Local Induction and Provably Total Computable Functions: A Case Study

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Abstract. Let $I\Pi_2^-$ denote the fragment of Peano Arithmetic obtained by restricting the induction scheme to parameter free Π_2 formulas. Answering a question of R. Kaye, L. Beklemishev showed that the provably total computable functions (p.t.c.f.) of $I\Pi_2^-$ are, precisely, the primitive recursive ones. In this work we give a new proof of this fact through an analysis of the p.t.c.f. of certain *local versions* of induction principles closely related to $I\Pi_2^-$. This analysis is essentially based on the equivalence between local induction rules and restricted forms of iteration. In this way, we obtain a more direct answer to Kaye's question, avoiding the metamathematical machinery (reflection principles, provability logic,...) needed for Beklemishev's original proof.

1 Introduction

An important notion in studying the computational content of a fragment of Arithmetic is that of its *provably total computable functions*. A number–theoretic computable function $f : \mathbb{N}^k \to \mathbb{N}$ is said to be a provably total computable function (p.t.c.f.) of a theory T, written $f \in \mathcal{R}(T)$, if there is a Σ_1 formula $\varphi(\vec{x}, y)$ such that:

1. φ defines the graph of f in the standard model of Arithmetic $\mathbb{N}$; and
2. $T \vdash \forall \vec{x}\, \exists! y\, \varphi(\vec{x}, y)$.

Observe that condition 1. amounts to the computability of f, whereas condition 2. yields an implicit measure of the complexity of f attending to the logical principles needed to prove that a Σ_1–definition of f defines a total function. Since it was introduced by G. Kreisel in the 1950s this notion has been widely studied, and nice recursion–theoretic and computational complexity characterizations of the sets $\mathcal{R}(T)$ have been obtained for a good number of theories T. For instance, by a classical result due independently to G. Mints, C. Parsons and G. Takeuti, the class of p.t.c.f. of the scheme of induction for Σ_1–formulas $I\Sigma_1$ equals to the class of the primitive recursive functions PR. Indeed, all classes $\mathcal{R}(I\Sigma_n)$, $n \geq 1$, can be characterized in terms of the Fast Growing Hierarchy up to the ordinal ε_0. As for weak fragments below $I\Sigma_1$, their p.t.c.f. have been characterized in terms of subrecursive operators (bounded recursion, bounded minimization, ...)

as well as in terms of computational complexity classes. In fact, their classes of p.t.c.f. have been intensively investigated in connection with important open problems in Complexity Theory, mainly in the context of Bounded Arithmetic.

In spite of the wide range of the theories considered, a number of uniform methods for characterizing the p.t.c.f. of an arithmetic theory are available. E.g. Herbrand analyses as developed by W. Sieg in [9], S. Buss' witnessing method [5] or, in general, proof–theoretic techniques using Cut elimination theorem. However, for some particular fragments of Peano Arithmetic none of these standard methods seems to be applicable. Of special interest is the case of the scheme of parameter free Π_2–induction, $I\Pi_2^-$, given by the induction scheme

$$I_\varphi : \quad \varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x\, \varphi(x)\,,$$

restricted to $\varphi(x) \in \Pi_2^-$ (as usual, we write $\varphi(x) \in \Gamma^-$ to mean that φ is in Γ and contains no other free variables than x). Since $I\Sigma_1^- \subseteq I\Pi_2^-$ and $I\Sigma_1$ is Σ_3–conservative over $I\Sigma_1^-$ [8], it follows that every primitive recursive function is provably total in $I\Pi_2^-$; and R. Kaye asked whether the p.t.c.f. of $I\Pi_2^-$ are exactly the primitive recursive ones. This question remained elusive until [4], where L. Beklemishev gave a positive answer using modal provability logic techniques. Although quite elegant, Beklemishev's answer only provides an indirect solution. Firstly, he reformulated $I\Pi_2^-$ in terms of local reflection principles (reflection principles in Arithmetic are axiom schemes expressing the statement that "if a formula φ is provable in a theory T then φ is valid"). Secondly, he derived the result as an application of a conservation theorem for local reflection principles whose proof leans upon properties of Gödel–Löb provability logic **GL**.

In this work we obtain a more direct answer to Kaye's question, avoiding the metamathematical machinery needed for Beklemishev's proof. In fact, our proof that $\mathcal{R}(I\Pi_2^-) = PR$ will follow the lines of standard arguments for characterizing classes $\mathcal{R}(T)$. Let us consider, for instance, a proof that $\mathcal{R}(I\Sigma_1) = PR$. Such a proof typically proceeds in two steps.

- Step 1: $I\Sigma_1$ is Π_2–conservative over the *inference rule* version of the principle of Σ_1–induction Σ_1–IR. So, $\mathcal{R}(I\Sigma_1) = \mathcal{R}(\Sigma_1\text{–IR})$.
- Step 2: Applications of Σ_1–IR correspond to applications of the primitive recursion operator.

