INDUCTION, MINIMIZATION AND COLLECTION FOR $\Delta_{n+1}(\mathbf{T})$–FORMULAS

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Abstract. For a theory $\mathbf{T}$, we study relationships among $\mathbf{I}\Delta_{n+1}(\mathbf{T})$, $\mathbf{L}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$. These theories are obtained restricting the schemes of induction, minimization and (a version of) collection to $\Delta_{n+1}(\mathbf{T})$ formulas. We obtain conditions on $\mathbf{T}$ ($\mathbf{T}$ is an extension of $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$ or $\Delta_{n+1}(\mathbf{T})$ is closed (in $\mathbf{T}$) under bounded quantification) under which $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{L}\Delta_{n+1}(\mathbf{T})$ are equivalent.

These conditions depend on $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$, the Π_{n+2}–consequences of $\mathbf{T}$. The first condition is connected with descriptions of $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ as $\mathbf{I}\Sigma_n$ plus a class of nondecreasing total Π_n–functions, and the second one is related with the equivalence between $\Delta_{n+1}(\mathbf{T})$–formulas and bounded formulas (of a language extending the language of Arithmetic). This last property is closely tied to a general version of a well known theorem of R. Parikh.

Using what we call Π_n–envelopes we give uniform descriptions of the previous classes of nondecreasing total Π_n–functions. Π_n–envelopes are a generalization of envelopes (see [10]) and are closely related to indicators (see [12]). Finally, we study the hierarchy of theories $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_m)$, $m \geq n$, and prove a hierarchy theorem.

1. Introduction

This paper is devoted to the study of two main topics: the relationship between induction and minimization, and the description of the class of Π_{n+2} consequences of a theory.

The first one is on Fragments of Arithmetic obtained restricting the schemes of induction, minimization and collection to Δ_{n+1}–formulas. These schemes for Σ_n and Π_n formulas have been thoroughly studied by J. Paris, L. Kirby and others (see [17] or [12]). The parameter free versions of those schemes have been studied by R. Kaye, J. Paris and C. Dimitracopoulos (see [11] and [14]). However, the relationships between those schemes for Δ_{n+1} formulas are not well known. About 1985, H. Friedman claimed that $\mathbf{L}\Delta_{n+1}$ and $\mathbf{I}\Delta_{n+1}$ are equivalent (see [10] pg. 398), but in [6] that equivalence appears as an open problem (problem 34) and it is credited to J. Paris. Here that equivalence will be called the Paris–Friedman's Conjecture. In [19], T. Slaman proves it for $n \geq 1$.

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In sections 2 and 6, we study those schemes restricted to $\Delta_{n+1}(\mathbf{T})$ formulas. If $\varphi \in \Sigma_{n+1}$ and $\psi \in \Pi_{n+1}$ then $\varphi \leftrightarrow \psi$ is a Π_{n+2} formula. So, the second topic is related to the first one. In sections 3–5, we analyse the class of Π_{n+2} consequences of a theory using a class of Π_n–functions and extensions of the language of Arithmetic related to that class of functions.

Now we present the main results obtained on these topics in this paper.

Part I: Induction and minimization for $\Delta_{n+1}(\mathbf{T})$ formulas.

In order to get a better insight on the Paris–Friedman's Conjecture we consider the theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$, $\mathbf{L}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$, where

$$\Delta_{n+1}(\mathbf{T}) = \{\varphi(x, \vec{v}) \in \Sigma_{n+1} : \text{ there exists } \psi(x, \vec{v}) \in \Pi_{n+1},\ \mathbf{T} \vdash \varphi \leftrightarrow \psi\}.$$

The idea is to change the semantic part of the axioms schemes on Δ_{n+1} formulas by a syntactic condition: the equivalence between a Σ_{n+1} formula and a Π_{n+1} formula is proved in a theory. Thus we obtain a relativization of Paris–Friedman's Conjecture. We study the following problem:

$(*)$ Under which conditions on $\mathbf{T}$ does $\mathbf{L}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$ hold?

We first observe that $\Longrightarrow$ always holds. In the other way, let us notice that the usual proof of $\mathbf{I}\Sigma_{n+1} \Longrightarrow \mathbf{L}\Sigma_{n+1}$ leans upon the closure of Σ_{n+1} under bounded quantification (this property is granted by the collection schemes, $\mathbf{B}\Sigma_{n+1}$). In fact, the closure under bounded quantification of the class of Δ_{n+1}–formulas is the main obstacle in order to adapt the refered proof to obtain that $\mathbf{I}\Delta_{n+1} \Longrightarrow \mathbf{L}\Delta_{n+1}$. So, to answer problem $(*)$ the above remarks suggest two natural properties: $\mathbf{T}$ has Δ_{n+1}–collection (that is, $\mathbf{T} \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$), and $\mathbf{T}$ is Δ_{n+1}–closed (that is, $\Delta_{n+1}(\mathbf{T})$ is closed in $\mathbf{T}$ under bounded quantification). We prove that if $\mathbf{T}$ satisfies one of the above conditions then $\mathbf{L}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$, see theorem 1.4.

We also study relationships among the above schemes, for distinct theories. The following theorem sums up the results obtained.

Theorem 1.1 *(see 2.1, 2.10, 2.17, 2.18, 6.12, 6.13, 6.14).* *For all $n \in \omega$*

$$\begin{array}{ccccccc}
\mathbf{I}\Sigma_n & \Longleftrightarrow & \mathbf{L}\Delta_{n+1}(\mathbf{I}\Sigma_n) & \Longleftrightarrow & \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n) & \Longleftrightarrow & \mathbf{B}^*\Delta_{n+1}(\mathbf{I}\Sigma_n) \\
 & & \underline{\Uparrow} & & \underline{\Uparrow} & & \Uparrow \\
\left.\begin{array}{c} \mathbf{B}\Sigma^-_{n+1},\ \mathbf{I}\Pi^-_{n+1} \\ \mathbf{I}\Delta_{n+1},\ \mathbf{B}\Sigma_{n+1} \end{array}\right\} & \not\Longleftrightarrow & \mathbf{L}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1}) & \Longleftrightarrow & \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1}) & \mathrel{|}\!\!\Longrightarrow & \mathbf{B}^*\Delta_{n+1}(\mathbf{I}\Sigma_{n+1}) \\
 & & \underline{\Uparrow} & & \underline{\Uparrow} & & \Uparrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \underline{\Uparrow} & & \underline{\Uparrow} & & \Uparrow \\
\begin{array}{r} \mathbf{I}\Sigma^-_{n+1} \\ \mathbf{I}\Pi^-_{n+1},\ \mathbf{B}\Sigma^-_{n+1} \end{array} & \begin{array}{c} \not\Longleftarrow \\ \not\Longleftrightarrow \end{array} & \mathbf{L}\Delta_{n+1}(\mathbf{PA}) & \Longleftrightarrow & \mathbf{I}\Delta_{n+1}(\mathbf{PA}) & \mathrel{|}\!\!\Longrightarrow & \mathbf{B}^*\Delta_{n+1}(\mathbf{PA}) \\
 & & \underline{\Uparrow} & & \underline{\Uparrow} & & \Uparrow \\
\begin{array}{c} \mathbf{B}\Sigma_{n+1} \\ \mathbf{I}\Pi^-_{n+1} \\ \mathbf{B}\Sigma^-_{n+1} \end{array} & \begin{array}{c} \not\Longleftrightarrow \\ \Longleftarrow\!\!\mathrel{|} \\ \not\Longrightarrow \end{array} & \mathbf{L}\Delta_{n+1}(\mathcal{N}) & \Longleftrightarrow & \mathbf{I}\Delta_{n+1}(\mathcal{N}) & \mathrel{|}\!\!\Longrightarrow & \mathbf{B}^*\Delta_{n+1}(\mathcal{N}) \\
 & & & & \underline{\Uparrow} & & \underline{\Uparrow} \\
 & & & & \mathbf{I}\Sigma_{n+1} & \mathrel{|}\!\!\Longrightarrow & \mathbf{B}\Sigma_{n+1}
\end{array}$$

(Some of those relations for parameter free schemes follow from results in [9]*, see also* [7] *and* [15]*).*

Part II: Π_{n+2} **consequences of a theory**.

Properties considered in part I ($\mathbf{T}$ has Δ_{n+1}–collection, $\mathbf{T}$ is Δ_{n+1}–closed and others that we call Δ_{n+1}–properties) depend on $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$, the class of Π_{n+2} consequences of $\mathbf{T}$. Here we give characterizations of these properties in a "functional" way. The idea is to describe $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ using $\mathbf{I}\Sigma_n$ and a class of Π_n–functions. To this end we introduce the concepts of Π_n–functional class (which provides a characterization of the theories having Δ_{n+1}–collection) and Π_n–Parikh pair (which corresponds with Δ_{n+1}–closed theories). Essentially, a Π_n–functional class is a set of nondecreasing Π_n–functions. The concept of Π_n–Parikh pair is suggested by the following well known result.

Theorem 1.2 *(Parikh). Let $\varphi(x,y) \in \Sigma_1$. If $\mathbf{I}\Delta_0 \vdash \forall x\, \exists y\, \varphi(x,y)$ then there exists $t(x) \in \mathbf{Term}(\mathcal{L})$ such that $\mathbf{I}\Delta_0 \vdash \forall x\, \exists y \leq t(x)\, \varphi(x,y)$.*

As a consequence of this result (see 3.27) each $\Delta_1(\mathbf{I}\Delta_0)$ formula is equivalent (in $\mathbf{I}\Delta_0$) to a Δ_0–formula. So, $\Delta_1(\mathbf{I}\Delta_0)$ is closed (in $\mathbf{I}\Delta_0$) under bounded quantification. We give a general version of this fact. If $\mathbf{T}$ is Δ_{n+1}–closed, then there is a conservative extension of $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ (in a language extending the language of Arithmetic) in which each $\Delta_{n+1}(\mathbf{T})$ formula is equivalent to a bounded formula. In particular, if $\mathbf{T}$ has Δ_{n+1}–collection then a strong Π_n–functional class provides such an extension.

One crutial result that relates the schemes of induction and collection is the Friedman–Paris' conservativeness theorem (see [10] or [12]):

Theorem 1.3. *For all $n \in \omega$, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1})$.*

Here we study a similar Π_{n+2}–conservativeness property, closely tied to Δ_{n+1}–collection: $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$. This property plays a central role in the study of Π_n–envelopes that will be developed in section 5. Roughly speaking, a Π_n–envelope is a Π_n–functional class given in an uniform way and generalizes the concept of envelope (see [10]). In section 6 we use results of sections 4 and 5 to separate the fragments $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_m)$, $m \geq n$ (see theorem 1.1). The following theorem sums up, for a consistent theory, $\mathbf{T}$, the relationships among the concepts introduced.

Theorem 1.4. *(see 2.10, 2.11, 3.8, 3.11, 3.28, 4.18, 5.11, 5.21)*

$$
\begin{array}{ccccccc}
\mathbf{T}\ \textit{is}\ \Delta_{n+1}\textit{-PF} & & \mathbf{T}\ \textit{has}\ \Delta_{n+1}\textit{-min.} & & \mathbf{T}\ \textit{is strong}\ \Pi_n\textit{-funct.} & & \mathbf{T}\ \textit{has}\ \Pi_n\textit{-s-env.} \\
\Uparrow & & \Updownarrow & & \Updownarrow & & \Updownarrow^{\mathbf{3}} \\
\mathbf{T}\ \textit{is}\ \Delta_{n+1}\textit{-closed} & \Longleftarrow & \mathbf{T}\ \textit{has}\ \Delta_{n+1}\textit{-coll.} & \Longleftrightarrow & \mathbf{T}\ \textit{is}\ \Pi_n\textit{-funct.} & \Longleftrightarrow^{\mathbf{2}}_{\mathbf{3}} & \mathbf{T}\ \textit{has}\ \Pi_n\textit{-env.} \\
\Updownarrow & & \Updownarrow & & \Updownarrow_{\mathbf{1}} & & \\
\mathbf{T}\ \textit{is}\ \Pi_n\textit{–Parikh} & & \left\{ \begin{array}{l} \mathbf{T}\ \textit{has}\ \Delta_{n+1}\textit{-ind.} \\ \mathbf{T}\ \textit{is}\ \Delta_{n+1}\textit{-closed} \end{array} \right. & & \mathbf{T}\ \textit{is}\ \Pi^{\mathbf{B}}_{n+2}\textit{-conserv.} & &
\end{array}
$$

Where: $\Longrightarrow_{\mathbf{1}}$ holds if $\mathbf{T}$ is Π_{n+2} axiomatizable; $\Longleftarrow^{\mathbf{2}}$ holds if the Π_n–envelope is given by a Π_n formula; and $\Longrightarrow_{\mathbf{3}}$ holds if $\mathbf{T}$ is recursively axiomatizable, and, for $n = 0$, $\mathbf{T} \vdash \mathbf{exp}$.

In order to simplify the statement of the above theorem we have used there the following notation: $\mathbf{T}$ has Π_n–envelope (Π_n–s–envelope) means that there exists a Π_n–envelope (strong Π_n–envelope) of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$; and $\mathbf{T}$ is $\Pi^{\mathbf{B}}_{n+2}$–conservative if $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$.

The analysis of theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$ that we develop in this paper is related with the work of L. D. Beklemishev in [2], [3] and [4], on induction and collection as inference rules. Some results in those papers, proved there using Proof Theoretic techniques, are similar to those given here for schemes on $\Delta_{n+1}(\mathbf{T})$–formulas. Now we give a more precise description of the relationship between Beklemishev's work and ours.

In the papers cited above, Beklemishev study the schemes of induction and collection as inference rules. The induction rule for a formula $\varphi(x)$ is:

$$\frac{\varphi(0),\ \forall x\,(\varphi(x)\rightarrow\varphi(x+1))}{\forall x\,\varphi(x)}$$

If Γ is a class of formulas, then Γ–IR is the class of induction rules for each formula in Γ. Given a theory $\mathbf{T}$, let $\mathbf{T}+\Sigma_{n+1}$–IR be the closure of $\mathbf{T}$ under first order logic and applications of Σ_{n+1}–IR. We also denote by $[\mathbf{T},\Sigma_{n+1}\text{–IR}]$ the closure of $\mathbf{T}$ under first order logic and unnested applications of Σ_{n+1}–IR; that is, the rule of induction can be applied only if the hypothesis of the rule are theorems of $\mathbf{T}$ (in first order logic). The rule of collection for a formula $\varphi(x,y)$ is:

$$\frac{\forall x\,\exists y\,\varphi(x,y)}{\forall z\,\exists\,u\,\forall\,x\le z\,\exists y\le u\,\varphi(x,y)}$$

Theories $\mathbf{T}+\Sigma_{n+1}$–CR and $[\mathbf{T},\Sigma_{n+1}\text{–CR}]$ are defined as for the induction rule. In [4] it is also considered the induction rule for Δ_{n+1} formulas: for each $\varphi(x)\in\Sigma_{n+1}$ and $\psi(x)\in\Pi_{n+1}$

$$\Delta_{n+1}\text{–IR}:\qquad \frac{\forall x\,(\varphi(x)\leftrightarrow\psi(x))}{\mathbf{I}_{\varphi,x}}$$

As we shall see in 2.19, a theory $\mathbf{T}$ (extension of $\mathbf{I}\Delta_0$) has Δ_{n+1}–collection if and only if $\mathbf{T}$ is closed under Σ_{n+1}–CR (that is, $[\mathbf{T},\Sigma_{n+1}\text{–CR}]\Longleftrightarrow\mathbf{T}$). For induction we have that $\mathbf{T}$ has Δ_{n+1}–induction if and only if $\mathbf{T}$ is closed under Δ_{n+1}–IR.

