Local Induction and Provably Total Computable Functions

Andrés Cordón–Franco, F. Félix Lara–Martín

Depto. Ciencias de la Computación e Inteligencia Artificial, University of Seville
C/ Tarfia, s/n, 41012 Sevilla (Spain)

Abstract

Let $I\Pi_2^-$ denote the fragment of Peano Arithmetic obtained by restricting the induction scheme to parameter free Π_2 formulas. Answering a question of R. Kaye, L. Beklemishev showed that the provably total computable functions of $I\Pi_2^-$ are, precisely, the primitive recursive ones. In this work we give a new proof of this fact through an analysis of certain *local variants* of induction principles closely related to $I\Pi_2^-$. In this way, we obtain a more direct answer to Kaye's question, avoiding the metamathematical machinery (reflection principles, provability logic,...) needed for Beklemishev's original proof.

Our methods are model–theoretic and allow for a general study of $I\Pi_{n+1}^-$ for all $n \geq 0$. In particular, we derive a new conservation result for these theories, namely that $I\Pi_{n+1}^-$ is Π_{n+2}–conservative over $I\Sigma_n$ for each $n \geq 1$.

Keywords: First order Arithmetic, conservation results, parameter free induction, primitive recursive functions.
2000 MSC: 03F30, 03D20

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1. Introduction

An important notion in studying the computational content of a fragment of Arithmetic is that of its *provably total computable functions.* A number–theoretic computable function $f : \mathbb{N}^k \to \mathbb{N}$ is said to be a provably total computable function (p.t.c.f.) of a theory T, written $f \in \mathcal{R}(T)$, if there is a Σ_1 formula $\varphi(\vec{x}, y)$ such that:

Email addresses: `acordon@us.es` (Andrés Cordón–Franco), `fflara@us.es` (F. Félix Lara–Martín)

Preprint submitted to Annals of Pure and Applied Logic *February 17, 2014*

1. φ defines the graph of f in the standard model of Arithmetic $\mathbb{N}$; and
2. $T \vdash \forall \vec{x}\, \exists! y\, \varphi(\vec{x}, y)$.

Since it was introduced by G. Kreisel in the 1950s this notion has been widely studied, and nice recursion–theoretic and computational complexity characterizations of the sets $\mathcal{R}(T)$ have been obtained for a good number of theories T. For instance, by a classical result due independently to G. Mints, C. Parsons and G. Takeuti, the class of p.t.c.f. of the scheme of induction for Σ_1–formulas $I\Sigma_1$ equals to the class of the primitive recursive functions PR. Indeed, all classes $\mathcal{R}(I\Sigma_n)$, $n \geq 1$, can be characterized in terms of the Fast Growing Hierarchy up to the ordinal ε_0. As for weak fragments below $I\Sigma_1$, their p.t.c.f. have been characterized in terms of subrecursive operators (bounded recursion, bounded minimization, ...) as well as in terms of computational complexity classes. In fact, their classes of p.t.c.f. have been intensively investigated in connection with important open problems in Complexity Theory, mainly in the context of Bounded Arithmetic.

In spite of the wide range of the theories considered, a number of uniform methods for characterizing the p.t.c.f. of an arithmetic theory are available. E.g. Herbrand analyses as developed by W. Sieg in [13], S. Buss' witnessing method [5] or, in general, proof–theoretic techniques using Cut elimination theorem. However, for some particular fragments of Peano Arithmetic none of these standard methods seems to be applicable. Of special interest is the case of the scheme of parameter free Π_2–induction, $I\Pi_2^-$, given by the induction scheme

$$I_\varphi : \quad \varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1)) \;\rightarrow\; \forall x\, \varphi(x)\,,$$

restricted to $\varphi(x) \in \Pi_2^-$ (as usual, we write $\varphi(x) \in \Gamma^-$ to mean that φ is in Γ and contains no other free variables than x). Since $I\Sigma_1^- \subseteq I\Pi_2^-$ and $I\Sigma_1$ is Σ_3–conservative over $I\Sigma_1^-$ [10], it follows that every primitive recursive function is provably total in $I\Pi_2^-$; and R. Kaye asked whether the p.t.c.f. of $I\Pi_2^-$ are exactly the primitive recursive ones. This question remained elusive until [4], where L. Beklemishev gave a positive answer using modal provability logic techniques. Although quite elegant, Beklemishev's answer only provides an indirect solution. Firstly, he reformulated $I\Pi_2^-$ in terms of local reflection principles (reflection principles in Arithmetic are axiom schemes expressing the statement that "if a formula φ is provable in a theory T then φ is valid"). Secondly, he derived the result as an application of a conservation theorem for local reflection principles whose proof leans upon properties of Gödel–Löb provability logic **GL**.

In this work we obtain a more direct answer to Kaye's question, avoiding the metamathematical machinery needed for Beklemishev's proof. In fact, our proof that $\mathcal{R}(I\Pi_2^-) = PR$ will follow the lines of standard arguments for characterizing classes $\mathcal{R}(T)$. Let us consider, for instance, a proof that $\mathcal{R}(I\Sigma_1) = PR$. Such a proof typically proceeds in two steps.

- Step 1: $I\Sigma_1$ is Π_2–conservative over the *inference rule* version of the principle of Σ_1–induction Σ_1–IR. So, $\mathcal{R}(I\Sigma_1) = \mathcal{R}(\Sigma_1\text{–IR})$.
- Step 2: Applications of Σ_1–IR correspond to applications of the primitive recursion operator.

The main obstacle to apply this argument to $I\Pi_2^-$ is that there is no simple, direct argument to reduce $I\Pi_2^-$ to an inference rule version of it. Here we solve this problem by showing that $I\Pi_2^-$ is equivalent to $I(\Sigma_2^-, \mathcal{K}_2)$, a certain *local* version of the parameter free Σ_2–induction scheme where the elements x for which the induction axiom claims $\varphi(x)$ to hold are restricted to be Σ_2–definable elements. Equipped with this result, it is easy to obtain that $I\Pi_2^-$ is Π_2 (in fact, Π_3) conservative over the corresponding *local* inference rule version $(\Sigma_2, \mathcal{K}_2)$–IR. Then, we show that applications of $(\Sigma_2, \mathcal{K}_2)$–IR correspond to (restricted forms) of the iteration operator and thus all functions in $\mathcal{R}(I\Pi_2^-)$ are primitive recursive.

Local induction schemes and local induction rules play a crucial role in our methods. Interestingly, these local subsystems can be applied in considerable generality to study fragments of arithmetic. Actually, in this work we also make use of these ideas to develop a general study of the theories $I\Pi_{n+1}^-$ for all $n \geq 1$. As a result, we are able to give new proofs of some well–known results on these fragments as well as to obtain a novel conservation result. Namely, we prove that $I\Pi_{n+1}^-$ is Π_{n+2}–conservative over $I\Sigma_n$ for all $n \geq 1$. This improves on a previous result by Beklemishev in [4] where conservativity between these theories with respect to boolean combinations of Σ_{n+1}–sentences was established, and closes a notable gap in our understanding of relationships between the standard fragments of arithmetic.

2. On Local Induction

In this section we give a precise definition of the auxiliary schemes that will be central in our analysis of the class of p.t.c.f. of $I\Pi_2^-$. We work in the language of first–order arithmetic $\mathcal{L} = \{0, S, +, \cdot, <\}$ and define the formula classes Δ_0, Σ_n and Π_n as usual. For a class Γ of formulas, $I\Gamma$ is the theory axiomatized over Robinson's Q by the induction scheme, I_φ, restricted to formulas $\varphi(x) \in \Gamma$. If free variables other that x are not allowed, we write $\varphi(x) \in \Gamma^-$ and, accordingly, $I\Gamma^-$ denotes the theory axiomatized over Q by the axioms I_φ, for $\varphi(x) \in \Gamma^-$.

The schemes we are interested in are *local* variants of the usual induction scheme in a sense that the conclusion of the induction principle is no longer assumed for every element in the universe but only for a certain subclass of the universe. More precisely, we define:

Definition 1. *For every $n \geq 1$, $I(\Sigma_n, \mathcal{K}_n)$ is the theory given by $I\Delta_0$ together with the scheme*

$$\begin{aligned} &\varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \\ &\qquad \rightarrow \forall x_1, x_2\, (\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2) \rightarrow \forall x\, (\delta(x) \rightarrow \varphi(x)) \end{aligned}$$

where $\varphi(x) \in \Sigma_n$ and $\delta(x) \in \Sigma_n^-$. The natural inference rule associated to this scheme, denoted $(\Sigma_n, \mathcal{K}_n)$–IR, is given by:

$$\frac{\varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1))}{\forall x_1, x_2\, (\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2) \rightarrow \forall x\, (\delta(x) \rightarrow \varphi(x))}$$

where $\delta(x) \in \Sigma_n^-$ and $\varphi(x) \in \Sigma_n$. Finally, if we restrict the scheme to $\varphi(x) \in \Sigma_n^-$, we obtain the parameter free counterpart of $I(\Sigma_n, \mathcal{K}_n)$, denoted $I(\Sigma_n^-, \mathcal{K}_n)$.