The main obstacle to apply this argument to $I\Pi_2^-$ is that there is no simple, direct argument to reduce $I\Pi_2^-$ to an inference rule version of it. Here we solve this problem by showing that $I\Pi_2^-$ is equivalent to $I(\Sigma_2^-, \mathcal{K}_2)$, a certain version of the parameter free Σ_2–induction scheme where the elements x for which the induction axiom claims $\varphi(x)$ to hold are restricted to be Σ_2–definable elements. Equipped with this result, it is easy to obtain that $I\Pi_2^-$ is Π_2 (in fact, Π_3) conservative over the corresponding inference rule version $(\Sigma_2, \mathcal{K}_2)$–IR. Then, we show that applications of $(\Sigma_2, \mathcal{K}_2)$–IR correspond to (restricted forms) of the iteration operator and thus all functions in $\mathcal{R}(I\Pi_2^-)$ are primitive recursive.

Our analysis also yields a new conservation theorem for fragments of Peano Arithmetic, which is of independent interest. Namely, we prove that $I\Pi_2^-$ is Π_3–conservative over $I\Sigma_1$. This improves on a previous result by Beklemishev in [4], where conservativity between these theories with respect to boolean combinations of Σ_2–sentences was established.

We close this section by giving a precise definition of the auxiliary scheme that will be central in our analysis of the class of p.t.c.f. of $I\Pi_2^-$.

Let $\mathcal{L} = \{0, S, +, \cdot, <\}$ denote the language of first order Arithmetic. If Γ is a set of formulas of $\mathcal{L}$, then $I\Gamma$ is the theory axiomatized over Robinson's Q by the induction scheme, I_φ, restricted to formulas $\varphi(x) \in \Gamma$. If free variables other that x are not allowed, we write $\varphi(x) \in \Gamma^-$ and, accordingly, $I\Gamma^-$ denotes the theory axiomatized over Q by the axioms I_φ, for $\varphi(x) \in \Gamma^-$.

Definition 1. *$I(\Sigma_2, \mathcal{K}_2)$ is the theory given by $I\Sigma_1^-$ together with the scheme*

$$\begin{aligned} &\varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \\ &\quad \rightarrow \forall x_1, x_2\, (\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2) \rightarrow \forall x\, (\delta(x) \rightarrow \varphi(x)) \end{aligned}$$

where $\varphi(x) \in \Sigma_2$ and $\delta(x) \in \Sigma_2^-$. The natural inference rule associated to this scheme, denoted $(\Sigma_2, \mathcal{K}_2)$–IR, is given by:

$$\frac{\varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1))}{\forall x_1, x_2\, (\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2) \rightarrow \forall x\, (\delta(x) \rightarrow \varphi(x))}$$

where $\delta(x) \in \Sigma_2^-$ and $\varphi(x) \in \Sigma_2$. Finally, if we restrict the scheme to $\varphi(x) \in \Sigma_2^-$, we obtain the parameter free counterpart of $I(\Sigma_2, \mathcal{K}_2)$, denoted $I(\Sigma_2^-, \mathcal{K}_2)$.

Remark 1. Firstly, let us recall that, given a model $\mathfrak{A}$, $\mathcal{K}_2(\mathfrak{A})$ denotes the set of elements of $\mathfrak{A}$ that are definable in $\mathfrak{A}$ by a formula $\delta(x) \in \Sigma_2$. This explains why $\mathcal{K}_2$ appears in our notation for these theories. Secondly, if $\mathfrak{A} \models I\Sigma_1^-$, then $\mathcal{K}_2(\mathfrak{A}) \prec_2 \mathfrak{A}$ (i.e., $\mathcal{K}_2(\mathfrak{A})$ is a Π_2–elementary substructure of $\mathfrak{A}$). This property plays an important role in what follows and it is because of it that $I(\Sigma_2, \mathcal{K}_2)$ is axiomatized over $I\Sigma_1^-$ instead of over a weaker system (such as Q or $I\Delta_0$).

A key fact is that $I(\Sigma_2^-, \mathcal{K}_2)$ provides an alternative formulation of $I\Pi_2^-$:

Lemma 2. *$I\Pi_2^- \equiv I(\Sigma_2^-, \mathcal{K}_2)$.*

Proof. We only prove that $I(\Sigma_2^-, \mathcal{K}_2)$ extends $I\Pi_2^-$. The converse is similar. Let $\mathfrak{A} \models I(\Sigma_2^-, \mathcal{K}_2)$ and $\varphi(x) \in \Pi_2^-$ such that $\mathfrak{A} \models \varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1))$. Assume $\mathfrak{A} \models \exists x\, \neg\varphi(x)$. Since $\mathfrak{A} \models I\Sigma_1^-$, $\mathcal{K}_2(\mathfrak{A}) \prec_2 \mathfrak{A}$ and there is $a \in \mathcal{K}_2(\mathfrak{A})$ such that $\mathfrak{A} \models \neg\varphi(a)$. Let $\delta(v)$ be a Σ_2 formula defining the element a and let $\theta(x)$ be $\exists v\, (\delta(v) \wedge \neg\varphi(v - x))$. Clearly, $\mathfrak{A} \models \theta(0) \wedge \forall x\, (\theta(x) \rightarrow \theta(x+1))$. By $I(\Sigma_2^-, \mathcal{K}_2)$, $\mathfrak{A} \models \theta(a)$ and so $\mathfrak{A} \models \neg\varphi(0)$, which is a contradiction.