Our analysis of theories with Δ_{n+1}–collection using Π_n–functional classes is also very similar (for $n=0$) to the one given by Beklemishev in [2] using what he call monotone formulas. In this way theorem **3.5** can be considered a generalization of theorem **5.4** of [2] (and it is linked with theorem **4.2** of [3]). Nevertheless, we must observe that one of the aims of Beklemishev's work in [3] is to obtain a proof of Friedman–Paris' conservativeness theorem. On the other hand, our analysis goes in a reverse direction, since we take that result as basic (due to its easy model theoretic proof) and relate it with a characterization of Π_n–envelopes using indicators (Π_n–IND property, see theorem 5.6).

The relationship of Σ_{n+1}–IR with the work developed here is not so obvious. But, as Beklemishev has noted (personal communication),

$$\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1})\Longleftrightarrow\mathbf{I}\Delta_0+\Sigma_{n+1}\text{–IR}.$$

This fact is closely tied to a conservativeness theorem of Parsons (see [18])

$$\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_{n+1})\Longleftrightarrow\mathbf{I}\Delta_0+\Sigma_{n+1}\text{–IR}.$$

These results are more deeply studied in [8] in connection with axiomatization properties of the theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$.

We conclude this section with some basic results and notation that we use through this paper. We work in the first–order language of Arithmetic, $\mathcal{L}=\{0,1,+,\cdot,<\}$ and $\mathcal{N}$ denotes the standard model of $\mathcal{L}$ whose universe is the set of the natural numbers, ω. As usual, bounded quantifiers are denoted by $\forall x \le t\, \varphi(x)$ and $\exists x \le t\, \varphi(x)$ (where x does not occur in t). $\Delta_0 = \Sigma_0 = \Pi_0$ is the class of bounded formulas and, for each $n \in \omega$,

$$\Sigma_{n+1} = \{\exists \vec{x}\, \varphi(\vec{x}) :\ \varphi(\vec{x}) \in \Pi_n\} \text{ and } \Pi_{n+1} = \{\forall \vec{x}\, \varphi(\vec{x}) :\ \varphi(\vec{x}) \in \Sigma_n\}.$$

Let $\varphi(x, \vec{v})$ be a formula of $\mathcal{L}$. We shall denote $\varphi(x, \vec{v}) \wedge \forall y < x\, \neg\varphi(y, \vec{v})$ by $\varphi_{\mu,x}(x, \vec{v})$. If $\mathfrak{A} \models \varphi_{\mu,x}(a, \vec{b})$ then we write $\mathfrak{A} \models a = (\mu x)[\varphi(x, \vec{b})]$. If there is no danger of misunderstanding we omit the subscript x and the parameters $\vec{v}$ and we shall write $\varphi_\mu(x)$.

We denote by $\mathbf{P}^-$ a finite set of Π_1 axioms such that if $\mathfrak{A} \models \mathbf{P}^-$ then $\mathfrak{A}$ is the nonnegative part of a commutative discretely ordered ring (see [12]).

Let $\varphi(x, \vec{v})$ be a formula. The induction and the least number principle axioms for $\varphi(x, \vec{v})$ with respect to x are, respectively, the following formulas

$$\mathbf{I}_{\varphi,x}(\vec{v}) \equiv \varphi(0, \vec{v}) \wedge \forall x\, [\varphi(x, \vec{v}) \rightarrow \varphi(x+1, \vec{v})] \rightarrow \forall x\, \varphi(x, \vec{v}),$$

$$\mathbf{L}_{\varphi,x}(\vec{v}) \equiv \exists x\, \varphi(x, \vec{v}) \rightarrow \exists x\, \varphi_{\mu,x}(x, \vec{v}).$$

Let $\varphi(x, y, \vec{v})$ be a formula. The collection axiom and the strong collection axiom for φ with respect to x, y are, respectively, the formulas

$$\mathbf{B}_{\varphi,x,y}(z, \vec{v}) \equiv \forall x \le z\, \exists y\, \varphi(x, y, \vec{v}) \rightarrow \exists u\, \forall x \le z\, \exists y \le u\, \varphi(x, y, \vec{v}),$$

$$\mathbf{S}_{\varphi,x,y}(z, \vec{v}) \equiv \exists u\, \forall x \le z\, [\exists y\, \varphi(x, y, \vec{v}) \rightarrow \exists y \le u\, \varphi(x, y, \vec{v})].$$

As usual, we write $\mathbf{I}_\varphi$ instead of $\mathbf{I}_{\varphi,x}$ and similarly we use $\mathbf{L}_\varphi$, $\mathbf{B}_\varphi$ and $\mathbf{S}_\varphi$. If Γ is a class of formulas of $\mathcal{L}$, then $\mathbf{I}\Gamma = \mathbf{P}^- + \{\mathbf{I}_\varphi :\ \varphi \in \Gamma\}$. The theory $\mathbf{L}\Gamma$ is defined similarly using $\mathbf{L}_\varphi$ instead of $\mathbf{I}_\varphi$. For collection, $\mathbf{B}\Gamma = \mathbf{I}\Delta_0 + \{\mathbf{B}_\varphi :\ \varphi \in \Gamma\}$ and using $\mathbf{S}_\varphi$ instead of $\mathbf{B}_\varphi$ we obtain $\mathbf{S}\Gamma$. Peano Arithmetic is the theory $\mathbf{PA} = \mathbf{P}^- + \{\mathbf{I}_\varphi :\ \varphi \text{ formula}\}$.

Now we consider schemes for parameter free formulas. Let Γ be a class of formulas. We write $\varphi(x_1, \ldots, x_n) \in \Gamma^-$ if $\varphi \in \Gamma$ and $x_1, \ldots, x_n$ are all the variables that occur free in φ. Then $\mathbf{I}\Gamma^- = \mathbf{P}^- + \{\mathbf{I}_{\varphi,x} :\ \varphi(x) \in \Gamma^-\}$ (similarly for $\mathbf{L}\Gamma^-$) and $\mathbf{B}\Gamma^- = \mathbf{I}\Delta_0 + \{\mathbf{B}^-_{\varphi,x,y} :\ \varphi(x, y) \in \Gamma^-\}$, where

$$\mathbf{B}^-_{\varphi,x,y} \equiv \forall x\, \exists y\, \varphi(x, y) \rightarrow \forall z\, \exists u\, \forall x \le z\, \exists y \le u\, \varphi(x, y).$$

The parameter free version of the strong collection scheme for Σ_n formulas is equivalent to $\mathbf{S}\Sigma_n$.

One of the basic functions used to describe metamathematical properties in the language of Arithmetic, such as truth predicates, is the exponential function. Let $\mathbf{E}(x, y, z)$ be a Δ_0–formula that defines in the standard model the exponential function, $\mathbf{I}\Delta_0$ proves its basic properties and $\mathbf{I}\Sigma_1$ proves that it is total (see [10] for details). We shall usually write $x^y = z$ instead of $\mathbf{E}(x, y, z)$ and shall denote by $\mathbf{exp}$ the Π_2 sentence $\forall x\, \forall y\, \exists z\, \mathbf{E}(x, y, z)$.

We shall write: $\mathbf{T} \Longrightarrow \mathbf{T}'$, if $\mathbf{T}$ is an extension of $\mathbf{T}'$; $\mathbf{T} \not\Longrightarrow \mathbf{T}'$, if $\mathbf{T}$ is not an extension of $\mathbf{T}'$; $\mathbf{T} \not\Longleftrightarrow \mathbf{T}'$, if $\mathbf{T} \not\Longrightarrow \mathbf{T}'$ and $\mathbf{T}' \not\Longrightarrow \mathbf{T}$; $\mathbf{T} \Longleftrightarrow \mathbf{T}'$, if $\mathbf{T}$ and $\mathbf{T}'$ are equivalent; and $\mathbf{T} \models\!\!\Longrightarrow \mathbf{T}'$, if $\mathbf{T}$ is a proper extension of $\mathbf{T}'$.

We recall some definitions and results which are important in the study of the above schemes. Let $\mathfrak{A} \models \mathbf{P}^-$, $n \in \omega$ and $X \subseteq \mathfrak{A}$. Then $\mathcal{K}_n(\mathfrak{A}, X)$ (if X is the empty set, we write $\mathcal{K}_n(\mathfrak{A})$) is the substructure of $\mathfrak{A}$ whose universe is $\{b \in \mathfrak{A} :\ b \text{ is } \Sigma_n \text{ definable in } (\mathfrak{A}, X)\}$. $\mathcal{I}_n(\mathfrak{A}, X)$ is the initial segment of $\mathfrak{A}$ determined by $\mathcal{K}_n(\mathfrak{A}, X)$. It holds the following results.

Theorem 1.5. (1) *Let $\mathfrak{A} \models \mathbf{I}\Sigma_n$ be nonstandard. Then for all $X \subseteq \mathfrak{A}$*

(a) *$\mathcal{K}_{n+1}(\mathfrak{A}, X) \prec_{n+1} \mathfrak{A}$ and $\mathcal{K}_{n+1}(\mathfrak{A}, X) \models \mathbf{I}\Sigma_n$.*

(b) *$\mathcal{K}_{n+1}(\mathfrak{A}, X) \prec^c_{n+1} \mathcal{I}_{n+1}(\mathfrak{A}, X) \prec^e_n \mathfrak{A}$. ($\subset^c$ and $\subset^e$ mean cofinal and initial substructure, respectively).*

(c) *If* $\mathcal{K}_{n+1}(\mathfrak{A}, X)$ *is not cofinal in* $\mathfrak{A}$ *then* $\mathcal{I}_{n+1}(\mathfrak{A}, X) \models \mathbf{B}\Sigma_{n+1}$.

(2) *Let* $\mathfrak{A} \models \mathbf{I}\Sigma_{n+1}$ *be nonstandard such that* $\mathcal{K}_{n+1}(\mathfrak{A})$ *is nonstandard. Then* $\mathcal{K}_{n+1}(\mathfrak{A}) \not\models \mathbf{B}\Sigma^-_{n+1}$ *and* $\mathcal{I}_{n+1}(\mathfrak{A}) \not\models \mathbf{I}\Sigma_{n+1}$.

Finally, we introduce the axiom schemes for Δ_{n+1} formulas.

$$\mathbf{I}\Delta_{n+1} = \mathbf{P}^- + \{\forall x\, [\varphi(x,\vec{v}) \leftrightarrow \psi(x,\vec{v})] \rightarrow \mathbf{I}_{\varphi,x}(\vec{v}) :\ \varphi \in \Sigma_{n+1},\ \psi \in \Pi_{n+1}\}.$$

Using $\mathbf{L}_\varphi$ instead of $\mathbf{I}_\varphi$, we obtain $\mathbf{L}\Delta_{n+1}$. Parameter free schemes, $\mathbf{I}\Delta^-_{n+1}$ and $\mathbf{L}\Delta^-_{n+1}$, are defined similarly. Uniform versions of the above fragments have been introduced by R. Kaye (see [11]). $\mathbf{UI}\Delta_{n+1}$ is $\mathbf{P}^-$ together with, for all $\varphi \in \Sigma_{n+1}$, $\psi \in \Pi_{n+1}$,

$$\forall x\, \forall \vec{v}\, [\varphi(x,\vec{v}) \leftrightarrow \psi(x,\vec{v})] \rightarrow \forall \vec{v}\, \mathbf{I}_{\varphi,x}(\vec{v}).$$

$\mathbf{UL}\Delta_{n+1}$ is defined accordingly using $\mathbf{L}_\varphi$. We introduce a uniform version of collection. $\mathbf{UB}\Delta_{n+1}$ is $\mathbf{I}\Delta_0$ together with, for all $\varphi \in \Pi_n$ and $\psi \in \Sigma_n$,

$$\forall x\, \forall \vec{v}\, [\exists y\, \varphi(x,y,\vec{v}) \leftrightarrow \forall w\, \psi(x,w,\vec{v})] \rightarrow \forall z\, \forall \vec{v}\, \mathbf{B}_{\varphi,x,y}(z,\vec{v}).$$

Theorem 1.6. *For all* $n \in \omega$, $\mathbf{I}\Delta^-_{n+1} \not\twoheadrightarrow \mathbf{UL}\Delta_{n+1}$ *and*

$$\begin{array}{ccccccccc}
 & & \mathbf{I}\Pi^-_{n+1} & \models\!\!\!\Longrightarrow & \mathbf{L}\Delta^-_{n+1} & \Longrightarrow & \mathbf{I}\Delta^-_{n+1} & & \\
 & & & & \Uparrow & & \Uparrow & & \\
\mathbf{UB}\Delta_{n+1} & \Longleftrightarrow & \mathbf{B}\Sigma^-_{n+1} & \Longleftrightarrow & \mathbf{UL}\Delta_{n+1} & \Longrightarrow & \mathbf{UI}\Delta_{n+1} & \models\!\!\!\Longrightarrow & \mathbf{I}\Sigma_n \\
 & & \Uparrow & & \Uparrow & & \Uparrow & & \\
 & & \mathbf{B}\Sigma_{n+1} & \Longleftrightarrow & \mathbf{L}\Delta_{n+1} & \Longrightarrow & \mathbf{I}\Delta_{n+1} & &
\end{array}$$

For $n \geq 1$, $\mathbf{L}\Delta^-_{n+1} \not\twoheadrightarrow \mathbf{UI}\Delta_{n+1}$ *and* $\mathbf{I}\Delta^-_{n+1} \not\Longleftrightarrow \mathbf{I}\Sigma_n$, *but* $\mathbf{I}\Delta^-_1 \models\!\!\!\Longrightarrow \mathbf{I}\Delta_0$.

R.O. Gandy (see [10]) proved the equivalence between $\mathbf{L}\Delta_{n+1}$ and $\mathbf{B}\Sigma_{n+1}$; and R. Kaye (see [11]) obtained a similar result for the uniform versions. See [9] for $\mathbf{UI}\Delta_{n+1} \models\!\!\!\Longrightarrow \mathbf{I}\Sigma_n$, $\mathbf{L}\Delta^-_{n+1} \not\twoheadrightarrow \mathbf{UI}\Delta_{n+1}$ and $\mathbf{I}\Delta^-_{n+1} \not\twoheadrightarrow \mathbf{UL}\Delta_{n+1}$; [7] for $\mathbf{I}\Delta_{n+1} \models\!\!\!\Longrightarrow \mathbf{UI}\Delta_{n+1}$; 2.14 for $\mathbf{UB}\Delta_{n+1} \Longleftrightarrow \mathbf{B}\Sigma^-_{n+1}$; and [4] for $\mathbf{UI}\Delta_1 \models\!\!\!\Longrightarrow \mathbf{I}\Delta^-_1$ (there $\mathbf{UI}\Delta_1$ is denoted by $s\mathbf{I}\Delta_1$). The above diagram contains the following open problems:

- **(–):** The Paris–Friedman's Conjecture: $\mathbf{L}\Delta_{n+1} \Longleftrightarrow \mathbf{I}\Delta_{n+1}$.
- **(–):** The Uniform Paris–Friedman's Conjecture: $\mathbf{UL}\Delta_{n+1} \Longleftrightarrow \mathbf{UI}\Delta_{n+1}$.
- **(–):** The Parameter Free Paris–Friedman's Conjecture: $\mathbf{L}\Delta^-_{n+1} \Longleftrightarrow \mathbf{I}\Delta^-_{n+1}$.