Remark 1. *Firstly, let us recall that, given a model $\mathfrak{A}$, $\mathcal{K}_n(\mathfrak{A})$ denotes the set of elements of $\mathfrak{A}$ that are definable in $\mathfrak{A}$ by a formula $\delta(x) \in \Sigma_n$. This explains why $\mathcal{K}_n$ appears in our notation for these theories. Secondly, if $\mathfrak{A} \models I\Sigma_{n-1}^-$, then $\mathcal{K}_n(\mathfrak{A}) \prec_n \mathfrak{A}$ (i.e. $\mathcal{K}_n(\mathfrak{A})$ is a Π_n–elementary substructure of $\mathfrak{A}$). This property plays an important role in what follows and it is because of it that some of our results on $I(\Sigma_n, \mathcal{K}_n)$ are obtained over $I\Sigma_{n-1}^-$ instead of over $I\Delta_0$.*

A key fact is that $I(\Sigma_n^-, \mathcal{K}_n)$ provides an alternative formulation of $I\Pi_n^-$ for every $n \geq 1$:

Lemma 1. *Over $I\Sigma_{n-1}^-$, $I\Pi_n^- \equiv I(\Sigma_n^-, \mathcal{K}_n)$.*

Proof. ($\vdash$): Suppose $\mathfrak{A} \models I\Pi_n^-$ and $\mathfrak{A} \models \varphi(0) \wedge \forall x\, (\varphi(x) \to \varphi(x+1))$, with $\varphi(x) \in \Sigma_n^-$. Let $\delta(v) \in \Sigma_n$ defining some element in $\mathfrak{A}$, say a. Towards a contradiction, assume $\mathfrak{A} \not\models \forall x\, (\delta(x) \to \varphi(x))$. Then, $\mathfrak{A} \models \neg\varphi(a)$. Define $\theta(x)$ to be $\forall v\, (\delta(v) \to \neg\varphi(x - v))$. Clearly, $\mathfrak{A} \models \theta(0) \wedge \forall x\, (\theta(x) \to \theta(x+1))$. By $I\Pi_n^-$, $\mathfrak{A} \models \theta(a)$ and so $\mathfrak{A} \models \neg\varphi(0)$, which is a contradiction.

($\dashv$): Suppose $\mathfrak{A} \models I(\Sigma_n^-, \mathcal{K}_n)$ and $\mathfrak{A} \models \varphi(0) \wedge \forall x\, (\varphi(x) \to \varphi(x+1))$, with $\varphi(x) \in \Pi_n^-$. Assume $\mathfrak{A} \models \exists x\, \neg\varphi(x)$. Since $\mathfrak{A} \models I\Sigma_{n-1}^-$, $\mathcal{K}_n(\mathfrak{A}) \prec_n \mathfrak{A}$ and there is $a \in \mathcal{K}_n(\mathfrak{A})$ such that $\mathfrak{A} \models \neg\varphi(a)$. Let $\delta(v)$ be a Σ_n formula defining the element a and let $\theta(x)$ be $\exists v\, (\delta(v) \wedge \neg\varphi(v - x))$. Clearly, $\mathfrak{A} \models \theta(0) \wedge \forall x\, (\theta(x) \to \theta(x+1))$. By $I(\Sigma_n^-, \mathcal{K}_n)$, $\mathfrak{A} \models \forall x\, (\delta(x) \to \theta(x))$ and so $\mathfrak{A} \models \theta(a)$. Thus $\mathfrak{A} \models \neg\varphi(0)$, which is a contradiction. $\square$

Given a theory T and an inference rule R, we denote by $[T, R]$ the closure of T under first order logic and unnested applications of R. We denote by $T + R$ the closure of T under first order logic and (nested) applications of R. Therefore, $T + R = \bigcup_{k \in \omega} [T, R]_k$, where $[T, R]_0 = T$ and $[T, R]_{k+1} = [[T, R]_k, R]$.

The first step in the analysis of $I\Pi_2^-$ is a suitable reduction of $I(\Sigma_2, \mathcal{K}_2)$ to a fragment defined by the rule $(\Sigma_2, \mathcal{K}_2)$–IR. Indeed, the following general result holds for each $n \geq 1$.

Proposition 1. *Let T be a Π_{n+2}–axiomatizable theory. Then, $T + I(\Sigma_n, \mathcal{K}_n)$ is Π_{n+1}–conservative over $T + (\Sigma_n, \mathcal{K}_n)$–IR.*

Very conveniently, this reduction can be carried out by the same tools used to derive the reduction of $I\Sigma_1$ to Σ_1–IR (e.g. by adapting the cut–elimination argument used in [3] to derive a similar reduction for the Collection scheme). Alternatively, here we give a model–theoretic proof following the methods developed by J. Avigad in [1], who in turn builds on previous ideas of A. Visser (unpublished) and D. Zambella [14]. In [1] Avigad introduced the notion of a *Herbrand saturated* model and showed that this notion provides us with an unified method to prove $\forall\exists$–conservation over universal theories. Here we consider a hierarchical version of that notion that yields an unified method to prove Π_{n+1}–conservation over Π_{n+2}–theories.

Definition 2. *We say that a model of a theory T, $\mathfrak{A}$, is a Σ_{n+1}–closed model of T if for every model of T, $\mathfrak{B}$,*

$$\mathfrak{A} \prec_n \mathfrak{B} \Longrightarrow \mathfrak{A} \prec_{n+1} \mathfrak{B}.$$

In words, $\mathfrak{A}$ is a Σ_{n+1}–closed model of T if every Π_n–formula that can be satisfied in a Π_n–elementary extension of $\mathfrak{A}$ which is a model of T can be already satisfied by an element of $\mathfrak{A}$. It is easy to show that Σ_{n+1}–closed models exist for every n. In fact, by a rather standard union of chain argument it follows that if T is a Π_{n+2}–axiomatizable theory, then every model of T can be Π_n–elementary extended to a Σ_{n+1}–closed model of T. As a consequence, the following version of theorem 3.4 of [1] holds.

Lemma 2. *Suppose* T_2 *is* Π_{n+2}*–axiomatizable. In order to prove that* T_1 *is* Π_{n+1}*–conservative over* T_2 *it is sufficient to show that every* Σ_{n+1}*–closed model of* T_2 *satisfies* T_1*.*

Next lemma is an analog of theorem 3.3 of [1] and states the key property of Σ_{n+1}–closed models for proving conservation results.

Lemma 3. *Suppose* $\mathfrak{A}$ *is a* Σ_{n+1}*–closed model of* T*,* $\varphi(v) \in \Pi_{n+1}$ *and* $a \in \mathfrak{A}$*. Then*

$$\mathfrak{A} \models \varphi(a) \quad \Longrightarrow \quad T \vdash \psi(v,w) \to \varphi(v),$$

for some $\psi(v,w) \in \Pi_n$ *such that* $\mathfrak{A} \models \psi(a,b)$ *for some* b *in* $\mathfrak{A}$*.*

Proof. It follows from the Σ_{n+1}–closedness condition that $T + D_{\Pi_n}(\mathfrak{A}) \vdash \varphi(a)$, where $D_{\Pi_n}(\mathfrak{A})$ denotes the Π_n–diagram of $\mathfrak{A}$, i.e. the set of all Π_n–formulas (possibly with parameters) valid in $\mathfrak{A}$. Now the result follows by compactness. □

We are now in a position to give a proof of Proposition 1.

Proof. Suppose that $\mathfrak{A}$ is a Σ_{n+1}–closed model of $T + (\Sigma_n, \mathcal{K}_n)$–IR and $\mathfrak{A} \models \varphi(0,b) \wedge \forall x\,(\varphi(x,b) \to \varphi(x+1,b))$, with $\varphi(x,v) \in \Sigma_n$. Consider $a \in \mathcal{K}_n(\mathfrak{A})$ and $\delta(x) \in \Sigma_n$ defining a. We must show that $\mathfrak{A} \models \varphi(a,b)$. It follows from Lemma 3 that

$$(T + (\Sigma_n, \mathcal{K}_n)\text{–IR}) \vdash \psi(v,w) \to \varphi(0,v) \wedge \forall x\,(\varphi(x,v) \to \varphi(x+1,v)),$$

with $\psi(v,w) \in \Pi_n$ and $\mathfrak{A} \models \psi(b,c)$ for some $c \in \mathfrak{A}$. Put $\theta(x,v,w) \equiv \psi(v,w) \to \varphi(x,v)$. Clearly, $\theta \in \Sigma_n$ and $(T + (\Sigma_n, \mathcal{K}_n)$–IR) proves the antecedent of the induction axiom for θ and so $\mathfrak{A} \models \forall v,w,x\,(\delta(x) \to \theta(x,v,w))$. Thus $\theta(a,b,c)$ is valid in $\mathfrak{A}$ and hence so is $\varphi(a,b)$. □

Combining Lemma 1 and Proposition 1, we get

Corollary 1. $I\Pi_2^-$ *is* Π_3*–conservative over* $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)$*–IR.*

3. Local Induction and Restricted Iteration

Next step in our analysis is to show that applications of $(\Sigma_2, \mathcal{K}_2)$–IR correspond to (a restricted form of) the iteration operator. To this end, we shall consider extensions of $\mathcal{L}$ obtained by adding a finite set of unary function symbols, $\mathcal{F} = \{f_1, \dots, f_n\}$, and a (finite or countable) set of new constant symbols, C. Through this section we consider a fixed set of constants, C, and we will denote by $\mathcal{L}_{\mathcal{F}}$ the language $\mathcal{L} + \{f_1, \dots, f_n\} + C$. If g is a new unary function symbol then $\mathcal{L}_{\mathcal{F},g}$ will denote the language $\mathcal{L}_{\{f_1,\dots,f_n,g\}}$.