Given a theory T and an inference rule R, we denote by $[T, R]$ the closure of T under first order logic and unnested applications of R. We denote by $T + R$ the closure of T under first order logic and (nested) applications of R. Therefore, $T + R = \bigcup_{k \in \omega} [T, R]_k$, where $[T, R]_0 = T$ and $[T, R]_{k+1} = [[T, R]_k, R]$.

The first step in the analysis of $I\Pi_2^-$ is a suitable reduction of $I(\Sigma_2, \mathcal{K}_2)$ to a fragment defined by the rule $(\Sigma_2, \mathcal{K}_2)$–IR. Indeed, we have:

Proposition 3. *$I(\Sigma_2, \mathcal{K}_2)$ is Π_3–conservative over $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)$–IR.*

Very conveniently, this reduction can be carried out by the same tools used to derive the reduction of $I\Sigma_1$ to Σ_1–IR (e.g., by adapting the cut–elimination argument used in [3] to derive a similar reduction for the Collection scheme). Alternatively, in [6], lemma 3.6, we gave a model–theoretic proof of this result using the notion of a Σ_{n+1}–closed model, following the methods introduced by Avigad in [1].

2 Local Induction and Restricted Iteration

Next step in our analysis is to show that applications of $(\Sigma_2, \mathcal{K}_2)$–IR correspond to (a restricted form of) the iteration operator. To this end, we shall consider extensions of $\mathcal{L}$ obtained by adding a finite set of unary function symbols, $\mathcal{F} = \{f_1, \dots, f_n\}$, and a (finite or countable) set of new constant symbols, C. Through this section we consider a fixed set of constants, C, and we shall denote by $\mathcal{L}_\mathcal{F}$ the language $\mathcal{L} + \{f_1, \dots, f_n\} + C$. If g is a new unary function symbol then $\mathcal{L}_{\mathcal{F},g}$ will denote the language $\mathcal{L}_{\{f_1,\dots,f_n,g\}}$.

Definition 4. *Let $f \in \mathcal{F}$ a unary function symbol and T an $\mathcal{L}_\mathcal{F}$–theory. We say that f is an* iterable non decreasing function *over T if the theory T proves:*

$$\forall x_1, x_2\, (x_1 \le x_2 \to f(x_1) \le f(x_2)), \ \textit{and}\ \forall x\, (x^2 < f(x))$$

Let $\Sigma_0^\mathcal{F}$ be the class of bounded formulas of $\mathcal{L}_\mathcal{F}$. Classes $\Sigma_{n+1}^\mathcal{F}$ and $\Pi_{n+1}^\mathcal{F}$ are defined as usual. The theory $I\Sigma_0^\mathcal{F}$ is the $\mathcal{L}_\mathcal{F}$–theory axiomatized over Q by

- The induction axiom I_φ for each formula $\varphi \in \Sigma_0^\mathcal{F}$, and
- Axioms for each $f \in \mathcal{F}$:
  $\forall x_1, x_2\, (x_1 \le x_2 \to f(x_1) \le f(x_2))$, and $\forall x\, (x^2 < f(x))$

This is a basic theory to deal with the *iteration* of f and to guarantee the usual properties of the iteration of a nondecreasing function with a $\Pi_0^\mathcal{F}$–definable graph. The basic facts provable in this theory were stated in [6]. Next result collects together the facts that we shall need in the present context.

Proposition 5. *For each $f \in \mathcal{F}$ there exists a formula $IT_f(z, x, y) \in \Sigma_0^\mathcal{F}$ such that the following formulas are theorems of $I\Sigma_0^\mathcal{F}$:*

1. $IT_f(z, x, y_1) \wedge IT_f(z, x, y_2) \to y_1 = y_2$.
2. $(IT_f(0, x, y) \leftrightarrow x = y) \wedge (IT_f(1, x, y) \leftrightarrow f(x) = y)$.
3. $IT_f(z+1, x, y) \leftrightarrow \exists y_0 \le y\, (IT_f(z, x, y_0) \wedge f(y_0) = y)$.
4. $IT_f(z, x, y) \to \forall z_0 < z\, \exists y_0 < y\, IT_f(z_0, x, y_0)$.
5. $z \ge 1 \wedge IT_f(z, x, y) \to x^2 < y \wedge z \le y$.
6. $z \ge 1 \wedge x_1 \le x_2 \wedge IT_f(z, x_1, y_1) \wedge IT_f(z, x_2, y_2) \to y_1 \le y_2$.
7. $IT_f(z_1, x, y_0) \wedge IT_f(z_2, y_0, y) \to IT_f(z_1 + z_2, x, y)$.

In what follows we use a more suggestive notation and write $f^z(x) = y$ instead of $IT_f(z, x, y)$.

Definition 6. *We say that $f \in \mathcal{F}$ is a* dominating function *over T if, for any term $t(x)$ of $\mathcal{L}_{\mathcal{F}}$, there exists $k \in \omega$ such that T proves*

$$\forall x\,(t(x) \leq f^k(x + \sigma(t)))$$

where $\sigma(t) = c_1 + \cdots + c_m$ and $c_1, \ldots, c_m$ are all the constants occurring in $t(x)$.