Recently, T. Slaman (see [19]) has obtained a partial answer. He has proved that

$$\mathbf{L}\Delta_{n+1} + \mathbf{exp} \Longleftrightarrow \mathbf{I}\Delta_{n+1} + \mathbf{exp}.$$

On the other hand, L. Beklemishev (see [4]) has proved that $\mathbf{I}\Delta_1 + \mathbf{exp}$ is a Σ_3–conservative extension of $\mathbf{UI}\Delta_1 + \mathbf{exp}$; hence, $\mathbf{UL}\Delta_1 + \mathbf{exp} \Longleftrightarrow \mathbf{UI}\Delta_1 + \mathbf{exp}$. Beklemishev's result seems to be easily extended to $n \geq 1$; so, only the case $n = 0$ seems to be open in the two first problems. However, Slaman's proof rests on the equivalence between $\mathbf{B}\Sigma_{n+1}$ and $\mathbf{L}\Delta_{n+1}$; therefore it can not be adapted to the parameter free problem.

2. The theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$, $\mathbf{L}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$

Through this paper $\mathbf{T}$ will denote a consistent theory in the first–order language of Arithmetic. For such a theory we introduce the classes of formulas

$$\Delta_{n+1}(\mathbf{T}) = \{\varphi(x,\vec{v}) \in \Sigma_{n+1} : \text{ there exists } \psi(x,\vec{v}) \in \Pi_{n+1},\ \mathbf{T} \vdash \varphi \leftrightarrow \psi\}.$$

When the schemes of induction and minimization are restricted to these classes of formulas we obtain the theories $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{L}\Delta_{n+1}(\mathbf{T})$. We also consider the following version of the collection schemes

$$\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) = \mathbf{I}\Delta_0 + \{\mathbf{B}_{\varphi,x,y}(z,\vec{v}) :\ \varphi \in \Pi_n,\ \exists y\, \varphi(x,y,\vec{v}) \in \Delta_{n+1}(\mathbf{T})\}.$$

Remark 2.1. We shall begin with some basic properties of the theories introduced above. First we observe that $\mathbf{I}\Sigma_{n+1} \Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{I}\Sigma_n$.

If $\varphi \in \Sigma_{n+1}$ and $\psi \in \Pi_{n+1}$ then $\varphi \leftrightarrow \psi$ is a Π_{n+2}–formula. So, it follows that (a similar result holds for minimization and collection)

Claim 2.2. *If* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}')$ *then* $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T}')$.

Let $\Delta^*_{n+1}(\mathbf{T})$ be the dual class of $\Delta_{n+1}(\mathbf{T})$. Since the negation of a $\Delta_{n+1}(\mathbf{T})$ formula (that is, a $\Delta^*_{n+1}(\mathbf{T})$–formula) is equivalent (in $\mathbf{T}$) to a $\Delta_{n+1}(\mathbf{T})$– formula, as in the proof of $\mathbf{I}\Pi_{n+1} \Longleftrightarrow \mathbf{I}\Sigma_{n+1}$ (see lemma **7.5** in [12]), we get that

Claim 2.3. $\mathbf{L}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{I}\Delta^*_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$.

For each $\psi(x,y) \in \Pi_{n-1}$, $\exists y\,[\psi(x,y) \vee (\neg\exists z\, \psi(x,z) \wedge y = 0)] \in \Delta_{n+1}(\mathbf{T})$. So, as in the proof of $\mathbf{B}\Sigma_{n+1} \Longrightarrow \mathbf{I}\Sigma_n$ (see **I.2.15** in [10]), we obtain

Claim 2.4. $\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{I}\Sigma_n$. *Hence, for* $n \geq 1$, $\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \not\Longrightarrow \mathbf{B}\Sigma_n$.

Suppose that $\mathbf{T}$ is an extension of $\mathbf{I}\Sigma_n$. Let $\varphi \in \Pi_n$ and $\psi \in \Sigma_n$ such that $\mathbf{T} \vdash \exists y\, \varphi(x,y) \leftrightarrow \forall y\, \psi(x,y)$. Let us consider the following formulas

$$\theta_1(x,w) \equiv x \leq w \wedge \exists u\,[\varphi_{\mu,u}(x,u) \wedge (\forall z)_{x \leq z \leq w}\, \exists y \leq u\, \varphi(z,y)],$$
$$\theta_2(x,w) \equiv x \leq w \wedge \forall y\, \psi(x,y) \wedge \forall u\,[\varphi(x,u) \rightarrow (\forall z)_{x \leq z \leq w} \exists y \leq u\, \varphi(z,u)].$$

Then $\mathbf{T} \vdash \theta_1(x,w) \leftrightarrow \theta_2(x,w)$ and $\theta_1 \in \Sigma_{n+1}$ and $\theta_2 \in \Pi_{n+1}$ in $\mathbf{T}$ and in $\mathbf{L}\Delta_{n+1}(\mathbf{T})$. From this, as in lemma **I.2.17** in [10], we obtain that

Claim 2.5. *If* $\mathbf{T}$ *is an extension of* $\mathbf{I}\Sigma_n$ *then* $\mathbf{L}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$.

Definition 2.6. *(*Δ_{n+1} **properties***) We say that*

1. $\mathbf{T}$ *is* Δ_{n+1}*–closed if* $\Delta_{n+1}(\mathbf{T})$ *is closed in* $\mathbf{T}$ *under bounded quantifiers.*
2. $\mathbf{T}$ *has* Δ_{n+1}*–collection if* $\mathbf{T} \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$.
3. $\mathbf{T}$ *has* Δ_{n+1}*–minimization if* $\mathbf{T} \Longrightarrow \mathbf{L}\Delta_{n+1}(\mathbf{T})$.
4. $\mathbf{T}$ *has* Δ_{n+1}*–induction if* $\mathbf{T} \Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$.
5. $\mathbf{T}$ *is* Δ_{n+1}*–PF if* $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{L}\Delta_{n+1}(\mathbf{T})$.

Remark 2.7. Let us consider some examples of theories having Δ_{n+1} properties. Since $\mathbf{B}\Sigma_{n+1} \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$, we get that every theory extending $\mathbf{B}\Sigma_{n+1}$ has Δ_{n+1}–collection. Now we improve this result.

Claim 2.8. *If* $\mathbf{T} \Longrightarrow \mathbf{B}\Sigma_{n+1}^-$ *then* $\mathbf{T}$ *has* Δ_{n+1}*–collection.*

Proof of Claim. Let $\varphi(x, y, v_1, \dots, v_m) \in \Pi_n^-$ and $\psi(x, w, \vec{v}) \in \Sigma_n^-$ such that $\mathbf{T} \vdash \exists y\, \varphi(x, y, \vec{v}) \leftrightarrow \forall w\, \psi(x, w, \vec{v})$. Let $\theta(x, y) \in \Sigma_{n+1}^-$ be

$$\varphi((x)_0, y, (x)_1, \dots, (x)_m) \vee [y = 0 \wedge \neg\forall w\, \psi((x)_0, w, (x)_1, \dots, (x)_m)].$$

Since $\mathbf{T} \vdash \forall x\, \exists y\, \theta(x, y)$, $\mathbf{T} \vdash \forall z\, \exists u\, \forall x \leq z\, \exists y \leq u\, \theta(x, y)$. Let $\mathfrak{A} \models \mathbf{T}$ and $a, \vec{b} \in \mathfrak{A}$ such that $\mathfrak{A} \models \forall x \leq a\, \exists y\, \varphi(x, y, \vec{b})$ and $c = \langle a, \vec{b}\rangle$. Then there exists $d \in \mathfrak{A}$ such that $\mathfrak{A} \models \forall x \leq c\, \exists y \leq d\, \theta(x, y)$. Since $a \leq c$, then $\mathfrak{A} \models \forall x \leq a\, \exists y \leq d\, \varphi(x, y, \vec{b})$; hence, $\mathfrak{A} \models \mathbf{B}_\varphi$, as required.

□

There exist theories, e.g. $\mathbf{I}\Sigma_n$ (see 2.17), that have Δ_{n+1}–collection and are not extension of $\mathbf{B}\Sigma_{n+1}^-$. Now we present a case in which both conditions are equivalent.

Claim 2.9. *If* $\mathbf{T}$ *is complete and has* Δ_{n+1}*–collection then* $\mathbf{T} \Longrightarrow \mathbf{B}\Sigma_{n+1}^-$.

Proof of Claim. Let $\mathfrak{A} \models \mathbf{T}$ and $\theta(x, y) \in \Sigma_{n+1}^-$ such that $\mathfrak{A} \models \forall x\, \exists y\, \theta(x, y)$. Since $\mathbf{T}$ is complete, $\mathbf{T} \vdash \forall x\, \exists y\, \theta(x, y)$; so, $\exists y\, \theta(x, y) \in \Delta_{n+1}(\mathbf{T})$. Since $\mathbf{T}$ has Δ_{n+1}–collection, $\mathfrak{A} \models \mathbf{B}_\theta$; hence, $\mathfrak{A} \models \forall z\, \exists u\, \forall x \leq z\, \exists y \leq u\, \theta(x, y)$. □

Next result was, chronologically, the main reason to introduce the theory $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$. This theory became one of the main tools in this work once we came to the concept of Π_n–functional theory (see subsection 3.1).

Theorem 2.10. (1) *If* $\mathbf{T}$ *is* Δ_{n+1}*–closed then* $\mathbf{T}$ *is* Δ_{n+1}*–PF.*
(2) *If* $\mathbf{T}$ *has* Δ_{n+1}*–collection then* $\mathbf{T}$ *is* Δ_{n+1}*–closed.*

Proof. (*(1)*): By 2.3, it is enough to see that $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{L}\Delta_{n+1}(\mathbf{T})$. Suppose that there exist $\mathfrak{A} \models \mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\varphi(x) \in \Delta_{n+1}(\mathbf{T})$ such that $\mathfrak{A} \models \exists x\, \varphi(x) \wedge \forall x\, \neg\varphi_\mu(x)$. Let $\theta(z) \in \Pi_{n+1}$ be $\forall x \leq z\, \neg\varphi(x)$. We have that $\mathfrak{A} \models \theta(0) \wedge [\theta(z) \rightarrow \theta(z+1)]$. Since $\mathbf{T}$ is Δ_{n+1}–closed, $\theta(z) \in \Delta_{n+1}^*(\mathbf{T})$. By 2.3, $\mathfrak{A} \models \mathbf{I}\Delta_{n+1}^*(\mathbf{T})$; so, $\mathfrak{A} \models \forall x\, \neg\varphi(x)$, a contradiction.
(*(2)*): Let $\varphi(x, y) \in \Pi_n$, $\psi(x, y) \in \Sigma_n$ such that $\mathbf{T} \vdash \exists y\, \varphi(x, y) \leftrightarrow \forall y\, \psi(x, y)$. By the closure properties under bounded quantification of $\mathbf{B}\Sigma_n$, there exists $\theta(z) \in \Sigma_{n+1}$ such that $\mathbf{B}\Sigma_n \vdash \theta(z) \leftrightarrow \exists u\, \forall x \leq z\, \exists y \leq u\, \varphi(x, y)$ (for $n = 0$, we do not need $\mathbf{B}\Sigma_n$). The following equivalences hold in the given theories.

$$\begin{array}{llll} \forall x \leq z\, \forall y\, \psi(x, y) & \leftrightarrow & \forall x \leq z\, \exists y\, \varphi(x, y) & [\text{in } \mathbf{T}] \\ & \leftrightarrow & \exists u\, \forall x \leq z\, \exists y \leq u\, \varphi(x, y) & [\text{in } \mathbf{B}^*\Delta_{n+1}(\mathbf{T})] \\ & \leftrightarrow & \theta(z) & [\text{in } \mathbf{B}\Sigma_n] \end{array}$$

Since $\mathbf{T}$ has Δ_{n+1}–collection, all the above equivalences hold in $\mathbf{T}$. Then, as $\forall x \leq z\, \forall y\, \psi(x, y) \in \Pi_{n+1}$, $\theta(z) \in \Delta_{n+1}(\mathbf{T})$. So, $\forall x \leq z\, \exists y\, \varphi(x, y)$ is equivalent in $\mathbf{T}$ to a $\Delta_{n+1}(\mathbf{T})$ formula. □

Remark 2.11. Now we describe others elementary relations among Δ_{n+1} properties of a theory.

Claim 2.12. *If* $\mathbf{T}$ *has* Δ_{n+1}*–collection then* $\mathbf{T}$ *has* Δ_{n+1}*–induction.*

Proof of Claim. Suppose that $\mathbf{T} \vdash \exists y\, \varphi(x,y) \leftrightarrow \forall y\, \psi(x,y)$, where $\varphi(x,y) \in \Pi_n$, $\psi(x,y) \in \Sigma_n$. Let $\theta(x,y) \in \Pi_n$ be $\varphi(x,y) \vee \neg\psi(x,y)$. Since $\mathbf{T} \vdash \forall x\, \exists y\, \theta(x,y)$, then $\exists y\, \theta(x,y) \in \Delta_{n+1}(\mathbf{T})$. Now the proof continues as in 2.4. □

Claim 2.13. *The following conditions are equivalent*

(i) $\mathbf{T}$ *has* Δ_{n+1}*–collection.*
(ii) $\mathbf{T}$ *is* Δ_{n+1}*–closed and has* Δ_{n+1}*–induction.*
(iii) $\mathbf{T}$ *has* Δ_{n+1}*–minimization.*

Proof of Claim. *(i)* $\Longrightarrow$ *(ii)* is 2.10 and 2.12. *(ii)* $\Longrightarrow$ *(iii)* follows from 2.10–*(1)*.
(*(iii)* $\Longrightarrow$ *(i)*): Suppose that $\mathbf{T}$ has Δ_{n+1}–minimization. Then $\mathbf{T} \Longrightarrow \mathbf{I}\Sigma_n$; so, by 2.5, $\mathbf{L}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$. Hence, $\mathbf{T} \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$. □

For each model $\mathfrak{A}$, $\mathbf{Th}(\mathfrak{A})$ has Δ_{n+1}–collection if and only if $\mathfrak{A} \models \mathbf{UB}\Delta_{n+1}$ (or $\mathfrak{A} \models \mathbf{B}\Sigma^-_{n+1}$, see 2.9); and $\mathbf{Th}(\mathfrak{A})$ has Δ_{n+1}–minimization if only if $\mathfrak{A} \models \mathbf{UL}\Delta_{n+1}$. So, as a consequence of 2.13, we obtain that

Claim 2.14. $\mathbf{B}\Sigma^-_{n+1} \Longleftrightarrow \mathbf{UB}\Delta_{n+1} \Longleftrightarrow \mathbf{UL}\Delta_{n+1}$.

Remark 2.15 *($\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ and Δ_{n+1} properties).* Here we shall see that a theory $\mathbf{T}$ has a Δ_{n+1}–property if and only if $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ has that property. This is easily seen for Δ_{n+1}–closed. Now we consider Δ_{n+1}–induction.

Claim 2.16. $\mathbf{T} \Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$ *if and only if* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$.