Definition 3. *Let $f \in \mathcal{F}$ be a unary function symbol and let T be an $\mathcal{L}_{\mathcal{F}}$–theory. We say that f is an* iterable non decreasing function *over T if the theory T proves:*

$$\forall x_1, x_2\, (x_1 \le x_2 \to f(x_1) \le f(x_2)), \quad \textit{and} \quad \forall x\, (x^2 < f(x))$$

Let $\Sigma_0^{\mathcal{F}} = \Pi_0^{\mathcal{F}}$ be the class of bounded formulas of $\mathcal{L}_{\mathcal{F}}$. Classes $\Sigma_{n+1}^{\mathcal{F}}$ and $\Pi_{n+1}^{\mathcal{F}}$ are defined as usual. The theory $I\Sigma_0^{\mathcal{F}}$ is the $\mathcal{L}_{\mathcal{F}}$–theory axiomatized over $I\Delta_0$ by

- The induction axiom I_φ for each formula $\varphi \in \Sigma_0^{\mathcal{F}}$, and
- Axioms for each $f \in \mathcal{F}$:

$$\forall x_1, x_2\, (x_1 \le x_2 \to f(x_1) \le f(x_2)), \text{ and } \forall x\, (x^2 < f(x))$$

This is a basic theory to deal with the *iteration* of f and to guarantee the usual properties of the iteration of a nondecreasing function with a $\Pi_0^{\mathcal{F}}$–definable graph. The basic facts provable in this theory were stated in [6]. Next result collects together the facts that we shall need in the present context.

Proposition 2. *For each $f \in \mathcal{F}$ there exists a formula $IT_f(z, x, y) \in \Sigma_0^{\mathcal{F}}$ such that the following formulas are theorems of $I\Sigma_0^{\mathcal{F}}$:*

1. $IT_f(z, x, y_1) \wedge IT_f(z, x, y_2) \to y_1 = y_2$.
2. $(IT_f(0, x, y) \leftrightarrow x = y) \wedge (IT_f(1, x, y) \leftrightarrow f(x) = y)$.
3. $IT_f(z + 1, x, y) \leftrightarrow \exists y_0 \le y\, (IT_f(z, x, y_0) \wedge f(y_0) = y)$.
4. $IT_f(z, x, y) \to \forall z_0 < z\, \exists y_0 < y\, IT_f(z_0, x, y_0)$.
5. $z \ge 1 \wedge IT_f(z, x, y) \to x^2 < y \wedge z \le y$.
6. $z \ge 1 \wedge x_1 \le x_2 \wedge IT_f(z, x_1, y_1) \wedge IT_f(z, x_2, y_2) \to y_1 \le y_2$.

7. $IT_f(z_1, x, y_0) \wedge IT_f(z_2, y_0, y) \rightarrow IT_f(z_1 + z_2, x, y)$.

In what follows we use a more suggestive notation and write $f^z(x) = y$ instead of $IT_f(z, x, y)$.

Definition 4. *We say that $f \in \mathcal{F}$ is a* dominating function *over T if, for each term $t(x)$ of $\mathcal{L}_{\mathcal{F}}$, there exists $k \in \omega$ such that T proves*

$$\forall x\,(t(x) \leq f^k(x + \sigma(t)))$$

where $\sigma(t) = c_1 + \cdots + c_m$ and $c_1, \ldots, c_m$ are all the constants occurring in $t(x)$.

Lemma 4. *Let T be an extension of $I\Sigma_0^{\mathcal{F}}$ and let $f \in \mathcal{F}$ be a (iterable non-decreasing) dominating function over T. Then, for each term $t(x_1, \ldots, x_m)$ of $\mathcal{L}_{\mathcal{F}}$ whose variables are among $x_1, \ldots, x_m$, there exists $k \in \omega$ such that*

$$T \vdash t(x_1, \ldots, x_m) < f^k(x_1 + \cdots + x_m + \sigma(t)).$$

Proof. We proceed by induction on terms of $\mathcal{L}_{\mathcal{F}}$. The most interesting case occurs when $t(x_1, \ldots, x_m)$ is a sum (or a product) of two terms, say $t_1(x_1, \ldots, x_m) + t_1(x_1, \ldots, x_m)$. By induction hypothesis,

$$t_1(\vec{x}) < f^k(x_1 + \cdots + x_m + \sigma(t_1)) \text{ and } t_2(\vec{x}) < f^l(x_1 + \cdots + x_m + \sigma(t_2)),$$

for some $k, l \in \omega$. Without loss of generality we may assume $k \geq \max(l, 2)$ (so, for every u, $f^k(u) \geq k \geq 2$.) Then,

$$\begin{array}{rcl} t(\vec{x}) & = & t_1(\vec{x}) + t_2(\vec{x}) \\ & < & f^k(x_1 + \cdots + x_m + \sigma(t_1)) + f^l(x_1 + \cdots + x_m + \sigma(t_2)) \\ & \leq & 2f^k(x_1 + \cdots + x_m + \sigma(t)) \\ & \leq & (f^k(x_1 + \cdots + x_m + \sigma(t)))^2 \\ & < & f^{k+1}(x_1 + \cdots + x_m + \sigma(t)). \end{array}$$

The remaining cases are similar. □

Languages $\mathcal{L}_{\mathcal{F}}$ and the notion of a dominating function are tailored to deal with the situation described in the following lemma.

Lemma 5. *Let $\Gamma = \{\theta_1(x, y), \ldots, \theta_m(x, y)\}$ be a finite set of Δ_0–formulas with only two free variables. For each $j = 1, \ldots, m$, let $\bar{\theta}_j(x, y)$ denote the formula $\forall u \leq x\, \exists v \leq y\, \theta_j(u, v)$. Let $\mathcal{F} = \{f_1, \ldots, f_m, f\}$ be a set of unary function symbols and let T be the $\mathcal{L}_{\mathcal{F}}$–theory extending $I\Delta_0$ with the following additional axioms:*

- *For each* $j = 1, \dots, m$*,*

$$\forall x\, (f_j(x) = y \leftrightarrow \exists y_0 \leq y\, (y_0 = \mu t.\, \bar{\theta}_j(x,t) \wedge y = (x+1)^2 + y_0)).$$

- $\forall x\, (f(x) = (x+1)^2 + f_1(x) + \cdots + f_m(x))$*.*

Then, T *extends* $I\Sigma_0^{\mathcal{F}}$ *and* f *is a dominating function over* T*.*

Proof. It is straighforward to check that each $h \in \mathcal{F}$ is an iterable nondecreasing function over T. In addition, by proposition V.1.3 of [8], T proves $\Sigma_0^{\mathcal{F}}$–induction. Thus we only must show that f is a dominating function over T. This fact can be proved by induction on terms of $\mathcal{L}_{\mathcal{F}}$. Again, the most interesting case occurs when $t(x)$ is a product (or sum) of two terms, say $t_1(x) \cdot t_2(x)$. By induction hypothesis, $t_1(x) \leq f^k(x + \sigma(t_1))$ and $t_2(x) \leq f^l(x+\sigma(t_2))$, for some $k \geq \max(l, 2)$ (so, for every u, $f^k(u) \geq k \geq 2$.) Then,

$$\begin{aligned} t(x) &\leq (t_1(x) + t_2(x))^2 \leq f(t_1(x) + t_2(x)) \\ &\leq f(f^k(x + \sigma(t_1)) + f^l(x + \sigma(t_2))) \\ &\leq f(2 \cdot f^k(x + \sigma(t))) \leq f((f^k(x + \sigma(t)))^2) \leq f^{k+2}(x + \sigma(t)). \end{aligned}$$

The remaining cases are similar. □

As a final step in the analysis of $(\Sigma_2, \mathcal{K}_2)$–IR and due to technical reasons, it will be convenient to denote the Σ_2–definable elements by closed terms of an extended language. This motivates the introduction of the following *local induction rules.*

Definition 5. *For each set of formulas* Γ *and each set of closed terms* Λ *of* $\mathcal{L}_{\mathcal{F}}$ *we consider the rules (where* $\varphi(x) \in \Gamma$ *and* $t \in \Lambda$*):*

$$(\Gamma, \Lambda)\text{–}IR : \quad \frac{\varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1))}{\varphi(t)}$$

$$(\Gamma, \Lambda)\text{–}IR_0 : \quad \frac{\forall x\, (\varphi(x) \rightarrow \varphi(x+1))}{\varphi(0) \rightarrow \varphi(t)}$$

These rules were first considered and intensively studied in [6]. There we proved that a number of results on classical induction rules are also true for the local ones. In what follows, we state two of these results that will be needed in the present paper. For the rest of the section, we assume that

T is an extension of $I\Sigma_0^{\mathcal{F}}$ obtained by adding a set of $\Pi_1^{\mathcal{F}}$ sentences, that Λ denotes the set of all closed terms of a sublanguage of $\mathcal{L}_{\mathcal{F}}$ extending $\mathcal{L}$ (and so Λ is closed under sum and product), and that there is $f \in \mathcal{F}$ which is a dominating function over T.