Lemma 7. *Let T be an extension of $I\Sigma_0^{\mathcal{F}}$ and let $f \in \mathcal{F}$ a (iterable nondecreasing) dominating function over T. Then, for each term $t(x_1, \ldots, x_m)$ of $\mathcal{L}_{\mathcal{F}}$ whose variables are among $x_1, \ldots, x_m$, there exists $k \in \omega$ such that*

$$T \vdash t(x_1, \ldots, x_m) < f^k(x_1 + \cdots + x_m + \sigma(t)).$$

Remark 2. Languages $\mathcal{L}_{\mathcal{F}}$ and the notion of a dominating function are tailored to deal with the following situation. Assume $\Gamma = \{\theta_1, \ldots, \theta_m\}$ is a finite set of Σ_0–formulas with only two free variables, say x and y, and for each $j = 1, \ldots, m$, $\bar{\theta}_j(x, y)$ denotes the formula $\forall u \leq x \exists v \leq y\, \theta_j(u, v)$. Let $\mathcal{F} = \{f_1, \ldots, f_m, f\}$ be a set of unary function symbols and let T be the extension of $I\Sigma_0^{\mathcal{F}}$ with the following additional axioms:

- For each $j = 1, \ldots, m$, $\forall x\,(f_j(x) = y \leftrightarrow \exists y_0 \leq y\,(\bar{\theta}_j(x, y_0) \wedge y = (x+1)^2 + y_0)$.
- $\forall x\,(f(x) = (x+1)^2 + f_1(x) + \cdots + f_m(x))$.

Then, every $h \in \mathcal{F}$ is an iterable nondecreasing function over T and f is a dominating function over T. This last fact can be proved by induction on terms. The most interesting case occurs when $t(x)$ is a product of two terms, $t_1(x) \cdot t_2(x)$. By induction hypothesis, $t_1(x) \leq f^k(x + \sigma(t_1))$ and $t_2(x) \leq f^l(x + \sigma(t_2))$, for some $k \geq \max(l, 2)$ (so, for every u, $f^k(u) \geq k \geq 2$.) Then,

$$\begin{aligned} t(x) &\leq (t_1(x) + t_2(x))^2 \leq f(t_1(x) + t_2(x)) \leq f(f^k(x + \sigma(t_1)) + f^l(x + \sigma(t_2))) \\ &\leq f(2 \cdot f^k(x + \sigma(t))) \leq f((f^k(x + \sigma(t)))^2) \leq f^{k+2}(x + \sigma(t)) \end{aligned}$$

and we conclude that $t(x) \leq f^{k+2}(x + \sigma(t))$. The remaining cases are similar.

As a final step in the analysis of $(\Sigma_2, \mathcal{K}_2)$–IR and due to technical reasons, it will be convenient to denote the Σ_2–definable elements by closed terms of an extended language. This motivates the introduction of the following *local induction rules.*

Definition 8. *For each set of formulas Γ and each set of closed terms Λ_0 of $\mathcal{L}_{\mathcal{F}}$ we consider the rules (where $\varphi(x) \in \Gamma$ and $t \in \Lambda_0$):*

$$(\Gamma, \Lambda_0)\text{–}IR: \quad \frac{\varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1))}{\varphi(t)}$$

$$(\Gamma, \Lambda_0)\text{–}IR_0: \quad \frac{\forall x\,(\varphi(x) \rightarrow \varphi(x+1))}{\varphi(0) \rightarrow \varphi(t)}$$

Definition 9. *We say that Λ_0 is exponentially closed over T if for every $t, s \in \Lambda_0$ there exists $t' \in \Lambda_0$ such that $[T, (\Sigma_1^{\mathcal{F}}, \Lambda_0)\text{–}IR] \vdash \exists y \leq t'\,(s^t = y)$.*

These rules were intensively studied in [6], where the following results were obtained. From now on, we assume that T is a fixed extension of $I\Sigma_0^{\mathcal{F}}$ obtained by adding a set of $\Pi_1^{\mathcal{F}}$ sentences, and Λ_0 denotes the set of all closed terms of a sublanguage of $\mathcal{L}_{\mathcal{F}}$ extending $\mathcal{L}$ and containing the set of constants C (and so Λ_0 is closed under sum and product).

Remark 3. Let us note that under these assumptions T satisfies a natural version of Parikh's theorem (see [7], chapter 5, theorem 1.4). This fact will be used extensively without further comments.

In addition, we assume that there is $f \in \mathcal{L}_{\mathcal{F}}$ a dominating function over T and Λ_0 is exponentially closed over T. Then, we have (see lemma 4.8, lemma 4.10 and theorem 4.14 of [6]):

Proposition 10. $T + (\Pi_2^{\mathcal{F}}, \Lambda_0)$–$IR \equiv T + \{\forall x\, \exists y\, (f^t(x) = y) :\ t \in \Lambda_0\}$.

Theorem 11. *$T + (\Pi_2^{\mathcal{F}}, \Lambda_0)$–$IR_0$ is $\Pi_2^{\mathcal{F}}$–conservative over $T + (\Pi_2^{\mathcal{F}}, \Lambda_0)$–$IR$.*

Here we extend our work in [6] and obtain a new theorem on these local induction systems that will be crucial to derive our main result. The ideas involved are similar to the ones used in [6] to obtain Proposition 10 and Theorem 11.