Proof of Claim. Let $\varphi \in \Sigma_{n+1}$ and $\psi \in \Pi_{n+1}$ such that $\mathbf{T} \vdash \varphi \leftrightarrow \psi$. Let $\mathbf{I}_{\varphi,\psi}$ be

$$\psi(0) \wedge \forall x\, [\varphi(x) \rightarrow \psi(x+1)] \rightarrow \forall x\, \psi(x).$$

Then, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \vdash \mathbf{I}_\varphi \leftrightarrow \mathbf{I}_{\varphi,\psi}$. Suppose that $\mathbf{T}$ has Δ_{n+1}–induction, then $\mathbf{T} \vdash \mathbf{I}_\varphi$; hence, $\mathbf{T} \vdash \mathbf{I}_{\varphi,\psi}$. Since $\mathbf{I}_{\varphi,\psi} \in \Pi_{n+2}$, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \vdash \mathbf{I}_{\varphi,\psi}$; so, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \vdash \mathbf{I}_\varphi$, as required. □

From this, 2.13 and 2.10 we get a similar result for $\mathbf{L}\Delta_{n+1}(\mathbf{T})$; and from 2.5, using again 2.13, also for $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$.

Claim 2.17. *If* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1})$, $\mathbf{T}$ *has* Δ_{n+1}*–collection. So,* $\mathbf{I}\Sigma_n$, $\mathbf{I}\Delta_{n+1}$ *and* $\mathbf{UI}\Delta_{n+1}$ *have* Δ_{n+1}*–collection.*

Remark 2.18. Now we study relations between $\mathbf{I}\Delta_{n+1}(\mathbf{T})$ and $\mathbf{I}\Sigma_n$, $\mathbf{B}\Sigma_{n+1}$ and $\mathbf{B}\Sigma^-_{n+1}$. By 2.17, $\mathbf{I}\Sigma_n$ has Δ_{n+1}–collection; so, by 2.10 and 2.4, it follows that $\mathbf{I}\Sigma_n \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n) \Longleftrightarrow \mathbf{L}\Delta_{n+1}(\mathbf{I}\Sigma_n) \Longleftrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{I}\Sigma_n)$. From this result, 1.3 and 2.2 we get that

$$\mathbf{I}\Sigma_n \Longleftrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{B}\Sigma_{n+1}) \Longleftrightarrow \mathbf{L}\Delta_{n+1}(\mathbf{B}\Sigma_{n+1}) \Longleftrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{B}\Sigma_{n+1}).$$

Since $\Sigma_0(\Sigma_n) \subseteq \Delta_{n+1}(\mathbf{B}\Sigma_{n+1})$, the above result gives a generalization of $\mathbf{I}\Sigma_n \Longleftrightarrow \mathbf{I}\Sigma_0(\Sigma_n)$ (see [10], theorem **I.2.4**).

Let $\mathbf{T}$ be a theory consistent with $\mathbf{I}\Sigma_{n+1}$, $\mathfrak{A} \models \mathbf{T}+\mathbf{I}\Sigma_{n+1}$ and $a \in \mathfrak{A}$ nonstandard. Then, by 1.5, $\mathcal{K}_{n+1}(\mathfrak{A}, a) \models \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{I}\Sigma_{n+1})$; hence, by 2.15, $\mathcal{K}_{n+1}(\mathfrak{A}, a) \models \mathbf{I}\Delta_{n+1}(\mathbf{T}) + \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$. So, again by 1.5, it holds that

$$\mathbf{I}\Sigma_{n+1} \models\!\!\!\Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T}) \not\Rightarrow\!\!\!\!\Rightarrow \mathbf{B}\Sigma_{n+1} \models\!\!\!\Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T}).$$

In particular, $\mathbf{B}\Sigma_{n+1} \models\!\!\!\Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathcal{N})$.

Now assume that there exists $\mathfrak{A} \models \mathbf{T}+\mathbf{I}\Sigma_{n+1}$ such that $\mathcal{K}_{n+1}(\mathfrak{A})$ is not standard. Then, by 1.5–*(2)*, $\mathcal{K}_{n+1}(\mathfrak{A}) \not\models \mathbf{B}\Sigma^-_{n+1}$. So, $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \not\Rightarrow\!\!\!\!\Rightarrow \mathbf{B}\Sigma^-_{n+1}$ and $\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \not\Rightarrow\!\!\!\!\Rightarrow \mathbf{B}\Sigma^-_{n+1}$.

Remark 2.19. (**On induction and collection rules**) In this remark we state some relations between Δ_{n+1}–properties and induction and collection rules.

Claim 2.20. *The following conditions are equivalent:*

(i) $\mathbf{T}$ *has* Δ_{n+1}*–induction.*

(ii) $\mathbf{T}$ *is closed under* Δ_{n+1}*–IR.*

Proof of Claim. The result follows from $[\mathbf{T}, \Delta_{n+1}\text{–IR}] \Longleftrightarrow \mathbf{T} + \mathbf{I}\Delta_{n+1}(\mathbf{T})$.

□

Claim 2.21. *Let* $\mathbf{T}$ *be an extension of* $\mathbf{I}\Delta_0$*. Then*

$$[\mathbf{T}, \Sigma_{n+1}\text{–}CR] \Longleftrightarrow \mathbf{T} + \mathbf{B}^*\Delta_{n+1}(\mathbf{T}).$$

Proof of Claim. ($\Longleftarrow$): Let $\varphi(x, y) \in \Pi_n$ such that $\mathbf{T} \vdash \forall x\, \exists y\, \varphi(x, y)$. Then $\exists y\, \varphi(x, y) \in \Delta_{n+1}(\mathbf{T})$. Hence, $\mathbf{T} + \mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \vdash \forall z\, \exists u\, \forall x \leq z\, \exists y \leq u\, \varphi(x, y)$.

($\Longrightarrow$): Let $\varphi(x, y) \in \Pi_n$ and $\psi(x, y) \in \Sigma_n$ such that $\mathbf{T} \vdash \exists y\, \varphi(x, y) \leftrightarrow \forall y\, \psi(x, y)$. Then $\mathbf{T} \vdash \forall x\, \exists y\, (\varphi(x, y) \vee \neg\psi(x, y))$. So,

$$[\mathbf{T}, \Sigma_{n+1}\text{–CR}] \vdash \forall z\, \exists u\, \forall x \leq z\, \exists y \leq u\, (\varphi(x, y) \vee \neg\psi(x, y)).$$

Hence, $[\mathbf{T}, \Sigma_{n+1}\text{–CR}] \vdash \forall x \leq z\, \exists y\, \varphi(x, y) \rightarrow \exists u\, \forall x \leq z\, \exists y \leq u\, \varphi(x, y)$.

□

Claim 2.22. *Let* $\mathbf{T}$ *be an extension of* $\mathbf{I}\Delta_0$*. The following conditions are equivalent:*

(i) $\mathbf{T}$ *has* Δ_{n+1}*–collection.*

(ii) $\mathbf{T}$ *is closed under* Σ_{n+1}*–CR.*

Proof of Claim. The result follows from 2.21.

□

3. Functional character of Δ_{n+1} properties

In this section we shall see that the Δ_{n+1}–properties of a theory $\mathbf{T}$ are connected with descriptions of $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ using $\mathbf{I}\Sigma_n$ and a class of Π_n functions.

In what follows Γ will denote a class of formulas of $\mathcal{L}$ with two free variables, x, y say. For a formula $\varphi(x, y)$, the conjunction of

(–): $\forall x\,\forall y_1\,\forall y_2\,[\varphi(x,y_1)\wedge\varphi(x,y_2)\rightarrow y_1=y_2]$, and

(–): $\forall x_1\,\forall x_2\,\forall y_1\,\forall y_2\,[x_1\le x_2\wedge\varphi(x_1,y_1)\wedge\varphi(x_2,y_2)\rightarrow y_1\le y_2]$,

will be denoted by $\mathrm{IPF}(\varphi)$. That is, $\mathrm{IPF}(\varphi)$ expresses that $\varphi(x,y)$ defines an increasing partial function. Let $\mathrm{IPF}(\Gamma)=\{\mathrm{IPF}(\varphi(x,y)):\ \varphi(x,y)\in\Gamma\}$, $\mathrm{Func}(\Gamma)=\{\forall x\,\exists! y\,\varphi(x,y):\ \varphi\in\Gamma\}$ and $\Gamma^*=\mathrm{Func}(\Gamma)+\mathrm{IPF}(\Gamma)$. Let us observe that for $\Gamma\subseteq\Pi_n$, $\Gamma^*\subseteq\Pi_{n+2}$. For a theory $\mathbf{T}$ let $\mathrm{Func}_n(\mathbf{T})=\mathrm{Func}(\mathrm{Gr}_n(\mathbf{T}))$, where $\mathrm{Gr}_n(\mathbf{T})=\{\varphi(x,y)\in\Pi_n^-:\ \mathbf{T}\vdash\forall x\,\exists! y\,\varphi(x,y)\}$.

Remark 3.1 *(The language $\mathcal{L}(\Gamma)$).* Let $\mathcal{L}(\Gamma)$ denotes the extension of $\mathcal{L}$ obtained by adding a function symbol G_φ for each $\varphi\in\Gamma$. Let Δ_0^Γ be the class of bounded formulas of $\mathcal{L}(\Gamma)$. The classes Σ_n^Γ and Π_n^Γ, $n\in\omega$, are defined as usually. Let us consider the following theories of language $\mathcal{L}(\Gamma)$.

$$\mathbf{I}\Delta_0^\Gamma=\mathbf{P}^-+\{\mathbf{I}_\varphi:\ \varphi\in\Delta_0^\Gamma\}+\{\varphi(x,y)\leftrightarrow G_\varphi(x)=y:\ \varphi\in\Gamma\},$$
$$\mathbf{I}\Delta_0^{\Gamma^*}=\mathbf{I}\Delta_0^\Gamma+\mathrm{IPF}(\Gamma).$$

Then $\mathbf{I}\Delta_0^\Gamma\Longleftrightarrow\mathbf{L}\Delta_0^\Gamma$ and, for $\Gamma\subseteq\Pi_n$, $\mathbf{I}\Delta_0^\Gamma$ and $\mathbf{I}\Delta_0^{\Gamma^*}$ are Π_{n+1}^Γ–axiomatizable.

In general, if $\mathbf{T}$ is a theory in the language of Arithmetic, $\mathbf{T}_\Gamma$ will denote the extension of $\mathbf{T}$ to $\mathcal{L}(\Gamma)$ obtained by adding to $\mathbf{T}$, as new axioms, the formulas $\varphi(x,y)\leftrightarrow G_\varphi(x)=y$, for each $\varphi\in\Gamma$. Observe that $(\mathbf{T}+\mathrm{Func}(\Gamma))_\Gamma$ is a conservative extension of $\mathbf{T}+\mathrm{Func}(\Gamma)$. It holds that

Claim 3.2. (i) *If $t(x)\in\mathbf{Term}(\mathcal{L}(\Gamma))$ then $(\mathbf{I}\Delta_0+\Gamma^*)_\Gamma\vdash x\le x'\rightarrow t(x)\le t(x')$.*

(ii) *Let $t(\vec{x})\in\mathbf{Term}(\mathcal{L}(\Gamma))$. There is $\varphi(\vec{x},y)\in\Delta_{n+1}(\mathbf{I}\Delta_0+Func(\Gamma))$ such that $(\mathbf{I}\Delta_0+Func(\Gamma))_\Gamma\vdash t(\vec{x})=y\leftrightarrow\varphi(\vec{x},y)$.*

Remark 3.3. By a standard argument on contraction of quantifiers (see [12]) for each $\varphi(\vec{x},\vec{y})\in\Pi_n^-$ there exists $\varphi^c(u,v,\vec{x},\vec{y})\in\Pi_n^-$ such that

(–): $\mathbf{I}\Delta_0\vdash\forall\vec{x}\,\exists\vec{y}\,\varphi(\vec{x},\vec{y})\leftrightarrow\forall u\,\exists v\,\forall\vec{x}\,\forall\vec{y}\,\varphi^c(u,v,\vec{x},\vec{y})$.

(–): $\mathbf{I}\Delta_0\vdash\varphi^c(u,v,\vec{x},\vec{y})\rightarrow\vec{x}\le u\wedge\vec{y}\le v$.

Claim 3.4. *Let $\psi(x,y)\in\Pi_n^-$. There exists $\psi_f(x,y)\in\Pi_n^-$ such that*

(i) $\mathbf{I}\Sigma_n\vdash\psi_f(x,y_1)\wedge\psi_f(x,y_2)\rightarrow y_1=y_2$.

(ii) $\mathbf{I}\Sigma_n\vdash\exists y\,\psi(x,y)\rightarrow\exists y\,\psi_f(x,y)$.

(iii) $\mathbf{I}\Sigma_n\vdash\psi_f(x,y)\rightarrow\exists z\le y\,\psi(x,z)$.

Proof of Claim. For $n=0$, let $\psi_f(x,y)$ be the formula $\psi(x,y)\wedge\forall z<y\,\neg\psi(x,z)$. It is clear that $\psi_f(x,y)$ satisfies the claim. For $n\ge1$, let $\psi_1(x,y,z)\in\Sigma_{n-1}$ such that $\psi(x,y)$ is $\forall z\,\psi_1(x,y,z)$. Let $\psi_f(x,y)$ be the following formula

$$\mathrm{Seq}(y)\wedge\psi(x,\mathrm{lg}(y))\wedge\forall j<\mathrm{lg}(y)\,[\neg\psi_1(x,j,(y)_j)\wedge\forall z<(y)_j\,\psi_1(x,j,z)].$$

To prove *(ii)* follow the proof of $\mathbf{I}\Sigma_n\Longrightarrow\mathbf{FAC}(\Sigma_n)$ (see lemma **I.2.35** in [10]). Parts *(i)* and *(iii)* are easy. □

From this result, using contraction of quantifiers, it follows that

Claim 3.5. *If $\mathbf{T}\Longrightarrow\mathbf{I}\Sigma_n$ then $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})=\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n+Func_n(\mathbf{T}))$.*

Below we shall see that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ can be described, for some theories, using $\mathbf{I}\Sigma_n$ and a family of Π_n–functions, $\Gamma \subseteq \mathrm{Gr}_n(\mathbf{T})$. In section 3.1, we prove that if $\mathbf{T}$ has Δ_{n+1}–collection then the functions in Γ are nondecreasing. In section 3.2, we shall see that if $\mathbf{T}$ is Δ_{n+1}–closed then every $\Delta_{n+1}(\mathbf{T})$ formula is equivalent to a bounded formula of $\mathcal{L}(\mathrm{Gr}_n(\mathbf{T}))$.

3.1. Π_n–functional classes.

Definition 3.6. (1) *Let $\Gamma \subseteq \Pi_n$. We say that Γ is a Π_n–functional class if $\mathbf{I}\Sigma_n + \Gamma^*$ is consistent.*

(2) *Let $\mathbf{T}$ be a theory. We say that $\mathbf{T}$ is Π_n–functional if there exists a Π_n–functional class Γ such that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma^*)$. In this case we say that Γ is a Π_n–functional class for $\mathbf{T}$.*

Let us notice that if Γ is a Π_n–functional class for $\mathbf{T}$, then, by 2.2, $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \iff \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*)$; and, since $\mathbf{I}\Sigma_n + \Gamma^*$ is Π_{n+2} axiomatizable, $\mathbf{T} \Longrightarrow \mathbf{I}\Sigma_n + \Gamma^*$.

Lemma 3.7. *Let Γ be a Π_n–functional class. Then*

(1) $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \Gamma^*)$.