Remark 2. *Let us note that under these assumptions T satisfies a natural version of Parikh's theorem (see [8], chapter 5, theorem 1.4). This fact will be used extensively without further comments.*

Firstly, next lemma can be seen as a local version of the well–known fact that $[I\Delta_0, \Sigma_1\text{–IR}] \equiv I\Delta_0 + exp$, where exp denotes a Π_2–axiom declaring that the exponential function is total.

Lemma 6. *The following theories are equivalent:*

1. $T + \{\forall x\, \exists y\, (f^t(x) = y) : \ t \in \Lambda\}$.
2. $[T, (\Sigma_1^{\mathcal{F}}, \Lambda)\text{–}IR]$
3. $T + (\Sigma_1^{\mathcal{F}}, \Lambda)\text{–}IR$.

Proof. The proof is a standard argument using Parikh's theorem. See lemma 4.8 of [6]. □

Observe that $(\Sigma_1^{\mathcal{F}}, \Lambda)$–IR collapses to unnested applications of the rule in contrast to the classical case, where the hierarchy $[I\Delta_0, \Sigma_1\text{–IR}]_k$, $k \in \omega$, is well–known to be proper.

Secondly, it is a theorem of Beklemishev (see [2], corollary 9.1) that $[T, \Sigma_1\text{–IR}] \equiv [T, \Pi_2\text{–IR}]$ for every $\Sigma_2 \cup \Pi_2$–extension of $I\Delta_0 + exp$. In lemma 4.10 of [6] we used a model–theoretic construction to prove a similar result for local induction rules under an additional assumption on the set of closed terms Λ.

Definition 6. *We say that Λ is exponentially closed over T if for every $t, s \in \Lambda$ there exists $t' \in \Lambda$ such that $[T, (\Sigma_1^{\mathcal{F}}, \Lambda)\text{–}IR] \vdash \exists y \leq t'\, (s^t = y)$.*

From now on, we also assume that Λ is exponentially closed over T. Then, we have

Lemma 7. *The following theories are equivalent:*

1. $[T, (\Sigma_1^{\mathcal{F}}, \Lambda)\text{–}IR]$
2. $[T, (\Pi_2^{\mathcal{F}}, \Lambda)\text{–}IR]$

3. $T + (\Pi_2^{\mathcal{F}}, \Lambda)$–*IR.*

Proof. See lemma 4.10 of [6]. $\square$

Again, note that $(\Pi_2^{\mathcal{F}}, \Lambda)$–IR collapses to unnested applications of the rule in contrast to the classical case. Finally, putting together Lemma 6 and Lemma 7 we get the useful fact that

Proposition 3. $T + (\Pi_2^{\mathcal{F}}, \Lambda)$–*IR* $\equiv T + \{\forall x\, \exists y\, (f^t(x) = y) :\ t \in \Lambda\}$.

We are now ready for the main result of this section. We extend our work in [6] by obtaining a new theorem on these local induction systems that will be crucial to derive the main results of the paper. Although $I\Delta_0 + \Sigma_2$–IR is known to be much stronger than $I\Sigma_1$ (indeed the former proves the consistency of the latter), in the local case we are able to show that $T + (\Sigma_2^{\mathcal{F}}, \Lambda)$–IR is contained in the theory $T + I\Sigma_1^{\mathcal{F}}$ or, even more, in the theory $T + B\Sigma_1^{\mathcal{F}} + I(\Sigma_1^{\mathcal{F}}, \Lambda)$. Here $B\Sigma_1^{\mathcal{F}}$ denotes the theory of language $\mathcal{L}_{\mathcal{F}}$ axiomatized by $I\Sigma_0^{\mathcal{F}}$ together with the *collection scheme*:

$$\forall x\, \exists y\, \varphi(x, y) \rightarrow \forall u \exists v\, \forall x \leq u\, \exists y \leq v\, \varphi(x, y)$$

for each $\varphi(x, y) \in \Sigma_1^{\mathcal{F}}$ (possibly containing parameters); and $I(\Sigma_1^{\mathcal{F}}, \Lambda)$ is the theory axiomatized over $I\Sigma_0^{\mathcal{F}}$ by the scheme

$$\varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x + 1)) \rightarrow \varphi(t)$$

for each $\varphi(x) \in \Sigma_1^{\mathcal{F}}$ (possibly containing parameters) and $t \in \Lambda$. Towards a proof, first we need the following lemma.

Lemma 8. $T + B\Sigma_1^{\mathcal{F}} + I(\Sigma_1^{\mathcal{F}}, \Lambda)$ *is* $\Pi_2^{\mathcal{F}}$*–conservative over* $T + (\Pi_2^{\mathcal{F}}, \Lambda)$*–IR.*

Proof. We adapt the proof of Proposition 1. The introduction of the notion of a $\Sigma_2^{\mathcal{F}}$–closed model and its use to obtain conservation results is straightforward. Hence, it is sufficient to show that every $\Sigma_2^{\mathcal{F}}$–closed model of $T + (\Pi_2^{\mathcal{F}}, \Lambda)$–IR is a model of $B\Sigma_1^{\mathcal{F}} + I(\Sigma_1^{\mathcal{F}}, \Lambda)$. To this end, let $\mathfrak{A}$ be a $\Sigma_2^{\mathcal{F}}$–closed model of $T + (\Pi_2^{\mathcal{F}}, \Lambda)$–IR. We can prove $\mathfrak{A} \models I(\Sigma_1^{\mathcal{F}}, \Lambda)$ reasoning as in the proof of Proposition 1; so, let us prove $\mathfrak{A} \models B\Sigma_1^{\mathcal{F}}$.

Let $\varphi(x, y, z) \in \Sigma_1^{\mathcal{F}}$ and $c \in \mathfrak{A}$ such that $\mathfrak{A} \models \forall x\, \exists y\, \varphi(x, y, c)$. By Lemma 3, there exist $d \in \mathfrak{A}$ and $\psi(u, v) \in \Pi_1^{\mathcal{F}}$ such that $\mathfrak{A} \models \psi(d, c)$ and

$$T + (\Pi_2^{\mathcal{F}}, \Lambda)\text{–IR} \vdash \psi(u, z) \rightarrow \forall x\, \exists y\, \varphi(x, y, z),$$

and hence, by Proposition 3, there are $t_1, \dots, t_n \in \Lambda$ such that

$$T + \{\forall x\, \exists y\, (f^{t_j}(x) = y) : j = 1, \dots, n\} \vdash \forall u, z, x\, \exists y\, (\psi(u, z) \to \varphi(x, y, z)).$$

Put $\mathcal{L}' = \mathcal{L}_{\mathcal{F}} \cup \{h_1, \dots, h_n\}$ and define T' to be the extension of T by the axioms $\forall x\, (f^{t_j}(x) = h_j(x))$, with $j = i, \dots, n$. By Parikh's theorem for T', there is a term $t(x, u, z)$ of $\mathcal{L}'$ such that

$$T' \vdash \forall u, z, x\, \exists y \le t(x, u, z)\, (\psi(u, z) \to \varphi(x, y, z)).$$

Then, we have

$$T' \vdash \forall x_0, u, z\, \forall x \le x_0\, \exists y \le t(x_0, u, z)\, (\psi(u, z) \to \varphi(x, y, z)),$$

for terms of $\mathcal{L}'$ define monotone functions. Since $\mathfrak{A}$ has a natural expansion to a model of T', we get that, for every $a \in \mathfrak{A}$,

$$\mathfrak{A} \models \forall x \le a\, \exists y \le t(a, d, c)\, (\psi(d, c) \to \varphi(x, y, c)).$$

As a consequence, there exists $b \in \mathfrak{A}$ such that

$$\mathfrak{A} \models \forall x \le a\, \exists y \le b\, (\psi(d, c) \to \varphi(x, y, c)).$$

But, recall $\mathfrak{A} \models \psi(d, c)$ and thus we get $\mathfrak{A} \models \forall x \le a\, \exists y \le b\, \varphi(x, y, c)$, as required. $\square$

Now for the main result.

