delta_0$ be a definition of C_2 in T. Then $y = \mu t.\, (\theta_1(\mathbf{x}, t) \vee \theta_2(\mathbf{x}, t))$ defines a p.t.c.f. of $T \vee I\Sigma_1$, say C_3. Note that $C_3 \in \mathcal{C} \cap PR$ and $f = \mathbf{BR}_{C_3}(g, h)$, which proves the claim.
Thus $\mathcal{E}^{\mathcal{C}} = \mathcal{E}^{\mathcal{C} \cap PR} \subseteq \mathbf{BR}(\mathcal{C} \cap PR) = \mathbf{M}(\mathcal{C} \cap PR) = \mathcal{R}(L\Delta_1(T))$, where in the last but one equality $\mathbf{BR}$ denotes the usual bounded recursion operator and we use that in presence of *exp*, bounded recursion can be reduced to bounded minimization. □

Exponentiation is used in two different ways in Theorem 7 above. On the one hand, *exp* is needed to prove $I\Delta_1(T)$ and $L\Delta_1(T)$ to be equivalent and thus share the same p.t.c.f.'s. On the other hand, *exp* is needed to reduce bounded recursion to bounded minimization in the proof that $\mathcal{E}^{\mathcal{C}} \subseteq \mathcal{R}(L\Delta_1(T))$. Eliminating this second use of *exp* seems to be a hard problem, for it is related to important problems in Complexity Theory. In fact, if $T = I\Delta_0$ then $\mathcal{R}(L\Delta_1(T)) = \mathcal{M}^2$ and $\mathcal{E}^{\mathcal{C}} = \mathcal{E}^2$. Thus if Theorem 7 holds for $T = I\Delta_0$ then the Linear Time Hierarchy coincides with LinSpace. Likewise, if Theorem 7 holds for $T = I\Delta_0 + \Omega_1$, where Ω_1 expresses "$x^{|x|}$ is total", then the Polynomial Time Hierarchy equals to PolySpace.

In the same spirit, we close this section with a reduction of the NE Problem (see Remark 2) to a purely recursion-theoretic question. Recall that $\mathcal{E}^3$ denotes the third level of the Grzegorczyk hierarchy, which is well-known to coincide with the set of Kalmár's elementary functions.

Proposition 12 *The following are equivalent.*

1. $Th_{\Pi_1}(\mathbb{N}) + \neg exp \vdash B\Sigma_1^-$.
2. *For each* $f \in \mathcal{E}^3$ *with a* Δ_0*-definable graph,* $\mathbf{C}(\mathcal{M}^2 \cup \{f\})$ *is closed under bounded minimization.*

Proof (*1*⇒*2*): Let $f \in \mathcal{E}^3$ whose graph is definable by a Δ_0-formula, say $\theta(\mathbf{x}, y)$, and put $T = Th_{\Pi_1}(\mathbb{N}) + \forall \mathbf{x}\, \exists y\, \theta(\mathbf{x}, y)$. It follows from Lemmas 4 and 5 that $\mathcal{R}(T) = \mathbf{C}(\mathcal{M}^2 \cup \{f\})$. Now observe that it follows from condition 1 that

Claim T is closed under Σ_1-*CR*.

On the one hand, since $\mathcal{R}(T) \subseteq \mathcal{E}^3 = \mathcal{R}(I\Delta_0 + exp)$, it follows from the proof of Lemma 5 that T is included in $Th_{\Pi_1}(\mathbb{N}) + exp$. But the latter theory is closed under Σ_1-CR and hence $T + exp$ implies $T + \Sigma_1$-CR. On the other hand, $T + \neg exp$ is an extension of $B\Sigma_1^-$ by condition 1 and so $T + \neg exp$ implies $T + \Sigma_1$-CR too. As a result, T implies $T + \Sigma_1$-CR, as required.

Thus $\mathbf{C}(\mathcal{M}^2 \cup \{f\})$ is closed under bounded minimization by Proposition 9.

(*2*⇒*1*): Observe that it follows from condition 2 that

Claim $Th_{\Pi_1}(\mathbb{N})$ implies $B\Delta_1(I\Delta_0 + exp)^-$.

To see this, assume that $I\Delta_0 + exp \vdash \forall x\, \exists y\, \varphi(x, y)$, with $\varphi(x, y) \in \Sigma_1^-$. Put $\varphi(x, y) \equiv \exists z\, \varphi_0(x, y, z)$, with $\varphi_0 \in \Delta_0$, and define $\theta(x, y)$ to be the Δ_0-formula $y = \mu t.\, \varphi_0(x, (t)_0, (t)_1)$. Then $\theta(x, y)$ defines a computable function $f \in \mathcal{E}^3$ since $I\Delta_0 + exp \vdash \forall x\, \exists y\, \theta(x, y)$. By condition 2, $Max(f) \in \mathbf{C}(\mathcal{M}^2 \cup \{f\}) = \mathcal{R}(I\Delta_0 + \forall x\, \exists y\, \theta(x, y))$. Note that $\forall x \le z\, \exists y \le u\, \theta(x, y) \wedge \exists x \le z\, \theta(x, u)$ is a Δ_0-formula defining $Max(f)$ in the standard model. Hence reasoning as in the proof of Lemma 5 we obtain that $Th_{\Pi_1}(\mathbb{N}) + \forall x\, \exists y\, \theta(x, y)$ proves $\forall z\, \exists u\, \forall x \le z\, \exists y \le u\, \theta(x, y)$ and so $Th_{\Pi_1}(\mathbb{N}) \vdash B_{\varphi(x,y)}$, as required.

Thus $Th_{\Pi_1}(\mathbb{N}) + \neg exp$ implies $B\Delta_1(I\Delta_0 + exp)^- + \neg exp$ which in turn implies $B\Sigma_1^-$ by Theorem 4. □

Corollary 5 *Assume that there exists some* $f \in \mathcal{E}^3$ *with a* Δ_0*-definable graph such that* $Max(f) \notin \mathbf{C}(\mathcal{M}^2 \cup \{f\})$. *Then* $I\Delta_0 + \neg exp$ *does not imply* $B\Sigma_1$.

Interestingly, Lemma 6.1 of [4] shows how to construct a function $f \in \mathcal{E}^4$ with an elementary graph such that $Max(f) \notin \mathbf{C}(\mathcal{E}^3 \cup \{f\})$. The construction uses Turing machines equipped with an internal clock. Although it is far from obvious how to adapt that construction to obtain a function satisfying the assumptions of Corollary 5, this approach gives us some new ideas to attack the NE Problem and to obtain, at least, a conditional negative answer under some complexity-theoretic assumption.

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