Theorem 12. *$T + I\Sigma_1^{\mathcal{F}}$ extends $T + (\Sigma_2^{\mathcal{F}}, \Lambda_0)$–$IR$.*

Proof. The arguments used in [2], proposition 2.1, can be easily adapted to yield that for every $k \in \omega$, $[T, (\Sigma_2^{\mathcal{F}}, \Lambda_0)\text{–IR}]_k \equiv [T, (\Pi_2^{\mathcal{F}}, \Lambda_0)\text{–IR}_0]_k$. So it is enough to prove that for every $k \in \omega$, $T + I\Sigma_1^{\mathcal{F}}$ extends $[T, (\Pi_2^{\mathcal{F}}, \Lambda_0)\text{–IR}_0]_k$. We proceed by induction on $k \in \omega$:

Case $k = 0$ is trivial; so, let us assume that $T + I\Sigma_1^{\mathcal{F}}$ extends $[T, (\Pi_2^{\mathcal{F}}, \Lambda_0)\text{–IR}_0]_k$. Let $t \in \Lambda_0$ and $\varphi(u, v) \in \Pi_2^{\mathcal{F}}$ such that

$$(\dagger) \qquad [T, (\Pi_2^{\mathcal{F}}, \Lambda_0)\text{–IR}]_k \vdash \forall u\, (\varphi(u, v) \to \varphi(u + 1, v)).$$

We must prove that $T + I\Sigma_1^{\mathcal{F}} \vdash \varphi(0, v) \to \varphi(t, v)$.

Without loss of generality, we can assume that $\varphi(u, v) \equiv \forall x\, \exists y\, \varphi_0(u, x, y, v)$, with $\varphi_0(u, x, y, v) \in \Sigma_0^{\mathcal{F}}$. Let g be a new unary function symbol and T^g the extension of $T + I\Sigma_0^{\mathcal{F},g}$ obtained by adding the sentences:

$$\forall x_1, x_2\, (x_1 \le x_2 \to g(x_1) \le g(x_2)), \quad \forall x\, (x^2 < g(x)) \quad \text{and} \quad \forall x\, (f(x) \le g(x)).$$

Thus, g is a dominating (iterable nondecreasing) function over T^g. By ($\dagger$), it follows that $[T^g, (\Pi_2^{\mathcal{F},g}, \Lambda_0)\text{–IR}]_k \vdash \varphi^g$, where φ^g is the following sentence:

$$\forall u\, (\forall x\, \exists y \le g(x + u + v)\, \varphi_0(u, x, y, v) \to \forall x\, \exists y\, \varphi_0(u + 1, x, y, v)).$$

Claim. There is a closed term $\tau_0 \in \Lambda_0$ such that $T^g + \forall x\, \exists y\, (g^{\tau_0}(x) = y)$ proves

$$\forall u\, (\forall x\, \exists y \le g(x + u + v)\, \varphi_0(u, x, y, v) \to \forall x\, \exists y \le g^{\tau_0}(u + x + v)\, \varphi_0(u + 1, x, y, v))$$

Proof of Claim: We distinguish two cases:

Case 1: $k = 0$. Then $T^g \vdash \varphi^g$. Hence, by Parikh's theorem, there exists a term $s(u, x, v)$ of $\mathcal{L}_{\mathcal{F},g}$ such that

$$T^g \vdash \forall u\,(\forall x\,\exists y \leq g(x+u+v)\,\varphi_0(u, x, y, v) \rightarrow \forall x\,\exists y \leq s(u, x, v)\,\varphi_0(u+1, x, y, v))$$

By Lemma 7, there is $m \in \omega$ such that $T^g \vdash s(u, x, v) < g^m(u + x + v + \sigma(s))$. By induction on z it can be proved that

$$T^g \vdash g^u(x + z) = y_1 \wedge g^{u+z}(x) = y_2 \rightarrow y_1 \leq y_2$$

and, thus, if $\tau_0 = m + \sigma(s)$ then $\tau_0 \in \Lambda_0$ and the result follows.

Case 2: $k \geq 1$. Since $[T^g, (\Pi_2^{\mathcal{F},g}, \Lambda_0)\text{–IR}]_k \vdash \varphi^g$ and φ^g is a $\Pi_2^{\mathcal{F},g}$–formula, by Theorem 11, $T^g + (\Pi_2^{\mathcal{F},g}, \Lambda_0)$–IR also proves φ^g. It follows from Proposition 10 that there exist $t_1, \ldots, t_n \in \Lambda_0$ such that

$$T^g + \{\forall x\,\exists y\,(g^{t_j}(x) = y) :\ j = 1, \ldots, n\} \vdash \varphi^g.$$

Let $r = t_1 + \cdots + t_n$. Then, by part (4) of Proposition 5, $T^g + \forall x\,\exists y\,(g^r(x) = y)$ extends $T^g + \{g^{t_j} \text{ is total} :\ j = 1, \ldots, n\}$. Let h be a new unary function symbol and let T^h be the extension of T^g obtained by adding to T^g the axiom $\forall x\,(g^r(x) = h(x))$. Then $T^h \vdash \varphi^g$ and T^h is conservative over T^g.