Theorem 1. *$T + B\Sigma_1^{\mathcal{F}} + I(\Sigma_1^{\mathcal{F}}, \Lambda)$ extends $T + (\Sigma_2^{\mathcal{F}}, \Lambda)$–IR.*

Proof. We shall prove, by induction on $k \ge 0$, that for every extension $\mathcal{L}_{\mathcal{F}}$ of $\mathcal{L}$, every theory $T \subseteq \Pi_1^{\mathcal{F}}$, and every Λ exponentially closed, it holds that

$$T + B\Sigma_1^{\mathcal{F}} + I(\Sigma_1^{\mathcal{F}}, \Lambda) \text{ extends } [T, (\Pi_2^{\mathcal{F}}, \Lambda)\text{–IR}_0]_k.$$

This suffices as the arguments used in [2], proposition 2.1, can be easily adapted to yield that for every $k \in \omega$, $[T, (\Sigma_2^{\mathcal{F}}, \Lambda)\text{–IR}]_k \equiv [T, (\Pi_2^{\mathcal{F}}, \Lambda)\text{–IR}_0]_k$.

Case $k = 0$ is trivial; so, let us assume that $T + B\Sigma_1^{\mathcal{F}} + I(\Sigma_1^{\mathcal{F}}, \Lambda)$ extends $[T, (\Pi_2^{\mathcal{F}}, \Lambda)\text{–IR}_0]_k$. Let $t \in \Lambda$ and $\varphi(u, v) \in \Pi_2^{\mathcal{F}}$ such that

$$(\dagger) \qquad [T, (\Pi_2^{\mathcal{F}}, \Lambda)\text{–IR}_0]_k \vdash \forall u\, (\varphi(u, v) \to \varphi(u + 1, v)).$$

We must prove that $T + B\Sigma_1^{\mathcal{F}} + I(\Sigma_1^{\mathcal{F}}, \Lambda) \vdash \varphi(0, v) \rightarrow \varphi(t, v)$.

Without loss of generality, we can assume $\varphi(u, v) \equiv \forall x\, \exists y\, \varphi_0(u, x, y, v)$, with $\varphi_0(u, x, y, v) \in \Sigma_0^{\mathcal{F}}$. Let g be a new unary function symbol and let T^g be the extension of $T + I\Sigma_0^{\mathcal{F},g}$ obtained by adding the axiom:

$$\forall x\, (f(x) \leq g(x)).$$

Thus, g is a dominating (iterable nondecreasing) function over T^g. By (†), it follows that $[T^g, (\Pi_2^{\mathcal{F},g}, \Lambda)\text{–IR}_0]_k \vdash \varphi^g$, where φ^g is the following sentence:

$$\forall u\, (\forall x\, \exists y \leq g(x + u + v)\, \varphi_0(u, x, y, v) \rightarrow \forall x\, \exists y\, \varphi_0(u + 1, x, y, v)).$$

Claim 1. *There exists a closed term $\tau \in \Lambda$ such that the theory $T^g + \forall x\, \exists y\, (g^\tau(x) = y)$ proves*

$$\forall u\, (\forall x\, \exists y \leq g(x{+}u{+}v)\, \varphi_0(u, x, y, v) \rightarrow \forall x\, \exists y \leq g^\tau(u{+}x{+}v)\, \varphi_0(u{+}1, x, y, v))$$

Proof of Claim: We distinguish two cases:

Case 1: $k = 0$. Then $T^g \vdash \varphi^g$. Hence, by Parikh's theorem, there exists a term $s(u, x, v)$ of $\mathcal{L}_{\mathcal{F},g}$ such that T^g proves

$$\forall u\, (\forall x\, \exists y \leq g(x + u + v)\, \varphi_0(u, x, y, v) \rightarrow \forall x\, \exists y \leq s(u, x, v)\, \varphi_0(u + 1, x, y, v)).$$

By Lemma 4, there is $m \in \omega$ such that $T^g \vdash s(u, x, v) < g^m(u{+}x{+}v{+}\sigma(s))$. By induction on z it can be proved that

$$T^g \vdash g^u(x + z) = y_1 \wedge g^{u+z}(x) = y_2 \rightarrow y_1 \leq y_2$$

and, thus, if $\tau = m + \sigma(s)$ then $\tau \in \Lambda$ and the result follows.

Case 2: $k \geq 1$. Since $[T^g, (\Pi_2^{\mathcal{F},g}, \Lambda)\text{–IR}_0]_k \vdash \varphi^g$ and φ^g is a $\Pi_2^{\mathcal{F},g}$–formula, by induction hypothesis, $T^g + B\Sigma_1^{\mathcal{F},g} + I(\Sigma_1^{\mathcal{F},g}, \Lambda) \vdash \varphi^g$ and, by Lemma 8 $T^g + (\Pi_2^{\mathcal{F},g}, \Lambda)$–IR also proves φ^g. It follows from Proposition 3 that there exist $t_1, \ldots, t_n \in \Lambda$ such that

$$T^g + \{\forall x\, \exists y\, (g^{t_j}(x) = y) :\ j = 1, \ldots, n\} \vdash \varphi^g.$$

Let $r = t_1{+}\cdots{+}t_n$. Then, by part (4) of Proposition 2, $T^g{+}\forall x\, \exists y\, (g^r(x) = y)$ extends $T^g + \{g^{t_j} \text{ is total} :\ j = 1, \ldots, n\}$. Let h be a new unary function symbol and let T^h be the extension of T^g obtained by adding to T^g the axiom $\forall x\, (g^r(x) = h(x))$. Then $T^h \vdash \varphi^g$ and T^h is conservative over T^g.

By Proposition 2, h is an iterable nondecreasing function over T^h and $T^h \vdash \forall x\,(g(x) \le h(x))$. Therefore, h is a dominating function over T^h and T^h extends $I\Sigma_0^{\mathcal{F},g,h}$. By Parikh's theorem, there is a term $s(u,x,v)$ of $\mathcal{L}_{\mathcal{F},g,h}$ such that T^h proves

$$\forall x\,\exists y \le g(x+u+v)\,\varphi_0(u,x,y,v) \to \forall x\,\exists y \le s(u,x,v)\,\varphi_0(u+1,x,y,v)$$

and, by Lemma 4, there is $m \in \omega$ such that

$$T^h \vdash s(u,x,v) < h^m(u+x+v+\sigma(s)).$$

Recall that $T^h \vdash h^u(x+z) = y_1 \wedge h^{u+z}(x) = y_2 \to y_1 \le y_2$ and, thus, if $\sigma_0 = m + \sigma(s)$ then $\sigma_0 \in \Lambda$ and $T^h + \forall x\,\exists y\,(h^{\sigma_0}(x) = y)$ proves

$$\forall x\,\exists y \le g(x+u+v)\,\varphi_0(u,x,y,v) \to \forall x\,\exists y \le h^{\tau}(u+x+v)\,\varphi_0(u+1,x,y,v).$$

Using part (7) of Proposition 2, we can prove, by $\Sigma_0^{\mathcal{F},g,h}$–induction, that

$$T^h \vdash h^z(x) = y \leftrightarrow g^{r\cdot z}(x) = y.$$

As a consequence, $T^h + \forall x\,\exists y\,(h^{\sigma_0}(x) = y)$ proves

$$\forall x\,\exists y \le g(x+u+v)\,\varphi_0(u,x,y,v) \to \forall x\,\exists y \le g^{r\cdot\sigma_0}(u+x+v)\,\varphi_0(u+1,x,y,v).$$

Hence, putting $\tau = r \cdot \sigma_0 \in \Lambda$, the result follows, concluding the proof of Claim.