(2) *$\mathbf{I}\Sigma_n + \Gamma^*$ has Δ_{n+1}–collection. So, it has Δ_{n+1}–induction.*

Proof. (*(1)*): Let $\theta(x,y) \in \Sigma_n$ such that $\mathbf{B}\Sigma_{n+1} + \Gamma^* \vdash \forall x\, \exists y\, \theta(x,y)$. Let us suppose that $\mathbf{I}\Sigma_n + \Gamma^* \nvdash \forall x\, \exists y\, \theta(x,y)$. Let $\mathfrak{A} \models \mathbf{I}\Sigma_n + \Gamma^* + \neg\forall x\, \exists y\, \theta(x,y)$ and $a \in \mathfrak{A}$ such that $\mathfrak{A} \models \forall y\, \neg\theta(a,y)$. Let

$$\mathbf{T} = \mathbf{ED}(\mathfrak{A}) + \{\exists! x\, \psi(x,a) \rightarrow \exists x < \mathbf{c}\, \psi(x,a) :\ \psi(x,y) \in \Sigma_{n+1}\}$$

(where $\mathbf{ED}(\mathfrak{A})$ is the elementary diagram of $\mathfrak{A}$ and $\mathbf{c}$ is a new constant symbol). By compactness, $\mathbf{T}$ is consistent. Let $\mathfrak{B} \models \mathbf{T}$. Then $\mathfrak{A} \prec \mathfrak{B}$, $\mathfrak{B} \models \mathbf{I}\Sigma_n + \Gamma^* + \neg\forall x\, \exists y\, \theta(x,y)$ and, by 1.5, $\mathcal{I}_{n+1}(\mathfrak{B}, a) \prec_n^e \mathfrak{B}$ and it is proper. So, $\mathcal{I}_{n+1}(\mathfrak{B}, a) \models \mathbf{B}\Sigma_{n+1} + \Gamma^*$. Since $\mathcal{I}_{n+1}(\mathfrak{B}, a) \models \forall y\, \neg\theta(a,y)$, this gives the desired contradiction.
(*(2)*): It follows from *(1)*, 2.8 and 2.15. □

Theorem 3.8. *Let $\mathbf{T}$ be a consistent theory. Then*

$$\mathbf{T} \text{ has } \Delta_{n+1}\text{–collection} \iff \mathbf{T} \text{ is } \Pi_n\text{–functional.}$$

Proof. ($\Longleftarrow$): Let Γ be a Π_n–functional class for $\mathbf{T}$. Hence, by 3.7–*(2)*, since $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma^*)$, $\mathbf{T}$ has Δ_{n+1}–collection.
($\Longrightarrow$): Let

$$\Gamma = \{\varphi(x,y) \in \Pi_n^- :\ \mathbf{T} \vdash \forall x\, \exists y\, \varphi(x,y),\ \mathbf{I}\Sigma_n \vdash \mathrm{IPF}(\varphi)\}.$$

Let us see that Γ is a Π_n–functional class for $\mathbf{T}$. It is enough to prove that for all $\theta(x,y) \in \Pi_n^-$ if $\mathbf{T} \vdash \forall x\, \exists y\, \theta(x,y)$ then $\mathbf{I}\Sigma_n + \Gamma^* \vdash \forall x\, \exists y\, \theta(x,y)$. We consider the following cases.
<u>Case 1</u>: $n = 0$: Let $\mathcal{C}_\theta(x,y)$ be the formula

$$\forall u \le x\, \exists v \le y\, \theta(u,v) \wedge \forall w < y\, \exists u \le x\, \forall v \le w\, \neg\theta(u,v).$$

That is, $y = \max(\{v :\ \exists u \le x\, [v = (\mu z)(\theta(u,z))]\})$. We have that

(i) $\mathbf{I}\Delta_0 \vdash \forall x\,\exists y\,\mathcal{C}_\theta(x,y) \to \forall x\,\exists y\,\theta(x,y)$,
(ii) $\mathbf{I}\Delta_0 \vdash \mathrm{IPF}(\mathcal{C}_\theta)$, and
(iii) $\mathbf{B}^*\Delta_1(\mathbf{T}) \vdash \forall x\,\exists y\,\mathcal{C}_\theta(x,y) \leftrightarrow \forall x\,\exists y\,\theta(x,y)$. (Since $\exists y\,\theta(x,y) \in \Delta_1(\mathbf{T})$).

Since $\mathbf{T}$ has Δ_1–collection and $\mathbf{T} \vdash \forall x\,\exists y\,\theta(x,y)$, by *(iii)*, $\mathbf{T} \vdash \forall x\,\exists y\,\mathcal{C}_\theta(x,y)$; hence, by *(ii)*, $\mathcal{C}_\theta(x,y) \in \Gamma$. So, $\mathbf{I}\Delta_0 + \Gamma^* \vdash \forall x\,\exists y\,\mathcal{C}_\theta(x,y)$, and, by *(i)*, $\mathbf{I}\Delta_0 + \Gamma^* \vdash \forall x\,\exists y\,\theta(x,y)$, as required.

Case 2: $n \geq 1$: Since $\mathbf{T}$ has Δ_{n+1}–collection, $\mathbf{T} \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{I}\Sigma_n$; hence, $(1 \leq n)$ we can use predicates and functions associated to Gödel's β function. For example, we will use the following $\Delta_1(\mathbf{I}\Sigma_1)$ formulas: $\mathrm{Seq}(x)$: "x is a sequence"; $\mathrm{lg}(x) = y$: "y is the length of x"; $(x)_j = y$: "y is the j–th projection of x". Let $\theta'(x,y,z) \in \Sigma_{n-1}$ such that $\theta(x,y)$ is $\forall z\,\theta'(x,y,z)$. Let $\mathcal{C}_\theta(x,y)$ be the formula

$$\begin{cases} \mathrm{Seq}(y) \wedge \mathrm{lg}(y) = 2 \wedge \mathrm{Seq}((y)_1) \wedge \mathrm{lg}((y)_1) = (y)_0 \wedge \\ \forall w < (y)_0\,[\mathrm{Seq}((y)_{1,w}) \wedge \mathrm{lg}((y)_{1,w}) = 2 \wedge (y)_{1,w,0} \leq x] \wedge \\ \forall u \leq x\,\exists v \leq (y)_0\,\forall z\,\theta'(u,v,z) \wedge \\ \forall w < (y)_0 \begin{cases} \forall u < (y)_{1,w,0}\,\exists v \leq w\,\forall z\,\theta'(u,v,z) \wedge \\ \forall v \leq w\,\exists z \leq (y)_{1,w,1}\,\neg\theta'((y)_{1,w,0},v,z) \wedge \\ \forall t < (y)_{1,w,1}\,\exists v \leq w\,\forall z \leq t\,\theta'((y)_{1,w,0},v,z) \end{cases} \end{cases}$$

We give an informal description of $\mathcal{C}_\theta(x,y)$. The interpretation of the first two lines is clear. The other parts of $\mathcal{C}_\theta(x,y)$ are developed to get

$$(y)_0 = (\mu w)[\forall u \leq x\,\exists v \leq w\,\forall z\,\theta'(u,v,z)]$$
$$\forall w < (y)_0 \begin{cases} (y)_{1,w,0} = (\mu u)[\neg\exists v \leq w\,\forall z\,\theta'(u,v,z)] \\ (y)_{1,w,1} = (\mu t)[\forall v \leq w\,\exists z \leq t\,\neg\theta'((y)_{1,w,0},v,z)] \end{cases}$$

Since $\mathcal{C}_\theta(x,y) \in \Pi_n(\mathbf{B}\Sigma_n)$, $\mathcal{C}_\theta(x,y) \in \Pi_n(\mathbf{T}), \Pi_n(\mathbf{I}\Sigma_n)$. We also have that

(i) $\mathbf{I}\Sigma_n \vdash \forall x\,\exists y\,\mathcal{C}_\theta(x,y) \to \forall x\,\exists y\,\theta(x,y)$,
(ii) $\mathbf{I}\Sigma_n \vdash \mathrm{IPF}(\mathcal{C}_\theta)$, and
(iii) $\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \vdash \forall x\,\exists y\,\mathcal{C}_\theta(x,y) \leftrightarrow \forall x\,\exists y\,\theta(x,y)$.

The proofs of *(i)* and *(ii)* are trivial, see the informal description of $\mathcal{C}_\theta(x,y)$ given above. To prove *(iii)* it is enough to see that

Claim 3.9. $\mathbf{B}^*\Delta_{n+1}(\mathbf{T}) \vdash \forall x\,\exists y\,\theta(x,y) \to \forall x\,\exists y\,\mathcal{C}_\theta(x,y)$.

Proof of Claim. Let $\mathfrak{A} \models \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$ such that $\mathfrak{A} \models \forall x\,\exists y\,\theta(x,y)$ and let $a \in \mathfrak{A}$. Since $\exists y\,\theta(x,y) \in \Delta_{n+1}(\mathbf{T})$, $\mathfrak{A} \models \exists y\,\forall u \leq a\,\exists v \leq y\,\theta(u,v)$. So, there exists $b \in \mathfrak{A}$ such that $\mathfrak{A} \models b = (\mu y)[\forall u \leq a\,\exists v \leq y\,\theta(u,v)]$. For each $d < b$ let

$$c_d = (\mu u)[u \leq a \wedge \neg\exists v \leq d\,\theta(u,v)].$$

So, for every $d < b$, $\mathfrak{A} \models \exists t\,\forall v \leq d\,\exists z \leq t\,\neg\theta'(c_d,v,z)$. Let

$$e_d = (\mu t)[\forall v \leq d\,\exists z \leq t\,\neg\theta'(c_d,v,z)].$$

In what follows we shall see that the elements $\langle c_0,e_0\rangle,\ldots,\langle c_{b-1},e_{b-1}\rangle$ can be given as a sequence. Let $\varphi(w,p,x) \in \Pi_n(\mathbf{B}\Sigma_n)$ be the formula

$$\begin{cases} \mathrm{Seq}(p) \wedge \mathrm{lg}(p) = 2 \wedge (p)_0 \leq x \wedge \forall u < (p)_0\,\exists v \leq w\,\forall z\,\theta'(u,v,z) \wedge \\ \forall v \leq w\,\exists z \leq (p)_1 \neg\,\theta'((p)_0,v,z) \wedge \forall t < (p)_1\,\exists v \leq w\,\forall z \leq t\,\theta'((p)_0,v,z) \end{cases}$$

We have that $\mathfrak{A} \models \forall w < b\,\exists p\,\varphi(w,p,a)$ (if $w < b$, take $p = \langle c_w,e_w\rangle$). Now let $\Psi(x,y',w,p) \in \Sigma_{n+1}(\mathbf{B}\Sigma_n)$ be the formula

$$[y' \leq w \wedge p = 0] \vee \begin{cases} [\exists y < y' \, \forall u \leq x \, \exists v \leq y \, \theta(u,v) \wedge p = 0] \vee \\ [\exists u \leq x \, \forall v \leq y' \, \neg\theta(u,v) \wedge p = 0] \vee \\ \varphi(w,p,x) \end{cases}$$

Since $\exists p \, \Psi(x, y', w, p) \in \Delta_{n+1}(\mathbf{T})$ and $\mathfrak{A} \models \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$, then there exists $\tilde{q} \in \mathfrak{A}$ such that $\mathfrak{A} \models \forall w < b \, \exists p \leq \tilde{q} \, \Psi(a, b, w, p)$. Let $s \in \mathfrak{A}$ such that $\mathfrak{A} \models \mathrm{Seq}(s) \wedge \mathrm{lg}(s) = b \wedge \forall j < b \, [(s)_j = \tilde{q}]$. Let $\delta(x, a, s, b) \in \Pi_n(\mathbf{B}\Sigma_n)$ be the formula $a < x \vee \exists y \leq \langle b, s \rangle \, \mathcal{C}_\theta(x, y)$. Then

$$\mathfrak{A} \models \delta(0, a, s, b) \wedge [\delta(x, a, s, b) \to \delta(x+1, a, s, b)].$$

So, $\mathfrak{A} \models \forall x \, \delta(x, a, s, b)$; hence, $\mathfrak{A} \models \exists y \leq \langle b, s \rangle \, \mathcal{C}_\theta(a, y)$. □

From *(i)–(iii)*, as in case $n = 0$, it follows $\mathbf{I}\Sigma_n + \Gamma^* \vdash \forall x \, \exists y \, \theta(x, y)$. □

Corollary 3.10. *Let $\mathbf{T} \Longrightarrow \mathbf{I}\Sigma_n$. The following conditions are equivalent:*

(1) *$\mathbf{T}$ is Π_n–functional.*
(2) *Every total Π_n–function of $\mathbf{T}$ is bounded by a total increasing function; that is, for every $\varphi(x, y) \in \Pi_n$ such that $\mathbf{T} \vdash \forall x \, \exists! y \, \varphi(x, y)$ there exists $\mathcal{C}_\varphi(x, y) \in \Pi_n$ such that $\mathbf{T} \vdash \forall x \, \exists y \, \mathcal{C}_\varphi(x, y) \wedge IPF(\mathcal{C}_\varphi)$ and $\mathbf{T} \vdash \mathcal{C}_\varphi(x, y) \to \exists y' \leq y \, \varphi(x, y')$.*

Remark 3.11. Here we study the relationship among being Π_n–functional, axiomatization and conservativeness, for consistent theories.