By Proposition 5, h is an iterable nondecreasing function over T^h and $T^h \vdash \forall x\,(g(x) \leq h(x))$. Therefore, h is a dominating function over T^h and T^h extends $I\Sigma_0^{\mathcal{F},g,h}$. By Parikh's theorem, there is a term $s(u, x, v)$ of $\mathcal{L}_{\mathcal{F},g,h}$ such that

$$T^h \vdash \forall u\,(\forall x\,\exists y \leq g(x+u+v)\,\varphi_0(u, x, y, v) \rightarrow \forall x\,\exists y \leq s(u, x, v)\,\varphi_0(u+1, x, y, v))$$

and, by Lemma 7, there is $m \in \omega$ such that $T^h \vdash s(u, x, v) < h^m(u+x+v+\sigma(s))$. Recall that $T^h \vdash h^u(x + z) = y_1 \wedge h^{u+z}(x) = y_2 \rightarrow y_1 \leq y_2$ and, thus, if $\sigma_0 = m + \sigma(s)$ then $\sigma_0 \in \Lambda_0$ and $T^h + \forall x\,\exists y\,(h^{\sigma_0}(x) = y)$ proves

$$\forall u\,(\forall x\,\exists y \leq g(x+u+v)\,\varphi_0(u, x, y, v) \rightarrow \forall x\,\exists y \leq h^{\tau_0}(u+x+v)\,\varphi_0(u+1, x, y, v))$$

Using part (7) of Proposition 5, we can prove, by $\Sigma_0^{\mathcal{F},g,h}$–induction, that

$$T^h \vdash h^z(x) = y \leftrightarrow g^{r \cdot z}(x) = y$$

As a consequence, $T^h + \forall x\,\exists y\,(h^{\sigma_0}(x) = y)$ proves

$$\forall u\,(\forall x\,\exists y \leq g(x+u+v)\,\varphi_0(u, x, y, v) \rightarrow \forall x\,\exists y \leq g^{r \cdot \sigma_0}(u+x+v)\,\varphi_0(u+1, x, y, v))$$

Hence, putting $\tau_0 = r \cdot \sigma_0 \in \Lambda_0$, the result follows concluding the proof of Claim.

Let $\mathfrak{A} \models T + I\Sigma_1^{\mathcal{F}}$ and $c \in \mathfrak{A}$ such that $\mathfrak{A} \models \varphi(0, c)$. We shall show that $\mathfrak{A} \models \varphi(t, c)$. Let $\psi(x, y, c) \in \Sigma_0^{\mathcal{F}}$ the formula

$$\forall z \leq x \exists w \leq y\,(\varphi_0(0, z, w, c) \wedge y = w + f(x)).$$

Then $\mathfrak{A} \models \forall x \exists y \psi(x, y, c)$ and the formula $\psi(x, y, c) \land \forall z < y \neg\psi(x, z, c)$ defines a total nondecreasing function $H : \mathfrak{A} \to \mathfrak{A}$. There is a $\Sigma_0^{\mathcal{F}}$ formula, that we denote by $H^z(x) = y$, defining the iteration of H and, since $\mathfrak{A} \models I\Sigma_1^{\mathcal{F}}$, we have

$$\mathfrak{A} \models \forall x \, \forall z \, \exists y \, (H^z(x) = y).$$

Let $\theta(u, v)$ be the following $\Pi_1^{\mathcal{F}}$ formula:

$$u > t \lor \forall x \, \forall y_1 \, \left[H^{\tau_0^u}(x + u + v) = y_1 \to \exists y \le y_1 \, \varphi_0(u, x, y, v) \right].$$

Since $\mathfrak{A} \models \forall x \, \exists y \, (H(x) = y)$, by definition of $\theta(u, v)$ we have $\mathfrak{A} \models \theta(0, v)$. Let us show that $\mathfrak{A} \models \forall u \, (\theta(u, v) \to \theta(u + 1, v))$.

Pick $a, b \in \mathfrak{A}$ such that $\mathfrak{A} \models a \le t \land \theta(a, b)$. Then, the formula $H^{\tau_0^a}(x) = y$ defines a total nondecreasing function in $\mathfrak{A}$ and we can use it to get an expansion of $\mathfrak{A}$ to a model $\mathfrak{A}^g$ of T^g such that $\mathfrak{A}^g \models \forall x \, \exists y \le g(x + a + b) \, \varphi_0(a, x, y, b)$. By part (7) of Proposition 5, we can prove by $\Sigma_0^{\mathcal{F}, g}$–induction on z that

$$\mathfrak{A}^g \models \forall z \le \tau_0 \, [g^z(x + a + b) = H^{\tau_0^a \cdot z}(x + a + b)]$$

In particular, $\mathfrak{A}^g \models \forall x \, (g^{\tau_0}(x+a+b) = H^{\tau_0^a \cdot \tau_0}(x+a+b))$ and, as a consequence, $\mathfrak{A}^g \models T^g + \forall x \, \exists y \, (g^{\tau_0}(x) = y)$. Hence, by the Claim, we conclude that $\mathfrak{A}^g \models \forall x \, \exists y \le g^{\tau_0}(x + a + b) \, \varphi_0(a + 1, x, y, b)$ and, therefore, $\mathfrak{A} \models \theta(a + 1, b)$.