Let $\mathfrak{A} \models T + B\Sigma_1^{\mathcal{F}} + I(\Sigma_1^{\mathcal{F}}, \Lambda)$ and $c \in \mathfrak{A}$ such that $\mathfrak{A} \models \varphi(0,c)$. We will show that $\mathfrak{A} \models \varphi(t,c)$. Let $\psi(x,y,c) \in \Sigma_0^{\mathcal{F}}$ be the formula

$$\forall z \le x\,\exists w \le y\,(\varphi_0(0,z,w,c) \wedge y = w + f(x)).$$

Then, bearing in mind that $\mathfrak{A} \models B\Sigma_1^{\mathcal{F}}$, it holds that $\mathfrak{A} \models \forall x\,\exists y\,\psi(x,y,c)$ and the formula $\psi(x,y,c) \wedge \forall z < y \neg\psi(x,z,c)$ defines a total nondecreasing function $H : \mathfrak{A} \to \mathfrak{A}$. Since Λ is exponentially closed, there exists $t' \in \Lambda$ such that

$$[T, (\Sigma_1^{\mathcal{F}}, \Lambda)\text{–IR}] \vdash \exists y \le t'\,(\tau^t = y).$$

On the other hand, there is a $\Sigma_0^{\mathcal{F}}$ formula, that we denote by $H^z(x) = y$, defining the iteration of H and, since $\mathfrak{A} \models I(\Sigma_1^{\mathcal{F}}, \Lambda)$, we have

$$\mathfrak{A} \models \forall x\,\exists y\,(H^{t'}(x) = y).$$

Let $\theta(u,v)$ be the following $\Pi_1^{\mathcal{F}}$ formula:

$$u > t \vee \forall x \forall y_1 \left[H^{\tau^u}(x+u+v) = y_1 \rightarrow \exists y \leq y_1 \, \varphi_0(u,x,y,v)\right].$$

Since $\mathfrak{A} \models \forall x \, \exists y \, (H(x) = y)$, by definition of $\theta(u,v)$ we have $\mathfrak{A} \models \theta(0,c)$. Let us show that $\mathfrak{A} \models \forall u \, (\theta(u,v) \rightarrow \theta(u+1,v))$.

Pick $a, b \in \mathfrak{A}$ such that $\mathfrak{A} \models a \leq t \wedge \theta(a,b)$. Then, the formula $H^{\tau^a}(x) = y$ defines a total nondecreasing function in $\mathfrak{A}$ and we can use it to get an expansion of $\mathfrak{A}$ to a model $\mathfrak{A}^g$ of T^g such that

$$\mathfrak{A}^g \models \forall x \, \exists y \leq g(x+a+b) \, \varphi_0(a,x,y,b).$$

By part (7) of Proposition 2, we can prove by $\Sigma_0^{\mathcal{F},g}$–induction on z that

$$\mathfrak{A}^g \models \forall z \leq \tau \, [g^z(x+a+b) = H^{\tau^a \cdot z}(x+a+b)]$$

In particular, $\mathfrak{A}^g \models \forall x \, (g^\tau(x+a+b) = H^{\tau^a \cdot \tau}(x+a+b))$ and, as a consequence, $\mathfrak{A}^g \models T^g + \forall x \, \exists y \, (g^\tau(x) = y)$. Hence, by the Claim, we conclude that $\mathfrak{A}^g \models \forall x \, \exists y \leq g^\tau(x+a+b) \, \varphi_0(a+1,x,y,b)$ and, therefore, $\mathfrak{A} \models \theta(a+1,b)$.

We have shown that $\mathfrak{A} \models \theta(0,c) \wedge \forall u \, (\theta(u,c) \rightarrow \theta(u+1,c))$, and we know that $\mathfrak{A} \models I(\Pi_1^{\mathcal{F}}, \Lambda)$ (because $I(\Sigma_1^{\mathcal{F}}, \Lambda) \equiv I(\Pi_1^{\mathcal{F}}, \Lambda)$), so, $\mathfrak{A} \models \theta(t,c)$. In particular, since

$$\mathfrak{A} \models \theta(t,c) \rightarrow \forall x \, \exists y \leq H^{\tau^t}(t+x+c) \, \varphi_0(t,x,y,c),$$

we conclude $\mathfrak{A} \models \varphi(t,c)$. $\square$

Note that theorem 4.14 of [6] is now a consequence of Theorem 1.

Corollary 2. *$T + (\Pi_2^{\mathcal{F}}, \Lambda)$–$IR_0$ is $\Pi_2^{\mathcal{F}}$–conservative over $T + (\Pi_2^{\mathcal{F}}, \Lambda)$–$IR$.*

Finally, as a direct corollary of Theorem 1, we get

Theorem 2. *$T + I\Sigma_1^{\mathcal{F}}$ extends $T + (\Sigma_2^{\mathcal{F}}, \Lambda)$–$IR$.*

This result will be a key ingredient in the analysis of the p.t.c.f. of $I\Pi_2^-$ in the following section, for in a sense it states that over a sufficiently weak base theory, applications of *local* Σ_2–*IR* are reducible to primitive recursion.

4. Provably Total Computable Functions of $I\Pi_2^-$

We are now in a position to give a proof that $\mathcal{R}(I\Pi_2^-) = PR$. Firstly, we need a version of Theorem 2 in the language of first–order Arithmetic.

Lemma 9. *$I\Sigma_1$ extends $I\Delta_0 + (\Sigma_2, \mathcal{K}_2)$–IR.*

Proof. Let $\mathfrak{A} \models I\Sigma_1$ and $\varphi(x) \in \Sigma_2$ such that

$$(\bullet) \quad I\Delta_0 + (\Sigma_2, \mathcal{K}_2)\text{–IR} \vdash \varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1)).$$

We must show that for every $\delta(u) \in \Sigma_2^-$,

$$(\star) \qquad \mathfrak{A} \models \forall x_1 \forall x_2 (\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2) \rightarrow \forall x\, (\delta(x) \rightarrow \varphi(x)).$$

By $(\bullet)$ there exist formulas $\varphi_1(x), \ldots, \varphi_r(x) \in \Sigma_2$ and $\delta_1(x), \ldots, \delta_r(x) \in \Sigma_2^-$ such that $I\Delta_0$ plus the sentences

$$\alpha_j : \quad \forall x_1 \forall x_2 (\delta_j(x_1) \wedge \delta_j(x_2) \rightarrow x_1 = x_2) \rightarrow \forall x\, (\delta_j(x) \rightarrow \varphi_j(x))$$

$(j = 1, \ldots, r)$ proves $\varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1))$. More precisely, for each $j \leq r$,

$$I\Delta_0 + \bigwedge_{1 \leq i < j} \alpha_i \vdash \varphi_j(0) \wedge \forall x\, (\varphi_j(x) \rightarrow \varphi_j(x+1)),$$

and $I\Delta_0 + \bigwedge_{i=1}^r \alpha_i \vdash \varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1))$.

Let $E = \{j : 1 \leq j \leq r, \mathfrak{A} \models \neg\exists x \delta_j(x)\}$ and, for each $j \in E$, let $\theta_j(x, y) \in \Pi_0$ such that $\neg\exists x\, \delta_j(x)$ is equivalent to $\forall x \exists y\, \theta_j(x, y)$. Let m be the cardinal of E and let $\mathcal{F} = \{f_1, \ldots, f_m, f\}$ be a set of new unary function symbols. From the set of Σ_0 formulas $\Gamma = \{\theta_j(x, y) : j \in E\}$, we define a theory T as in Lemma 5. Let $\mathcal{L}(\mathfrak{A})$ denote the language obtained by adding to $\mathcal{L}$ a constant symbol $\underline{a}$, for each $a \in \mathfrak{A}$. Put $T' = T + D_{\Pi_1}(\mathfrak{A})$, where $D_{\Pi_1}(\mathfrak{A})$ is the Π_1–diagram of $\mathfrak{A}$. Let Λ be the set of closed terms of $\mathcal{L}(\mathfrak{A})$ containing only constants of the form $\underline{a}$ for $a \in \mathcal{K}_2(\mathfrak{A})$. Then $\mathfrak{A}$ has a natural expansion $\mathfrak{A}_{\mathcal{F}}$ to the language $\mathcal{L}_{\mathcal{F}} \cup \mathcal{L}(\mathfrak{A})$ such that $\mathfrak{A}_{\mathcal{F}} \models T' + I\Sigma_1^{\mathcal{F}}$. By Theorem 2, $\mathfrak{A}_{\mathcal{F}} \models T' + (\Sigma_2^{\mathcal{F}}, \Lambda)$–IR. Given $\delta(x) \in \Sigma_2^-$, we distinguish several cases:

If $\mathfrak{A} \models \neg\exists x\, \delta(x)$ then $(\star)$ obviously holds. On the other hand, if $\mathfrak{A} \models \neg\forall x_1 \forall x_2 (\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2)$, since this is a Σ_2–sentence and T' extends $D_{\Pi_1}(\mathfrak{A})$, $T' \vdash \neg\forall x_1 \forall x_2 (\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2)$. So,

$$T' \vdash \forall x_1 \forall x_2 (\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2) \rightarrow \forall x\, (\delta(x) \rightarrow \varphi(x)).$$

In that way $(\star)$ holds again. We must deal with a last case: $\mathfrak{A} \models \exists! x\, \delta(x)$.

Then there exists $d \in \mathcal{K}_2(\mathfrak{A})$ such that $\mathfrak{A} \models \delta(d)$ and $\underline{d} \in \Lambda$. In order to verify $(\star)$ it is enough to show that $T' + (\Sigma_2^{\mathcal{F}}, \Lambda)$–IR $\vdash \varphi(\underline{d})$.