Claim 3.12. *Let $\mathbf{T}$ be such that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$. Then*

(i) *$\mathbf{T}$ is Π_n–functional.*
(ii) *If $\mathbf{T}'$ is Σ_{n+2}–axiomatizable then*

$$\mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \mathbf{T} + \mathbf{T}') = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{T}').$$

Proof of Claim. (*(i)*): By 2.8, $\mathbf{B}\Sigma_{n+1} + \mathbf{T}$ has $\Delta_{n+1}(\mathbf{T})$–collection; so, by the hypothesis, **2.15** and 3.8, $\mathbf{T}$ is Π_n–functional.
(*(ii)*): Follows from $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$. □

Claim 3.13. *Let $\mathbf{T}$ be Π_{n+2}–axiomatizable and let $\mathbf{T}'$ be Σ_{n+2}–axiomatizable.*

(i) *$\mathbf{T}$ is Π_n–functional if and only if $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$.*
(ii) *If $\mathbf{T}$ is Π_n–functional and $\mathbf{T} + \mathbf{T}'$ is consistent, $\mathbf{T} + \mathbf{T}'$ is Π_n–functional.*

Proof of Claim. *(ii)* follows from *(i)* and 3.12. Let us see *(i)*. Let Γ be a Π_n–functional class for $\mathbf{T}$. Then

$$\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \Gamma^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \mathbf{T}).$$

Where the first identity follows from 3.7–*(1)* and the last one, since $\mathbf{T}$ is Π_{n+2}–axiomatizable, from $\mathbf{T} \Longleftrightarrow \mathbf{I}\Sigma_n + \Gamma^*$. □

Claim 3.14. *If $\mathbf{T}$ is Σ_2–axiomatizable and $\mathbf{T} \Longrightarrow \mathbf{I}\Delta_0$ then $\mathbf{T}$ is Π_0–functional. In particular, $\mathbf{I}\Pi_1^-$, $\mathbf{Th}_{\Pi_1}(\mathcal{N})$, $\mathbf{I}\Delta_1^-$ and $\mathbf{L}\Delta_1^-$ are Π_0–functional.*

By a result of D. Leivant (see [16]), for $n \geq 1$, if $\mathbf{T}$ is a sound and Σ_{n+2}–axiomatizable theory, then $\mathbf{T}$ does not extend $\mathbf{I}\Sigma_n$. By **2.1** in [8], this is also true for any consistent theory. As $\mathbf{B}^*\Delta_{n+1}(\mathbf{T})$ extends $\mathbf{I}\Sigma_n$ we get that

Claim 3.15. ($n \geq 1$) *If* $\mathbf{T}$ *is* Σ_{n+2}*–axiomatizable and consistent then* $\mathbf{T}$ *is not* Π_n*–functional. In particular, for* $n \geq 1$, $\mathbf{I}\Pi^-_{n+1}$, $\mathbf{I}\Delta^-_{n+1}$ *and* $\mathbf{L}\Delta^-_{n+1}$ *are not* Π_n*–functional.*

Remark 3.16. Now we give some examples of Π_{n+2}–axiomatizable theories which do not have Δ_{n+1}–collection. Suppose that $n \geq 1$, $\mathbf{T} \Longrightarrow \mathbf{I}\Sigma^-_n$ and $\mathbf{Th}_{\Pi_{n+1}}(\mathbf{T}) \neq \mathbf{Th}_{\Pi_{n+1}}(\mathcal{N})$. Then (see [9], theorem **3.7**), it holds that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \not\Longrightarrow \mathbf{L}\Delta^-_{n+1}$. Hence, there exist $\varphi(x) \in \Sigma^-_n$ and $\psi(x) \in \Pi^-_n$ such that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \nvdash \forall x\,(\varphi(x) \leftrightarrow \psi(x)) \rightarrow \mathbf{L}_\varphi$. Let $\theta \in \Pi_{n+2}$ be the sentence $\forall x\,(\varphi(x) \leftrightarrow \psi(x))$. Then $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) + \theta$ does not have Δ_{n+1}–minimization; hence, it does not have Δ_{n+1}–collection. From this we get:

Claim 3.17. *($n \geq 1$) Every* Σ_1*–theory,* Π_{n+2}*–axiomatizable and (consistent) extension of* $\mathbf{I}\Sigma^-_n$ *has a* Π_{n+2}*–axiomatizable extension that does not have* Δ_{n+1}*–collection.*

We now consider case $n = 0$. By theorem **3.2** in [9], if $\mathbf{T}$ is an extension of $\mathbf{I}\Delta_0$ such that $\mathbf{T} + \mathbf{exp}$ is consistent and $\mathbf{Th}_{\Pi_1}(\mathbf{T} + \mathbf{exp}) \neq \mathbf{Th}_{\Pi_1}(\mathcal{N})$ then $\mathbf{Th}_{\Pi_2}(\mathbf{T}) \not\Longrightarrow \mathbf{L}\Delta^-_1$. This gives the following result (related with some results of Beklemishev (see [2], theorems **6.1** and **6.2**)).

Claim 3.18. *Let* $\mathbf{T}$ *be a* Σ_1*–theory,* Π_2*–axiomatizable, extension of* $\mathbf{I}\Delta_0$ *such that* $\mathbf{T}+\mathbf{exp}$ *is consistent. Then there exists a* Π_2*–axiomatizable extension of* $\mathbf{T}$ *which does not have* Δ_1*–collection.*

We now consider Δ_{n+1}–induction. Next result has also been obtained by Beklemishev for $n = 0$ (see [4]).

Claim 3.19. *Every theory,* $\mathbf{T}$*,* Σ_{n+1}*–definable in* $\mathcal{N}$*,* Π_{n+2}*–axiomatizable and consistent with* $\mathbf{PA} + \mathbf{Th}_{\Pi_n}(\mathcal{N})$ *has a* Π_{n+2}*–axiomatizable extension which does not have* Δ_{n+1}*–induction.*

Proof of Claim. Follows from $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) \not\Longrightarrow \mathbf{I}\Delta^-_{n+1}$, see corollary **4.6** in [9]. □

3.2. Π_n–Parikh pairs.

Definition 3.20. *Let* $\Gamma \subseteq \Pi_n$ *and* $\Gamma_1 \subseteq \Pi_{n+2}$ *such that Func*$(\Gamma) \subseteq \Gamma_1$*. We say that* (Γ, Γ_1) *is a* Π_n*–Parikh pair if* $\mathbf{I}\Sigma_n + \Gamma_1$ *is consistent and*

1. *for all* $\psi(\vec{x}, \vec{y}) \in \Pi_n$ *such that* $\mathbf{I}\Sigma_n + \Gamma_1 \vdash \forall \vec{x}\, \exists \vec{y}\, \psi(\vec{x}, \vec{y})$ *there exists a term of* $\mathcal{L}(\Gamma)$*,* $t(\vec{x})$*, such that* $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists \vec{y} \leq t(\vec{x})\, \psi(\vec{x}, \vec{y})$*, and*
2. *for all* $\theta(\vec{x}) \in \Delta^\Gamma_0$ *there exists* $\psi(\vec{x}) \in \Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma_1)$ *such that*
$$(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \theta(\vec{x}) \leftrightarrow \psi(\vec{x}).$$

Definition 3.21. *We say that* $\mathbf{T}$ *is a* Π_n*–Parikh theory if there exists a* Π_n*–Parikh pair* (Γ, Γ_1) *such that* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma_1)$*.*

Remark 3.22. Here we give some basic facts on Π_n–Parikh pairs. We first observe that $(\mathrm{Gr}_n(\mathbf{T}), \mathrm{Func}_n(\mathbf{T}))$ satisfies condition *(1)* of definition 3.20.

Claim 3.23. *Let $\psi(\vec{x},\vec{y}) \in \Pi_n^-$ such that $\mathbf{T} \vdash \forall \vec{x}\, \exists \vec{y}\, \psi(\vec{x},\vec{y})$. There is a term of $\mathcal{L}(Gr_n(\mathbf{T}))$, $t(\vec{x})$, such that $(\mathbf{I}\Sigma_n + Func_n(\mathbf{T}))_{Gr_n(\mathbf{T})} \vdash \forall \vec{x}\, \exists \vec{y} \le t(\vec{x})\, \psi(\vec{x},\vec{y})$.*

Proof of Claim. Let $\psi'(u,v)$ be $\forall \vec{x} \le u\, \forall \vec{y} \le v\, \psi^c(u,v,\vec{x},\vec{y})$, where $\psi^c(u,v,\vec{x},\vec{y})$ is as in 3.3. Then $\mathbf{T} \vdash \forall u\, \exists v\, \psi'(u,v)$. Let $\theta(u,v) \in \Pi_n^-$ be $\psi'_f(u,v)$, see 3.4, and let $t(\vec{x})$ be the term $G_\theta(\mathbf{J}_k(x_1,\dots,x_k))$ (where $\mathbf{J}_k(x_1,\dots,x_k)$ is a term of $\mathcal{L}(\mathrm{Gr}_n(\mathbf{T}))$ associated to Cantor's function used in contraction of quantifiers). Then $(\mathbf{I}\Sigma_n + \mathrm{Func}_n(\mathbf{T}))_{\mathrm{Gr}_n(\mathbf{T})} \vdash \forall \vec{x}\, \exists \vec{y} \le t(\vec{x})\, \psi(\vec{x},\vec{y})$. □

In what follows (Γ, Γ_1) shall denote a Π_n–Parikh pair.

Claim 3.24. *($n \ge 1$) Let $\varphi(\vec{x},\vec{y}) \in \Pi_{n-1}$ and $\psi(\vec{x},\vec{y}) \in \Sigma_{n-1}$. Then there exist terms of $\mathcal{L}(\Gamma)$, $t(\vec{x})$, $t'(\vec{x})$, such that*

$$(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \exists \vec{y}\, \varphi(\vec{x},\vec{y}) \leftrightarrow \exists \vec{y} \le t(\vec{x})\, \varphi(\vec{x},\vec{y}),$$
$$(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{y}\, \psi(\vec{x},\vec{y}) \leftrightarrow \forall \vec{y} \le t'(\vec{x})\, \psi(\vec{x},\vec{y}).$$

Proof of Claim. Let $\varphi_1(\vec{x},\vec{y}) \in \Pi_n$ be the formula $\varphi(\vec{x},\vec{y}) \vee (\forall \vec{z}\, \neg\varphi(\vec{x},\vec{z}) \wedge \vec{y} = 0)$. Since $\mathbf{I}\Sigma_n + \Gamma_1 \vdash \forall \vec{x}\, \exists \vec{y}\, \varphi_1(\vec{x},\vec{y})$, by 3.20–*(1)*, there exists a term of $\mathcal{L}(\Gamma)$, $t(\vec{x})$, such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\ \exists \vec{y} \le t(\vec{x})\, \varphi_1(\vec{x},\vec{y})$; hence,

$$(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \exists \vec{y}\, \varphi(\vec{x},\vec{y}) \leftrightarrow \exists \vec{y} \le t(\vec{x})\, \varphi(\vec{x},\vec{y}).$$

For $\psi \in \Sigma_{n-1}$ we obtain the result, from the above one, using $\neg\psi$. □

Claim 3.25. *Let $\varphi(\vec{x},\vec{y}) \in \Delta_0^\Gamma$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists \vec{y}\, \varphi(\vec{x},\vec{y})$. There is a term $t(\vec{x})$ of $\mathcal{L}(\Gamma)$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists \vec{y} \le t(\vec{x})\, \varphi(\vec{x},\vec{y})$.*

Proof of Claim. By 3.20–*(2)*, there exists $\psi(\vec{x},\vec{y},z) \in \Pi_n$ such that $\exists z\, \psi(\vec{x},\vec{y},z) \in \Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma_1)$ and $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \varphi(\vec{x},\vec{y}) \leftrightarrow \exists z\, \psi(\vec{x},\vec{y},z)$. Let $t(\vec{x})$ be a term of $\mathcal{L}(\Gamma)$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists \vec{y}, z \le t(\vec{x})\, \psi(\vec{x},\vec{y},z)$. Then $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists \vec{y} \le t(\vec{x})\, \varphi(\vec{x},\vec{y})$. □

Claim 3.26. *Let $\varphi(\vec{x}) \in \Delta_1^\Gamma((\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma)$. Then there exists $\theta(\vec{x}) \in \Delta_0^\Gamma$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \varphi(\vec{x}) \leftrightarrow \theta(\vec{x})$.*

Proof of Claim. Assume $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \exists y\, \varphi'(\vec{x},y) \leftrightarrow \forall y\, \psi'(\vec{x},y)$, where $\varphi'(\vec{x},y)$ and $\psi'(\vec{x},y)$ are Δ_0^Γ and $\varphi(\vec{x})$ is $\exists y\, \varphi'(\vec{x},y)$. Let $\delta(\vec{x},y) \in \Delta_0^\Gamma$ the formula $\varphi'(\vec{x},y) \vee \neg\, \psi'(\vec{x},y)$. Then $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists y\, \delta(\vec{x},y)$; so, by 3.25, there exists a term $t(\vec{x})$ of $\mathcal{L}(\Gamma)$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{x}\, \exists y \le t(\vec{x})\, \delta(\vec{x},y)$. Hence, $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \varphi(\vec{x}) \leftrightarrow \exists y \le t(\vec{x})\, \varphi'(\vec{x},y)$. □

Theorem 3.27. *Let (Γ, Γ_1) be a Π_n–Parikh pair, $\varphi(\vec{x}) \in \Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma_1)$. Then there exists $\theta(\vec{x}) \in \Delta_0^\Gamma$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \varphi(\vec{x}) \leftrightarrow \theta(\vec{x})$.*

Proof. For $n = 0$ the result follows from 3.26. Suppose that $n \ge 1$. Let $\varphi_0(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_n)$, $\psi_0(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_n) \in \Delta_0$ such that (assume n even)

- **(−):** $\varphi(\vec{x}) \equiv \exists \vec{y}\, \forall \vec{z}_1\, \exists \vec{z}_2 \dots \exists \vec{z}_n\, \varphi_0(\vec{x},\vec{y},\vec{z}_1,\vec{z}_2,\dots,\vec{z}_n)$,
- **(−):** $\psi(\vec{x}) \equiv \forall \vec{y}\, \exists \vec{z}_1\, \forall \vec{z}_2 \dots \forall \vec{z}_n\, \psi_0(\vec{x},\vec{y},\vec{z}_1,\vec{z}_2,\dots,\vec{z}_n)$, and
- **(−):** $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \varphi(\vec{x}) \leftrightarrow \psi(\vec{x})$.

By 3.24, there exist $t_1(\vec{x},\vec{y})$, $t_2(\vec{x},\vec{y},\vec{z}_1),\dots,t_n(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_{n-1})$ terms of $\mathcal{L}(\Gamma)$ such that the following formulas are equivalent in $(\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma$

(–): $\forall\vec{z}_1\,\exists\vec{z}_2\dots\exists\vec{z}_n\,\varphi_0(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_n)$.

(–): $\forall\vec{z}_1\le t_1(\vec{x},\vec{y})\,\exists\vec{z}_2\le t_2(\vec{x},\vec{y},\vec{z}_1)\dots\exists\vec{z}_n\le t_n(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_{n-1})\,\varphi_0$.

Let $\varphi'(\vec{x},\vec{y})\in\Delta_0^\Gamma$ be the last formula. Analogously, we get that there exist $t'_1(\vec{x},\vec{y})$, $t'_2(\vec{x},\vec{y},\vec{z}_1),\dots,t'_n(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_{n-1})$ terms of $\mathcal{L}(\Gamma)$ such that the following formulas are equivalent in $(\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma$

(–): $\exists\vec{z}_1\forall\vec{z}_2\dots\forall\vec{z}_n\,\psi_0(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_n)$.

(–): $\exists\vec{z}_1\le t'_1(\vec{x},\vec{y})\,\forall\vec{z}_2\le t'_2(\vec{x},\vec{y},\vec{z}_1)\dots\forall\vec{z}_n\le t'_n(\vec{x},\vec{y},\vec{z}_1,\dots,\vec{z}_{n-1})\,\psi_0$.

Let $\psi'(\vec{x},\vec{y})\in\Delta_0^\Gamma$ be the last formula. Then

$$(\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma\vdash\exists\vec{y}\,\varphi'(\vec{x},\vec{y})\leftrightarrow\forall\vec{y}\,\psi'(\vec{x},\vec{y}).$$

So, $\exists\vec{y}\,\varphi'(\vec{x},\vec{y})\in\Delta_1^\Gamma((\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma)$ and, by 3.26, there is $\theta(\vec{x})\in\Delta_0^\Gamma$ such that $(\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma\vdash\exists\vec{y}\,\varphi'(\vec{x},\vec{y})\leftrightarrow\theta(\vec{x})$; hence, $(\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma\vdash\varphi(\vec{x})\leftrightarrow\theta(\vec{x})$, as required. □

Theorem 3.28. *Let* $\mathbf{T}$ *be an extension of* $\mathbf{I}\Sigma_n$. *Then*

$$\mathbf{T}\text{ is }\Pi_n\text{–Parikh}\iff\mathbf{T}\text{ is }\Delta_{n+1}\text{–closed.}$$

Proof. ($\Longrightarrow$): Let $\varphi(x,\vec{v})\in\Delta_{n+1}(\mathbf{T})$ and $t(\vec{v})\in\mathbf{Term}(\mathcal{L})$. Let us see that $\forall x\le t(\vec{v})\,\varphi(x,\vec{v})\in\Delta_{n+1}(\mathbf{T})$. Let (Γ,Γ_1) be a Π_n–Parikh pair for $\mathbf{T}$. Then, using 3.27 and 3.20–*(2)*, there exist $\theta(x,\vec{v})\in\Delta_0^\Gamma$ and $\psi(\vec{v})\in\Delta_{n+1}(\mathbf{T})$ such that $(\mathbf{I}\Sigma_n+\Gamma_1)_\Gamma$ proves $\varphi(x,\vec{v})\leftrightarrow\theta(x,\vec{v})$ and $\forall x\le t(\vec{v})\,\varphi(x,\vec{v})\leftrightarrow\psi(\vec{v})$.