We have shown that $\mathfrak{A} \models \theta(0, v) \land \forall u \, (\theta(u, v) \to \theta(u + 1, v))$, and we know that $\mathfrak{A} \models I\Pi_1^{\mathcal{F}}$ (because $I\Sigma_1^{\mathcal{F}} \equiv I\Pi_1^{\mathcal{F}}$), so, $\mathfrak{A} \models \forall u \, \theta(u, b)$. In particular, since

$$\mathfrak{A} \models \theta(t, v) \to \forall x \, \exists y \le H^{\tau_0^t}(t + x + v) \, \varphi_0(t, x, y, v),$$

we conclude $\mathfrak{A} \models \varphi(t, v)$.

3 Main Result

We are now ready to obtain the main results. Firstly, we need a version of Theorem 12 in the language of first–order Arithmetic.

Lemma 13. *$I\Sigma_1$ extends $I\Delta_0 + (\Sigma_2, \mathcal{K}_2)$–IR.*

Proof. Let $\mathfrak{A} \models I\Sigma_1$ and $\varphi(x) \in \Sigma_2$ such that

$$(\bullet) \quad I\Delta_0 + (\Sigma_2, \mathcal{K}_2)\text{–IR} \vdash \varphi(0) \land \forall x \, (\varphi(x) \to \varphi(x + 1)).$$

We must show that for every $\delta(u) \in \Sigma_2^-$,

$$(\star) \qquad \mathfrak{A} \models \forall x_1 \, \forall x_2 (\delta(x_1) \land \delta(x_2) \to x_1 = x_2) \to \forall x \, (\delta(x) \to \varphi(x)).$$

By $(\bullet)$ there exist formulas $\varphi_1(x), \ldots, \varphi_r(x) \in \Sigma_2$ and $\delta_1(x), \ldots, \delta_r(x) \in \Sigma_2^-$ such that $I\Delta_0$ plus the sentences

$$\alpha_j : \quad \forall x_1 \, \forall x_2 (\delta_j(x_1) \land \delta_j(x_2) \to x_1 = x_2) \to \forall x \, (\delta_j(x) \to \varphi_j(x))$$

$(j = 1\ldots, r)$ proves $\varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1))$. More precisely for each $j \leq r$,

$$I\Delta_0 + \bigwedge_{1\leq i<j} \alpha_i \vdash \varphi_j(0) \wedge \forall x\,(\varphi_j(x) \rightarrow \varphi_j(x+1)),$$

and $I\Delta_0 + \bigwedge_{i=1}^{r} \alpha_i \vdash \varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1))$.

Let $E = \{j : 1 \leq j \leq r,\ \mathfrak{A} \models \neg\exists x \delta_j(x)\}$ and, for each $j \in E$, let $\theta_j(x,y) \in \Pi_0$ such that $\neg\exists x\,\delta_j(x)$ is equivalent to $\forall x \exists y\,\theta_j(x,y)$. Let m the cardinal of E and let $\mathcal{F} = \{f_1, \ldots, f_m, f\}$ a set of new unary function symbols. From the set of Σ_0 formulas $\Gamma = \{\theta_j(x,y) : \ j \in E\}$, we define a theory T as in Remark 2. Let $\mathcal{L}(\mathfrak{A})$ denote the language obtained by adding to $\mathcal{L}$ a constant symbol $\underline{a}$, for each $a \in \mathfrak{A}$. Put $T' = T + D_{\Pi_1}(\mathfrak{A})$, where $D_{\Pi_1}(\mathfrak{A})$ is the Π_1–diagram of $\mathfrak{A}$. Let Λ_0 be the set of closed terms of $\mathcal{L}(\mathfrak{A})$ containing only constants of the form $\underline{a}$ for $a \in \mathcal{K}_2(\mathfrak{A})$. Then $\mathfrak{A}$ has a natural expansion $\mathfrak{A}_{\mathcal{F}}$ to the language $\mathcal{L}_{\mathcal{F}} \cup \mathcal{L}(\mathfrak{A})$ such that $\mathfrak{A}_{\mathcal{F}} \models T' + I\Sigma_1^{\mathcal{F}}$. By Proposition 12, $\mathfrak{A}_{\mathcal{F}} \models T' + (\Sigma_2^{\mathcal{F}}, \Lambda_0)$–IR. Given $\delta(x) \in \Sigma_2^-$, we can distinguish several cases:

If $\mathfrak{A} \models \neg\exists x\,\delta(x)$ then $(\star)$ obviously holds. On the other hand, if $\mathfrak{A} \models \neg \forall x_1\,\forall x_2(\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2)$, since this is a Σ_2–sentence and T' extends $D_{\Pi_1}(\mathfrak{A})$, we have that $T' \vdash \neg\forall x_1\,\forall x_2(\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2)$. So,

$$T' \vdash \forall x_1\,\forall x_2(\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2) \rightarrow \forall x\,(\delta(x) \rightarrow \varphi(x)).$$

In that way $(\star)$ holds again. We must deal with a last case: $\mathfrak{A} \models \exists! x\,\delta(x)$.

Then there exists $d \in \mathcal{K}_2(\mathfrak{A})$ such that $\mathfrak{A} \models \delta(d)$ and $\underline{d} \in \Lambda_0$. In order to verify $(\star)$ it is enough to show that $T' + (\Sigma_2^{\mathcal{F}}, \Lambda_0)$–IR $\vdash \varphi(\underline{d})$.