We prove, by induction on j, that for all $j = 1, \ldots, r$, $T' + (\Sigma_2^{\mathcal{F}}, \Lambda)$–IR $\vdash \alpha_j$. Let $j \leq r$, and assume that $T' + (\Sigma_2^{\mathcal{F}}, \Lambda)$–IR $\vdash \bigwedge_{1 \leq i < j} \alpha_i$. Then

$$(\bullet)_j \quad T' + (\Sigma_2^{\mathcal{F}}, \Lambda)\text{–IR} \vdash \varphi_j(0) \wedge \forall x\, (\varphi_j(x) \rightarrow \varphi_j(x+1)).$$

If $j \in E$ or $\mathfrak{A} \models \neg \forall x_1 \forall x_2 (\delta_j(x_1) \wedge \delta_j(x_2) \rightarrow x_1 = x_2)$ then, reasoning as in previous cases, we conclude that $T' \vdash \alpha_j$. If $\mathfrak{A} \models \exists! x\, \delta_j(x)$, then there exists $b \in \mathcal{K}_2(\mathfrak{A})$ such that $\mathfrak{A} \models \delta_j(b)$ and $\underline{b} \in \Lambda$. Using $(\bullet)_j$ we obtain $T' + (\Sigma_2^{\mathcal{F}}, \Lambda)$–IR $\vdash \varphi_j(\underline{b})$. Therefore, $T' + (\Sigma_2^{\mathcal{F}}, \Lambda)$–IR $\vdash \exists x\, (\delta_j(x) \wedge \varphi_j(x))$, and it follows that $T' + (\Sigma_2^{\mathcal{F}}, \Lambda)$–IR $\vdash \alpha_j$, as required.

We have proved that $T' + (\Sigma_2^{\mathcal{F}}, \Lambda)$–IR $\vdash \bigwedge_{j=1}^{r} \alpha_j$ and so

$$T' + (\Sigma_2^{\mathcal{F}}, \Lambda)\text{–IR} \vdash \varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1)).$$

Thus, $T' + (\Sigma_2^{\mathcal{F}}, \Lambda)$–IR $\vdash \varphi(\underline{d})$ and, as a consequence, $(\star)$ holds. ☐

Next theorem extends a previous conservation result obtained in [4] and, as a direct corollary, yields the characterization of the p.t.c.f. of $I\Pi_2^-$.

Theorem 3. *$I\Pi_2^-$ is Π_3–conservative over $I\Sigma_1$.*

Proof. Let θ be a Π_3 sentence provable in $I\Pi_2^-$. Then $I(\Sigma_2, \mathcal{K}_2) \vdash \theta$ by Lemma 1 and $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)$–IR $\vdash \theta$ by Proposition 1. We need the following fact:

Claim 2. *$I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)$–IR $\equiv I\Sigma_1^- + (I\Delta_0 + (\Sigma_2, \mathcal{K}_2)$–IR$)$.*

Proof of Claim: Each axiom of $I\Sigma_1^-$ is a Σ_3 sentence, so it is enough to prove that for every $\sigma_0(u) \in \Pi_2$,

$$[I\Delta_0, (\Sigma_2, \mathcal{K}_2)\text{–IR}] + \exists u\, \sigma_0(u) \text{ extends } [I\Delta_0 + \exists u\, \sigma_0(u), (\Sigma_2, \mathcal{K}_2)\text{–IR}].$$

Assume $I\Delta_0 + \exists u\, \sigma_0(u) \vdash \varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1))$, where $\varphi(x) \in \Sigma_2$, and let $\psi(x, u) \in \Sigma_2$ be $\sigma_0(u) \rightarrow \varphi(x)$. Then, $I\Delta_0$ proves

$$\psi(0, u) \wedge \forall x\, (\psi(x, u) \rightarrow \psi(x+1, u))$$

and, therefore, $[I\Delta_0, (\Sigma_2, \mathcal{K}_2)\text{–IR}] \vdash U_\delta \rightarrow \forall x\,(\delta(x) \rightarrow \psi(x,u))$, where $\delta(x) \in \Sigma_2^-$ and U_δ denotes the sentence $\forall x_1 \forall x_2 (\delta(x_1) \wedge \delta(x_2) \rightarrow x_1 = x_2)$. Then $[I\Delta_0, (\Sigma_2, \mathcal{K}_2)\text{–IR}]$ also proves

$$\exists u\, \sigma_0(u) \rightarrow (U_\delta \rightarrow \forall x\,(\delta(x) \rightarrow \varphi(x)))$$

and so $[I\Delta_0, (\Sigma_2, \mathcal{K}_2)\text{–IR}] + \exists u\, \sigma_0(u) \vdash U_\delta \rightarrow \forall x\,(\delta(x) \rightarrow \varphi(x))$, as required.

It follows from Claim and Lemma 9 that $I\Sigma_1$ implies $I\Sigma_1^- + (\Sigma_2, \mathcal{K}_2)$–IR and, therefore, $I\Sigma_1 \vdash \theta$. $\square$

Corollary 3. *The class of provably total computable functions of* $I\Pi_2^-$ *is the class of primitive recursive functions.*

5. Relativization and Concluding Remarks

It is natural to ask ourselves whether Theorem 3 is also true for $I\Pi_{n+1}^-$ and $I\Sigma_n$ for an arbitrary $n \geq 1$. We have already seen that the reduction of $I\Pi_{n+1}^-$ to $I\Sigma_n^- + (\Sigma_{n+1}, \mathcal{K}_{n+1})$–IR works for all n and it is immediate to check that the claim in the proof of Theorem 3 can be generalized too. Thus, the key point is to prove that Lemma 9 also holds for $n > 1$, i.e. to prove that $I\Sigma_n$ implies $I\Sigma_{n-1} + (\Sigma_{n+1}, \mathcal{K}_{n+1})$–IR for all $n \geq 1$. Our proof of Lemma 9 for $n = 1$ leans upon Theorem 2 reducing $(\Sigma_2^{\mathcal{F}}, \Lambda)$–IR to $I\Sigma_1^{\mathcal{F}}$. Interestingly, the result for $n > 1$ can also be derived from Theorem 2 by using some standard *relativization* techniques. Building on previous work of Kaye [9], in [7] it is shown that, for each $n \geq 1$, there is a Π_n–formula $y = \mathbb{K}_n(x)$ satisfying that

(a) $I\Sigma_n \equiv I\Delta_0 + \forall x\, \exists! y\, (y = \mathbb{K}_n(x))$,
(b) $y = \mathbb{K}_n(x)$ is iterable and non decreasing over $I\Sigma_n$, and
(c) initial segments of $\mathfrak{A} \models I\Sigma_n$ closed under function $y = \mathbb{K}_n(x)$ are Π_n–elementary substructures of $\mathfrak{A}$.

Using functions $\mathbb{K}_n$ one can reformulate $I\Sigma_n$ as a $\Pi_1^{\mathcal{F}}$–theory in an extended language $\mathcal{L} \cup \{g_1, \ldots, g_n\}$ so that Σ_{n+m} formulas of $\mathcal{L}$ correspond to $\Sigma_m^{\mathcal{F}}$ formulas of the extended language (a similar treatment of relativization was also developed by Z. Ratajczyk in [11] via the notion of a *conditionally absolute* formula.)

Lemma 10. *Let* $n \geq 1$ *and let* $\mathcal{F} = \{g_1, \ldots, g_n\}$. *There is a* $\Pi_1^{\mathcal{F}}$*–theory* T^n *satisfying that*

1. *T^n extends $I\Sigma_n$,*
2. *every model of $I\Sigma_n$ has a (canonical) extension to a model of T^n,*
3. *every $\Sigma_m^{\mathcal{F}}$ formula is equivalent in T^n to a Σ_{n+m}–formula of $\mathcal{L}$, and*
4. *every Σ_{n+m} formula is equivalent in T^n to a $\Sigma_m^{\mathcal{F}}$–formula.*

Proof. (Sketch)
$\underline{n=1}$: Put $T^1 \equiv I\Sigma_0^{\mathcal{F}} + (y = g_1(x) \rightarrow y = \mathbb{K}_1(x))$.