($\Longleftarrow$): Let us prove that $(\mathrm{Gr}_n(\mathbf{T}),\mathrm{Func}_n(\mathbf{T}))$ is a Π_n–Parikh pair for $\mathbf{T}$. By 3.23, we only need to prove 3.20–*(2)*. The proof is by induction on the length of Δ_0^Γ–formulas. Let $\theta(\vec{x})\in\Delta_0^\Gamma$, we only consider the case where $\theta(\vec{x})$ is $\exists y\le t(\vec{x})\,\theta_0(\vec{x},y)$. By induction hypothesis there exists $\psi_0(\vec{x},y)\in\Delta_{n+1}(\mathbf{T})$ such that $(\mathbf{I}\Sigma_n+\mathrm{Func}_n(\mathbf{T}))_{\mathrm{Gr}_n(\mathbf{T})}\vdash\psi_0(\vec{x},y)\leftrightarrow\theta_0(\vec{x},y)$. Then, by 3.2–*(ii)*, there exists $\delta(\vec{x},v)\in\Delta_{n+1}(\mathbf{I}\Sigma_n+\mathrm{Func}_n(\mathbf{T}))$ such that

$$(\mathbf{I}\Sigma_n+\mathrm{Func}_n(\mathbf{T}))_{\mathrm{Gr}_n(\mathbf{T})}\vdash\exists v\,[\delta(\vec{x},v)\wedge\exists y\le v\,\psi_0(\vec{x},y)]\leftrightarrow\exists y\le t(\vec{x})\,\theta_0(\vec{x},y).$$

As $\mathbf{T}$ is Δ_{n+1}–closed, there exists $\psi(\vec{x},v)\in\Delta_{n+1}(\mathbf{T})$ such that

$$\mathbf{T}\vdash\exists y\le v\,\psi_0(\vec{x},y)\leftrightarrow\psi(\vec{x},v).$$

Since $\exists v\,[\delta(\vec{x},v)\wedge\psi(\vec{x},v)]\in\Delta_{n+1}(\mathbf{T})$, this proves the result. □

4. Extended Parikh's Theorem

In this section, we shall see that for some kind of Π_n–functional class Γ there exists an extension of $\mathcal{L}$ such that each $\Delta_{n+1}(\mathbf{I}\Sigma_n+\Gamma^*)$ formula is equivalent to a bounded formula of $\mathcal{L}(\Gamma)$.

4.1. Δ_0^Γ formulas as Δ_{n+1} formulas.

Lemma 4.1. *Let* $\Gamma\subseteq\Pi_n$ *and* $\Gamma_1\subseteq\Pi_{n+2}$ *such that Func*$(\Gamma)\subseteq\Gamma_1$ *and*

for all $s(\vec{x}),t(\vec{x},y)\in\mathbf{Term}(\mathcal{L}(\Gamma))$ *there exists* $t_s(\vec{x})\in\mathbf{Term}(\mathcal{L}(\Gamma))$ *such that* $(\mathbf{B}\Sigma_n+\Gamma_1)_\Gamma\vdash y\le s(\vec{x})\to t(\vec{x},y)\le t_s(\vec{x})$.

(1) *Let* $\varphi(\vec{x}) \in \Delta_0^\Gamma$. *Then there exist* $\psi(\vec{x}, z) \in \Sigma_n$, $\theta(\vec{x}, z) \in \Pi_n$ *and* $t(\vec{x}) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ *such that*

$$(\mathbf{B}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall z \geq t(\vec{x})\,[\varphi(\vec{x}) \leftrightarrow \psi(\vec{x}, z) \leftrightarrow \theta(\vec{x}, z)].$$

(2) *Let* $\varphi(\vec{x}) \in \Delta_0^\Gamma$. *Then there exists* $\delta(\vec{x}) \in \Delta_{n+1}(\mathbf{B}\Sigma_n + \Gamma_1)$ *such that* $(\mathbf{B}\Sigma_n + \Gamma_1)_\Gamma \vdash \varphi(\vec{x}) \leftrightarrow \delta(\vec{x})$.

For $n = 0$, $\mathbf{B}\Sigma_0$ *can be replaced by* $\mathbf{I}\Delta_0$ *(collection is not needed).*

Proof. By induction on the length of $\varphi(\vec{x})$ as in lemma **I.1.30** in [10]. □

Remark 4.2. Let Γ be a Π_n–functional class. We have the following results.

Claim 4.3. *For every* $t(\vec{x}, y), s(\vec{x}) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ *there exists a term* $t_s(\vec{x})$ *such that* $(\mathbf{I}\Delta_0 + \Gamma^*)_\Gamma \vdash y \leq s(\vec{x}) \rightarrow t(\vec{x}, y) \leq t_s(\vec{x})$. *So, lemma 4.1 holds for* $(\mathbf{B}\Sigma_n + \Gamma^*)_\Gamma$ *and* (Γ, Γ^*) *satisfies part (2) of definition 3.20.*

Proof of Claim. By 3.2–*(i)*, the result follows taking $t_s(\vec{x})$ as $t(\vec{x}, s(\vec{x}))$. □

Claim 4.4. $(\mathbf{I}\Sigma_n + \Gamma^*)_\Gamma \Longrightarrow \mathbf{I}\Delta_0^{\Gamma^*}$.

Proof of Claim. Let $\varphi(x) \in \Delta_0^\Gamma$ and $\mathfrak{A} \models (\mathbf{I}\Sigma_n + \Gamma^*)_\Gamma$ such that $\mathfrak{A} \models \exists x\, \varphi(x)$. By 4.1–*(1)*, there exist $\psi(x, z) \in \Sigma_n$ and $t(x)$ term of $\mathcal{L}(\Gamma)$ such that $(\mathbf{B}\Sigma_n + \Gamma^*)_\Gamma \models \forall z \geq t(x)\,[\varphi(x) \leftrightarrow \psi(x, z)]$. Let $a \in \mathfrak{A}$ such that $\mathfrak{A} \models \varphi(a)$ and let $b = t(a)$. Then $\mathfrak{A} \models \psi(a, b)$. Since $\mathfrak{A} \models \mathbf{L}\Sigma_n$, there is $c \in \mathfrak{A}$ such that $\mathfrak{A} \models c = (\mu x)[\psi(x, b)]$. Since Γ is a Π_n–functional class, by 3.2–*(i)* $\mathfrak{A} \models c = (\mu x)[\varphi(x)]$; hence, $\mathfrak{A} \models \mathbf{L}_\varphi$. □

Remark 4.5. Here we prove that Π_0–functional classes provide examples of Π_0–Parikh pairs. In the next subsection we shall see that for $n \geq 1$ this is also true for some kind of Π_n–functional classes. In what follows Γ shall denote a Π_0–functional class. As in 4.4, using 4.1–*(1)* for $n = 0$, we get

Claim 4.6. $\mathbf{I}\Delta_0^{\Gamma^*} \Longleftrightarrow (\mathbf{I}\Delta_0 + \Gamma^*)_\Gamma$.

Claim 4.7 *(Parikh's theorem)***.** *Let* $\Gamma' \subseteq \Pi_1^\Gamma$. *For each* $\varphi(\vec{x}, \vec{y}) \in \Delta_0^\Gamma$ *such that* $\mathbf{I}\Delta_0^{\Gamma^*} + \Gamma' \vdash \forall \vec{x}\, \exists \vec{y}\, \varphi(\vec{x}, \vec{y})$ *there exists a term* $t(\vec{x})$ *of* $\mathcal{L}(\Gamma)$ *such that* $\mathbf{I}\Delta_0^{\Gamma^*} + \Gamma' \vdash \forall \vec{x}\, \exists \vec{y} \leq t(\vec{x})\, \varphi(\vec{x}, \vec{y})$.

Claim 4.8. (Γ, Γ^*) *is a* Π_0*–Parikh pair.*

Proof of Claim. By 4.7, part *(1)* of definition 3.20 holds for (Γ, Γ^*). So, the result follows from 4.3. □

4.2. Δ_{n+1} **formulas as** Δ_0^Γ **formulas. Strong** Π_n**–functional classes.** In order to improve 4.1, 4.4 and 4.6–4.8, we consider a special kind of Π_n–functional classes. Let Γ be a Π_n–functional class and $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*$. We shall also denote by $\mathfrak{A}$ the expansion of $\mathfrak{A}$ to $\mathcal{L}(\Gamma)$ given by: for every $a, b \in \mathfrak{A}$ and $\varphi \in \Gamma$

$$\mathfrak{A}(G_\varphi(a)) = b \quad \Longleftrightarrow \quad \mathfrak{A} \models \varphi(a, b).$$

Let $a_1, \dots, a_k \in \mathfrak{A}$. The simple initial segment of $\mathfrak{A}$ determined by $a_1, \dots, a_k$ is $\mathcal{S}_\Gamma(\mathfrak{A}, a_1, \dots, a_k) = \{b : b \le t(\vec{a}),\ t(\vec{x}) \in \mathbf{Term}(\mathcal{L}(\Gamma))\}$. Observe that if $\mathfrak{A} \models (\mathbf{I}\Delta_0 + \Gamma^*)_\Gamma$ then $\mathcal{S}_\Gamma(\mathfrak{A}, \vec{a})$ is an $\mathcal{L}(\Gamma)$–structure.

Definition 4.9. *Let Γ be a Π_n–functional class. We say that Γ is a strong Π_n–functional class if for every $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*$ and every $\mathfrak{I}$*

if $\mathfrak{I} \subset^e \mathfrak{A}$ as $\mathcal{L}(\Gamma)$ structures then $\mathfrak{I} \prec_n^e \mathfrak{A}$ as $\mathcal{L}$–structures.

Let us observe that every Π_0–functional class is a strong Π_0–functional class. Moreover, if Γ is a strong Π_n–functional class and Γ' is a Π_n–functional class such that $\Gamma \subseteq \Gamma'$, then Γ' is a strong Π_n–functional class.

Lemma 4.10. *($n \ge 1$) Let Γ be a strong Π_n–functional class. Then for every $k < n$,* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{k+2} + \Gamma^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_k + \Gamma^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Delta_0 + \Gamma^*)$.

Proof. Suppose that $\mathbf{B}\Sigma_{k+2} + \Gamma^* \vdash \forall x\, \exists y\, \varphi(x, y)$, where $\varphi(x, y) \in \Pi_n$ and $\mathbf{I}\Sigma_k + \Gamma^* \nvdash \forall x\, \exists y\, \varphi(x, y)$. By compacteness, $\mathbf{T}$ is consistent, where

$$\mathbf{T} = (\mathbf{I}\Sigma_k + \Gamma^*)_\Gamma + \forall y\, \neg\varphi(\mathbf{c}, y) + \{t(\mathbf{c}) < \mathbf{d} :\ t(x) \text{ term of } \mathcal{L}(\Gamma)\}.$$

Let $\mathfrak{A} \models \mathbf{T}$, $a = \mathfrak{A}(\mathbf{c})$ and $\mathfrak{B} = \mathcal{S}_\Gamma(\mathfrak{A}, a)$. Since Γ is a Π_n–functional class, $\mathfrak{B} \subset^e \mathfrak{A}$ as $\mathcal{L}(\Gamma)$–structures and, by the last group of axioms of $\mathbf{T}$, it is proper. Also, for all $\theta(x, y) \in \Gamma$ and $b \in \mathfrak{B}$ there exists $d \in \mathfrak{B}$ such that $\mathfrak{A} \models \theta(b, d)$. Since Γ is a strong Π_n–functional class and $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*$, $\mathfrak{B} \prec_n^e \mathfrak{A}$ as $\mathcal{L}$–structures. So, from $\mathfrak{A} \models \forall y\, \neg\varphi(a, y)$ we get that $\mathfrak{B} \models \forall y\, \neg\varphi(a, y)$. Since, $\mathfrak{B} \models \mathbf{I}\Delta_0 + \Gamma^*$; and, for $k < n$, $\mathfrak{B} \prec_{k+1}^e \mathfrak{A}$, then $\mathfrak{B} \models \mathbf{B}\Sigma_{k+2}$. So, $\mathfrak{B} \models \mathbf{B}\Sigma_{k+2} + \Gamma^*$ and $\mathfrak{B} \models \exists y\, \varphi(a, y)$. Contradiction. This proves the first identity. The second one follows from the first by induction on $k < n$. □

Remark 4.11. (Strength of 4.4, 4.6, 3.7, 4.1) In what follows let Γ be a strong Π_n–functional class.

Claim 4.12. *(i)* $\mathbf{I}\Delta_0^{\Gamma^*} \iff (\mathbf{I}\Sigma_n + \Gamma^*)_\Gamma \iff (\mathbf{I}\Delta_0 + \Gamma^*)_\Gamma$.
(ii) $\mathbf{I}\Delta_0 + \Gamma^* \iff \mathbf{I}\Sigma_n + \Gamma^*$.

Proof of Claim. By 4.4, $(\mathbf{I}\Sigma_n + \Gamma^*)_\Gamma \implies \mathbf{I}\Delta_0^{\Gamma^*} \implies (\mathbf{I}\Delta_0 + \Gamma^*)_\Gamma$. Let $\theta \in \Sigma_n$. Then $\mathbf{B}\Sigma_{n+1} \vdash \mathbf{I}_\theta$; so, by 4.10 (for $k = n - 1$), $\mathbf{I}\Delta_0 + \Gamma^* \vdash \mathbf{I}_\theta$. This proves *(i)*. Part *(ii)* follows from *(i)*. □

By 4.12, we can rewrite 3.7 as follows

Claim 4.13. *(i)* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Delta_0 + \Gamma^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B}\Sigma_{n+1} + \Gamma^*)$.
(ii) $\mathbf{I}\Delta_0 + \Gamma^* \implies \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*) \implies \mathbf{B}^*\Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma^*)$.

Claim 4.14. *(i) Let $\varphi(\vec{x}) \in \Delta_0^\Gamma$. There are $\psi(\vec{x}, z) \in \Sigma_n$, $\theta(\vec{x}, z) \in \Pi_n$ and $t(\vec{x})$ such that $\mathbf{I}\Delta_0^{\Gamma^*} \vdash \forall z \ge t(\vec{x})\, [\varphi(\vec{x}) \leftrightarrow \psi(\vec{x}, z) \leftrightarrow \theta(\vec{x}, z)]$.*
(ii) Let $\varphi(\vec{x}) \in \Delta_0^\Gamma$. Then there exists $\delta(\vec{x}) \in \Delta_{n+1}(\mathbf{B}\Sigma_n + \Gamma^*)$ such that $\mathbf{I}\Delta_0^{\Gamma^*} \vdash \varphi(\vec{x}) \leftrightarrow \delta(\vec{x})$.