We prove, by induction on j, that for all $j = 1, \ldots, r$, $T' + (\Sigma_2^{\mathcal{F}}, \Lambda_0)$–IR $\vdash \alpha_j$. Let $j \leq r$, and assume that $T' + (\Sigma_2^{\mathcal{F}}, \Lambda_0)$–IR $\vdash \bigwedge_{1\leq i<j} \alpha_i$. Then

$$(\bullet)_j \quad T' + (\Sigma_2^{\mathcal{F}}, \Lambda_0)\text{–IR} \vdash \varphi_j(0) \wedge \forall x\,(\varphi_j(x) \rightarrow \varphi_j(x+1)).$$

If $j \in E$ or $\mathfrak{A} \models \neg\forall x_1\,\forall x_2(\delta_j(x_1) \wedge \delta_j(x_2) \rightarrow x_1 = x_2)$ then, reasoning as in previous cases, we conclude that $T' \vdash \alpha_j$. If $\mathfrak{A} \models \exists! x\,\delta_j(x)$, then there exists $b \in \mathcal{K}_2(\mathfrak{A})$ such that $\mathfrak{A} \models \delta_j(b)$ and $\underline{b} \in \Lambda_0$. Using $(\bullet)_j$ we get $T' + (\Sigma_2^{\mathcal{F}}, \Lambda_0)$–IR $\vdash \varphi_j(\underline{b})$. As a consequence, $T' + (\Sigma_2^{\mathcal{F}}, \Lambda_0)$–IR $\vdash \exists x\,(\delta(x) \wedge \varphi_j(x))$, and it follows that $T' + (\Sigma_2^{\mathcal{F}}, \Lambda_0)$–IR $\vdash \alpha_j$, as required.

We have proved that $T' + (\Sigma_2^{\mathcal{F}}, \Lambda_0)$–IR $\vdash \bigwedge_{j=1}^{r} \alpha_j$; hence

$$T' + (\Sigma_2^{\mathcal{F}}, \Lambda_0)\text{–IR} \vdash \varphi(0) \wedge \forall x\,(\varphi(x) \rightarrow \varphi(x+1))$$

It follows that $T' + (\Sigma_2^{\mathcal{F}}, \Lambda_0)$–IR $\vdash \varphi(\underline{d})$ and, as a consequence, $(\star)$ holds.

Our last theorem extends a previous conservation result obtained in [4] and, as a direct corollary, yields the characterization of the p.t.c.f. of $I\Pi_2^-$.

Theorem 14. *$I\Pi_2^-$ is Π_3–conservative over $I\Sigma_1$.*

Proof. Let θ be a Π_3 sentence provable in $I\Pi_2^-$. Then $I(\Sigma_2, \mathcal{K}_2) \vdash \theta$ by Lemma 2 and $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)$–IR $\vdash \theta$ by Proposition 3. We need the following fact:

Claim. $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)$–IR $\equiv I\Sigma_1^- + (I\Delta_0 + (\Sigma_2, \mathcal{K}_2)$–IR)

Proof of Claim: Each axiom of $I\Sigma_1^-$ is a Σ_3 sentence, so it is enough to prove that for every $\sigma_0(u) \in \Pi_2$,

$$[I\Delta_0, (\Sigma_2, \mathcal{K}_2)\text{–IR}] + \exists u\, \sigma_0(u) \text{ extends } [I\Delta_0 + \exists u\, \sigma_0(u), (\Sigma_2, \mathcal{K}_2)\text{–IR}].$$

Assume $I\Delta_0 + \exists u\, \sigma_0(u) \vdash \varphi(0) \wedge \forall x\, (\varphi(x) \to \varphi(x+1))$, with $\varphi(x) \in \Sigma_2$, and let $\psi(x,u) \in \Sigma_2$ be $\sigma_0(u) \to \varphi(x)$. Then, $I\Delta_0 \vdash \psi(0,u) \wedge \forall x\, (\psi(x,u) \to \psi(x+1,u))$, and, therefore, $[I\Delta_0, (\Sigma_2, \mathcal{K}_2)\text{–IR}] \vdash U_\delta \to \forall x\, (\delta(x) \to \psi(x,u))$, where $\delta(x)$ is in Σ_2^- and U_δ denotes the sentence $\forall x_1 \forall x_2 (\delta(x_1) \wedge \delta(x_2) \to x_1 = x_2)$. Then it holds that $[I\Delta_0, (\Sigma_2, \mathcal{K}_2)\text{–IR}]$ also proves

$$\exists u\, \sigma_0(u) \to (U_\delta \to \forall x\, (\delta(x) \to \varphi(x)))$$

and so $[I\Delta_0, (\Sigma_2, \mathcal{K}_2)\text{–IR}] + \exists u\, \sigma_0(u) \vdash U_\delta \to \forall x\, (\delta(x) \to \varphi(x))$, as required.

It follows from this Claim and Lemma 13 that $I\Sigma_1$ extends $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)$–IR and, therefore, $I\Sigma_1 \vdash \theta$.

Corollary 15. *The class of provably total computable functions of $I\Pi_2^-$ is the class of primitive recursive functions.*

Acknowledgements. This work was partially supported by grant MTM2008-06435 of Ministerio de Ciencia e Innovación, Spain.

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