Conditions (1), (2) and (3) are easy to verify, for we know that allowing monotone functions instead of only variables as the bounds in $\Sigma_0^{\mathcal{F}}$ formulas does not increase the strength of $\Sigma_0^{\mathcal{F}}$–induction (see, e.g. proposition V.1.3 of [8]). As for (4), since $I\Sigma_1$ contains the *strong collection* scheme for Π_0–formulas

$$\forall z\, \exists u\, \forall x \leq z\, (\exists y\, \varphi(x,y) \rightarrow \exists y \leq u\, \varphi(x,y)),$$

by a Parikh–like argument (available thanks to condiction (c) above) it follows that for each $\theta(\vec{x}, y) \in \Pi_0$ there is some $k \in \omega$ such that

$$I\Sigma_1 \vdash \exists y\, \theta(\vec{x}, y) \leftrightarrow \exists y \leq \mathbb{K}_1^k(x_1 + \ldots + x_p)\, \theta(\vec{x}, y),$$

and the result follows.
$\underline{n \rightarrow n+1}$: Let $y = \mathbb{K}'_{n+1}(x)$ denote a $\Pi_1^{\mathcal{F}}$–formula equivalent in T^n to $y = \mathbb{K}_{n+1}(x)$ and put $T^{n+1} \equiv T^n + (y = g_{n+1}(x) \rightarrow y = \mathbb{K}'_{n+1}(x))$. □

Equipped with this result, it is not hard to check that everything in the proof of Lemma 9 relativizes. Indeed, let $n \geq 2$ and suppose $\mathfrak{A}$ is a model of $I\Sigma_n$ and $\varphi(x)$ is in Σ_{n+1}. As in Lemma 9 let $\delta_1(x), \ldots, \delta_r(x)$ be the Σ_{n+1}^-–formulas occurring in a proof of $\varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1))$ in $I\Sigma_{n-1} + (\Sigma_{n+1}, \mathcal{K}_{n+1})$–IR. Let $E = \{j : 1 \leq j \leq r, \mathfrak{A} \models \neg\exists x \delta_j(x)\}$ and let $\mathcal{F} = \{f_1, \ldots, f_m, g_1, \ldots, g_{n-1}, f\}$, where m is the cardinal of E. For each $j \in E$, let $\theta'_j(x,y) \in \Pi_0^{\mathcal{F}}$ such that $\neg\exists x\, \delta_j(x)$ is equivalent in T^{n-1} to $\forall x \exists y\, \theta_j(x,y)$. From this set of $\Sigma_0^{\mathcal{F}}$ formulas define a $\Pi_1^{\mathcal{F}}$–theory T extending T^{n-1} as in Lemma 5. Finally, put $T' \equiv T + D_{\Pi_1^g}(\mathfrak{A})$, where $D_{\Pi_1^g}(\mathfrak{A})$ is the Π_1–diagram of $\mathfrak{A}$ in the language of T^{n-1}, and take $\Lambda = \mathcal{K}_{n+1}(\mathfrak{A})$. Then, $\mathfrak{A} \models T' + I\Sigma_1^{\mathcal{F}}$. So, applying Theorem 2 and reasoning as in Lemma 9 we get $\mathfrak{A} \models I\Sigma_{n-1} + (\Sigma_{n+1}, \mathcal{K}_{n+1})$–IR, as desired.

Thus, we have

Theorem 4. *For every $n \geq 1$, $I\Pi_{n+1}^-$ is Π_{n+2}–conservative over $I\Sigma_n$.*

A straightforward consequence of this result is a characterization of the class of p.t.c.f. of $I\Pi^-_{n+1}$ in terms of the extended Grzegorczyk Hierarchy $\{\mathcal{E}^\alpha :\ \alpha < \varepsilon_0\}$, see [12] for precise definitions.

Corollary 4. *For every* $n \geq 1$, $\mathcal{R}(I\Pi^-_{n+1}) = \mathcal{R}(I\Sigma_n) = \mathcal{E}^{\omega_n}$, *where* $\omega_0 = 1$, $\omega_{n+1} = \omega^{\omega_n}$.

An important ingredient in this analysis of the class of Π_{n+2}–consequences of $I\Pi^-_{n+1}$ is the study of the closure a weak theory, such as $I\Sigma^-_n$ (or even $I\Delta_0$), under $(\Sigma_{n+1}, \mathcal{K}_{n+1})$–IR. This analysis can be extended to stronger base theories providing us with similar conservation results for theories of the form $T + I\Pi^-_{n+1}$, where T is a Π_{n+2}–axiomatizable extension $I\Sigma_n$. In the following proposition we obtain this kind of conservation results when T is closed under Σ_{n+1}–collection rule:

$$\Sigma_{n+1}\text{–CR} : \qquad \frac{\forall x\, \exists y\, \varphi(x,y)}{\forall u\, \exists v\, \forall x \leq u\, \exists y \leq v\, \varphi(x,y)}$$

for $\varphi(x,y) \in \Sigma_{n+1}$.

Proposition 4. *Let T be a Π_{n+2}–axiomatizable extension of $I\Sigma_n$, closed under Σ_{n+1}–CR. Then:*

1. *$T + I\Pi^-_{n+1}$ is Π_{n+2}–conservative over $[T, \Sigma_{n+1}$–IR$]$*
2. *$T + I\Pi^-_{n+1}$ is Π_{n+1}–conservative over $T + \Pi_{n+1}$–IR.*

Proof. These results were proved for $n = 0$ in [6]. The proof for $n \geq 1$ is very similar, modulo relativization. Here we discuss the proof for $n = 1$.

(1) First of all, let us recall that, over $I\Sigma_1$, $I\Pi^-_2 \equiv I(\Sigma^-_2, \mathcal{K}_2)$ and that, by Proposition 1, $T + I(\Sigma_2, \mathcal{K}_2)$ is Π_3–conservative over $T + (\Sigma_2, \mathcal{K}_2)$–IR. So it is enough to show that $[T, \Sigma_2$–IR$]$ extends this last theory. But observe that

$$(\bullet) \qquad T + (\Sigma_2, \mathcal{K}_2)\text{–IR} \equiv [T, (\Sigma_2, \mathcal{K}_2)\text{–IR}].$$

This can be obtained from Lemma 6, by using the relativization device that we have developed (see the proof of lemma 3.7 in [6] for details). By $(\bullet)$, $[T, \Sigma_2$–IR$]$ obviously extends $T + (\Sigma_2, \mathcal{K}_2)$–IR and the result follows.

(2) By part (1) it suffices to show that $[T, \Sigma_2$–IR$]$ is Π_2–conservative over $T + \Pi_2$–IR. By proposition 2.1 of [2], $[T, \Sigma_2$–IR$]$ is equivalent to $[T, \Pi_2$–IR$_0]$ and it is straightforward to show (using Lemma 3) that every Σ_2–closed model of $T + \Pi_2$–IR is a model of $[T, \Pi_2$–IR$_0]$. By Lemma 2 it follows that $[T, \Pi_2$–IR$_0]$ is Π_2–conservative over $T + \Pi_2$–IR, as required. □

The interest of Proposition 4 is twofold. On the one hand, part (1) provides a generalization of a similar result obtained in [10]:

Theorem 5 (Kaye–Paris–Dimitracopoulos).
$I\Pi_1^-$ is Π_2–conservative over $I\Delta_0 + \exp$ ($\equiv [I\Delta_0, \Sigma_1$–IR]).

We can think of this result as a counterpart of Theorem 4 for $I\Pi_1^-$. However, a generalization of Theorem 5 for every $n \geq 1$ must take into consideration two different scenarios, since $I\Delta_0 \equiv I\Delta_0^-$, but $I\Sigma_n$ is a proper extension of $I\Sigma_n^-$. Together Proposition 4 and Theorem 4 show that both generalizations are correct. For $T = I\Sigma_n$, Proposition 4 shows that Theorem 5 also holds for every $n \geq 1$ (essentially, this result was obtained by Kaye in [9]):

Corollary 5. *$I\Sigma_n + I\Pi_{n+1}^-$ is Π_{n+2}–conservative over $[I\Sigma_n, \Sigma_{n+1}$–IR].*

In turn, Theorem 4 shows that this corollary also holds for $I\Sigma_n^-$, since for every $n \geq 1$, $I\Sigma_n \equiv [I\Sigma_n^-, \Sigma_{n+1}\text{–IR}]$ and, obviously $I\Pi_{n+1}^-$ extends $I\Sigma_n^-$.

On the other hand, Proposition 4 reduces the question about the class of p.t.c.f. of $I\Sigma_1 + I\Pi_2^-$ to the study of the closure of $I\Sigma_1$ under Π_2–IR. In a similar vein, by combining parts (1) and (2), we obtain that, for every $k \geq 1$,

$$[I\Sigma_1, \Sigma_2\text{–IR}]_{k+1} \text{ is } \Pi_2\text{–conservative over } [I\Sigma_1, \Sigma_2\text{–IR}]_k + \Pi_2\text{–IR}.$$

These reductions suggest that local induction can be a useful tool in obtaining new proofs of some of the already known characterizations of classes of p.t.c.f. in terms of the extended Grzegorczyk hierarchy; for instance, $\mathcal{R}(I\Sigma_1 + I\Pi_2^-)$ (studied by Beklemishev in [4]), $\mathcal{R}([I\Sigma_1, \Sigma_2\text{–IR}]_k)$ or $\mathcal{R}(I\Sigma_2)$ and, more generally, $\mathcal{R}(I\Sigma_n + I\Pi_{n+1}^-)$ and $\mathcal{R}(I\Sigma_n)$. This points out natural extensions of the results and methods we have introduced in this paper.

Acknowledgement

This work was partially supported by grants MTM2008–06435 and MTM2011–26840 of Ministerio de Ciencia e Innovación, Spain. Cofinanced with FEDER funds, EU.

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