Proof of Claim. For $n \geq 1$, $\mathbf{I}\Sigma_n \Longrightarrow \mathbf{B}\Sigma_n$. So, $(\mathbf{I}\Sigma_n + \Gamma^*)_\Gamma \Longrightarrow (\mathbf{B}\Sigma_n + \Gamma^*)_\Gamma$. Then the result follows from 4.1 and 4.12. □

Theorem 4.15 *(Extended Parikh's theorem (Strength of 4.7)).*
Let Γ be a strong Π_n–functional class and $\Gamma' \subseteq \Pi_{n+1} \cup \Pi_1^\Gamma$. For each $\varphi(x,y) \in \Pi_n \cup \Delta_0^\Gamma$ such that $(\mathbf{B}\Sigma_{n+1} + \Gamma' + \Gamma^*)_\Gamma \vdash \forall x \, \exists y \, \varphi(x,y)$ there exists a term $t(x)$ of $\mathcal{L}(\Gamma)$ such that $\mathbf{I}\Delta_0^{\Gamma^*} + \Gamma' \vdash \forall x \, \exists y \leq t(x) \, \varphi(x,y)$.

Proof. Deny the proposition's conclusion. We proceed as in 4.10. By compacteness the following theory is consistent ($\mathbf{c}$ and $\mathbf{d}$ are new constants)

$$\mathbf{T} = \begin{cases} \mathbf{I}\Delta_0^{\Gamma^*} + \Gamma' & + \quad \{\forall y \leq t(\mathbf{c}) \, \neg\varphi(\mathbf{c}, y) : \ t(x) \text{ term of } \mathcal{L}(\Gamma)\} \\ & + \quad \{t(\mathbf{c}) < \mathbf{d} : \ t(x) \text{ term of } \mathcal{L}(\Gamma)\} \end{cases}$$

Let $\mathfrak{A} \models \mathbf{T}$, $a = t(\mathbf{c})$ and $\mathfrak{B} = \mathcal{S}_\Gamma(\mathfrak{A}, a)$. Since Γ is a Π_n–functional class, $\mathfrak{B} \subset^e \mathfrak{A}$ as $\mathcal{L}(\Gamma)$–structures. Then $\mathfrak{B} \prec_n^e \mathfrak{A}$ as $\mathcal{L}$–structures. So, $\mathfrak{B} \models \forall y \, \neg\varphi(a,y)$ and, since $\mathfrak{A} \models \mathbf{I}\Sigma_n$ and $\mathfrak{A}$ is a proper extension of $\mathfrak{B}$ (last set of axioms of $\mathbf{T}$), $\mathfrak{B} \models (\mathbf{B}\Sigma_{n+1} + \Gamma' + \Gamma^*)_\Gamma$. Contradiction. □

Corollary 4.16. *(Strength of 4.8) If Γ is a strong Π_n–functional class then (Γ, Γ^*) is a Π_n–Parikh pair.*

4.3. Existence theorems of strong Π_n–functional classes.

Theorem 4.17. $(n \geq 1)$ *There is a strong Π_n–functional class, $\mathbb{H}_n$, such that*
for all $\varphi \in \mathbb{H}_n$, $\mathbf{I}\Sigma_{n-1} \vdash IPF(\varphi)$ and $\mathbf{I}\Sigma_n \Longleftrightarrow \mathbf{I}\Sigma_n + \mathbb{H}_n^*$.

Proof. For each $\theta(v,y) \in \Pi_{n-1}^-$ let $\theta'(x,w)$ be the following formula

$$\begin{cases} [\neg\exists v \leq x \, \exists y \, \theta(v,y) \wedge w = 0] \vee \\ \exists w_1, w_2 \leq w \begin{cases} w = \langle w_1, w_2 \rangle \wedge w_1 \leq x \wedge \\ \forall v \leq x \, [\exists y \, \theta(v,y) \rightarrow \exists y \leq w_2 \, \theta(v,y)] \wedge \\ \theta_{\mu,w_2}(w_1, w_2) \wedge \forall v \leq x \, [\theta_{\mu,w_2}(v, w_2) \rightarrow v \leq w_1] \end{cases} \end{cases}$$

It is clear that there is $\theta^*(x,w) \in \Pi_n$ such that $\mathbf{I}\Sigma_{n-1} \vdash \theta'(x,w) \leftrightarrow \theta^*(x,w)$. Let $\mathbb{H}_n = \{\theta^*(x,w) : \ \theta(v,y) \in \Pi_{n-1}\}$. Let $\theta(v,y) \in \Pi_{n-1}$. It holds that $\mathbf{I}\Sigma_{n-1} \vdash \text{IPF}(\theta^*)$ and $\mathbf{I}\Sigma_n \vdash \forall x \, \exists w \, \theta^*(x,w)$; so, $\mathbb{H}_n$ is a Π_n–functional class and $\mathbf{I}\Sigma_n \Longleftrightarrow \mathbf{I}\Sigma_n + \mathbb{H}_n^*$. Let us observe that $\mathbb{H}_1 \subseteq \mathbb{H}_2 \subseteq \cdots \subseteq \mathbb{H}_n \subseteq \ldots$. Now, by induction on $n \geq 1$, we prove that $\mathbb{H}_n$ is a strong Π_n–functional class. Let $\mathfrak{A} \models \mathbf{I}\Delta_0 + \mathbb{H}_n^*$ and $\mathfrak{I} \subset^e \mathfrak{A}$ such that

$(*)$ for all $\varphi(x,w) \in \mathbb{H}_n$, $a \in \mathfrak{I}$ there is $b \in \mathfrak{I}$ such that $\mathfrak{A} \models \varphi(a,b)$.

By induction on $n \geq 1$, using Tarski–Vaught's test, we prove that $\mathfrak{I} \prec_n \mathfrak{A}$.
$(n = 1)$: Let us see that $\mathfrak{I} \prec_1 \mathfrak{A}$. Let $\theta(v,y) \in \Pi_0$ and $a \in \mathfrak{I}$ such that $\mathfrak{A} \models \exists y \, \theta(a,y)$. Since $\theta^*(x,w) \in \mathbb{H}_1$ and $\mathfrak{A} \models \mathbf{I}\Delta_0 + \mathbb{H}_1^*$, then $\mathfrak{A} \models \forall x \, \exists y \, \theta^*(x,w)$. Since $a \in \mathfrak{I}$, by $(*)$, there exists $d \in \mathfrak{I}$ such that $\mathfrak{A} \models \theta^*(a,d)$. Since $\mathbf{I}\Delta_0 \vdash \theta'(x,w) \leftrightarrow \theta^*(x,w)$, $\mathfrak{A} \models \theta'(a,d)$; so, there exists $b \in \mathfrak{A}$ such that $b \leq d$ and $\mathfrak{A} \models \theta(a,b)$. Since $d \in \mathfrak{I}$ and $\mathfrak{I} \subset^e \mathfrak{A}$, then $b \in \mathfrak{I}$, as required.
$(n \rightarrow n+1)$: Since $\mathfrak{A} \models \mathbf{I}\Delta_0 + \mathbb{H}_n^*$ and, by induction hypothesis, $\mathbb{H}_n$ is a strong Π_n–functional class, we get that $\mathfrak{A} \models \mathbf{I}\Sigma_n + \mathbb{H}_n^*$. Let $\theta(x,y) \in \Pi_n$ and $a \in \mathfrak{I}$ such that

$\mathfrak{A} \models \exists y\, \theta(a, y)$. Now as in the case $n = 1$, using that $\mathfrak{A} \models \mathbf{I\Sigma}_n + \mathbb{H}_n^*$, we obtain that there exists $b \in \mathfrak{I}$ such that $\mathfrak{A} \models \theta(a, b)$. $\square$

Proposition 4.18 *(Strength of 3.8). If $\mathbf{T}$ has Δ_{n+1}–collection, there is a strong Π_n–functional class Γ such that $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Delta}_0 + \Gamma^*)$.*

Proof. Suppose that $n \geq 1$. By 3.8, there is a Π_n–functional class Γ_1 such that $\mathbf{Th}_{n+2}(\mathbf{T}) = \mathbf{Th}_{n+2}(\mathbf{I\Sigma}_n + \Gamma_1^*)$. Let $\Gamma = \mathbb{H}_n + \Gamma_1$. Then, Γ is a strong Π_n–functional class; so, by 4.17 and 4.12, $\mathbf{I\Sigma}_n + \Gamma_1^* \iff \mathbf{I\Delta}_0 + \Gamma^*$. Hence $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Sigma}_n + \Gamma_1^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I\Delta}_0 + \Gamma^*)$. $\square$

Lemma 4.19. *Let $\mathbf{T} \implies \mathbf{I\Sigma}_n$, $\mathfrak{A} \models \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$ and $a \in \mathfrak{A}$. If (Γ, Γ_1) and (Γ', Γ_1') are Π_n–Parikh pairs for $\mathbf{T}$ then $\mathcal{S}_\Gamma(\mathfrak{A}, a) = \mathcal{S}_{\Gamma'}(\mathfrak{A}, a)$.*

Proof. Let $b \in \mathcal{S}_\Gamma(\mathfrak{A}, a)$. There are $t(x) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ such that $b \leq t(a)$ and $\varphi(x, y) \in \Delta_{n+1}(\mathbf{T})$ such that $(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash t(x) = y \leftrightarrow \varphi(x, y)$. Let $s(x)$ be a term of $\mathcal{L}(\Gamma')$ such that $(\mathbf{I\Sigma}_n + \Gamma_1')_{\Gamma'} \vdash \forall x\, \exists y \leq s(x)\, \varphi(x, y)$. So, $b \leq s(a)$; hence, $b \in \mathcal{S}_{\Gamma'}(\mathfrak{A}, a)$. $\square$

Theorem 4.20. *Let $\mathbf{T}$ be an extension of $\mathbf{I\Sigma}_n$ and (Γ, Γ_1) a Π_n–Parikh pair for $\mathbf{T}$ (so, $\mathbf{T}$ is Δ_{n+1}–closed). The following conditions are equivalent*

1. *$\mathbf{T}$ has Δ_{n+1}–collection.*
2. *$(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \implies \mathbf{I\Delta}_0^\Gamma$.*
3. *For each $s(\vec{v}), t(\vec{v}, x) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ there exists $t_s(\vec{v}) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ such that $(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash x \leq s(\vec{v}) \to t(\vec{v}, x) \leq t_s(\vec{v})$.*
4. *$\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{B\Sigma}_{n+1} + \Gamma_1)$.*
5. *For every $\mathfrak{A} \models (\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma$ and $a \in \mathfrak{A}$, $\mathcal{S}_\Gamma(\mathfrak{A}, a) \prec_n \mathfrak{A}$ as $\mathcal{L}$–structures and $\mathcal{S}_\Gamma(\mathfrak{A}, a) \models \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$.*

Proof. From 2.8 and 2.15, it follows *(4) $\implies$ (1)*.

(*(1) $\implies$ (5)*): By 4.19 and 4.16 we may assume that Γ is a strong Π_n–functional class for $\mathbf{T}$ (and that $\Gamma_1 = \Gamma^*$). So, by 4.9, $\mathcal{S}_\Gamma(\mathfrak{A}, a) \prec_n \mathfrak{A}$ as $\mathcal{L}$–structures. Let $\varphi(x, y) \in \Pi_n$ such that $\mathbf{T} \vdash \forall x\, \exists y\, \varphi(x, y)$. Then there exists $t(x) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ such that $(\mathbf{I\Sigma}_n + \Gamma^*)_\Gamma \vdash \forall x\, \exists y \leq t(x)\, \varphi(x, y)$. Let $b \in \mathcal{S}_\Gamma(\mathfrak{A}, a)$. Then, it holds that there exist $c \in \mathfrak{A}$ and $s(x) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ such that $\mathfrak{A} \models c \leq t(b) \wedge \varphi(b, c) \wedge b \leq s(a)$. Since Γ is Π_n–functional, $c \leq t(b) \leq t(s(a))$; hence, $c \in \mathcal{S}_\Gamma(\mathfrak{A}, a)$. So, $\mathcal{S}_\Gamma(\mathfrak{A}, a) \models \exists y\, \varphi(b, y)$.

(*(5) $\implies$ (4)*): Let $\varphi(x, y) \in \Pi_n$ such that $\mathbf{B\Sigma}_{n+1} + \Gamma_1 \vdash \forall x\, \exists y\, \varphi(x, y)$. Suppose that $\mathbf{I\Sigma}_n + \Gamma_1 \nvdash \forall x\, \exists y\, \varphi(x, y)$. Let $\mathbf{T}'$ be the theory

$$(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma + \forall y\, \neg\varphi(\mathbf{c}, y) + \{t(\mathbf{c}) < \mathbf{d} :\ t(x) \in \mathbf{Term}(\mathcal{L}(\Gamma))\}.$$

By compacteness, $\mathbf{T}'$ is consistent. Let $\mathfrak{A} \models \mathbf{T}'$ and $a = \mathfrak{A}(\mathbf{c})$. Since $\mathcal{S}_\Gamma(\mathfrak{A}, a) \prec_n^e \mathfrak{A}$ as $\mathcal{L}$–structures and is proper, $\mathcal{S}_\Gamma(\mathfrak{A}, a) \models \forall y\, \neg\varphi(a, y)$ and $\mathcal{S}_\Gamma(\mathfrak{A}, a) \models \mathbf{B\Sigma}_{n+1}$. Then, $\mathcal{S}_\Gamma(\mathfrak{A}, a) \models \mathbf{B\Sigma}_{n+1} + \Gamma_1$. Contradiction.

(*(1) $\implies$ (2)*): Let $\varphi(x) \in \Delta_0^\Gamma$. There exists $\psi(x) \in \Delta_{n+1}(\mathbf{T})$ such that $(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash \varphi(x) \leftrightarrow \psi(x)$. Since $\mathbf{T}$ has Δ_{n+1}–induction, by 2.15, $\mathbf{I\Sigma}_n + \Gamma_1$ has Δ_{n+1}–induction; hence, $\mathbf{I\Sigma}_n + \Gamma_1 \vdash \mathbf{I}_\psi$. So, $(\mathbf{I\Sigma}_n + \Gamma_1)_\Gamma \vdash \mathbf{I}_\varphi$.

(*(2) $\Longrightarrow$ (1)*): Let $\varphi(x) \in \Delta_{n+1}(\mathbf{T})$. By 3.27, there exists $\theta(x) \in \Delta_0^\Gamma$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \theta(x) \leftrightarrow \varphi(x)$. Then, by *(2)*, $\mathbf{T} \vdash \mathbf{I}_\varphi$; so, $\mathbf{T}$ has Δ_{n+1}–induction. Since $\mathbf{T}$ is Δ_{n+1}–closed, by 2.13, $\mathbf{T}$ has Δ_{n+1}–collection.

(*(1) $\Longrightarrow$ (3)*): Let $s(\vec{v}), t(\vec{v}, x) \in \mathbf{Term}(\mathcal{L}(\Gamma))$. By 3.2 there exist $\varphi(\vec{v}, x)$, $\theta(\vec{v}, x, z) \in \Delta_{n+1}(\mathbf{I}\Sigma_n + \Gamma_1)$ such that

$$(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash [s(\vec{v}) = x \leftrightarrow \varphi(\vec{v}, x)] \wedge [t(\vec{v}, x) = z \leftrightarrow \theta(\vec{v}, x, z)].$$

Let Γ_0 be a strong Π_n–functional class for $\mathbf{T}$. By 4.16, there exist $t_0(\vec{v}, x)$ and $s_0(\vec{v})$ terms of $\mathcal{L}(\Gamma_0)$ and $\psi(\vec{v}, z) \in \Delta_{n+1}(\mathbf{T})$ such that

$$(\mathbf{I}\Sigma_n + \Gamma_0^*)_{\Gamma_0} \vdash \forall \vec{v}\, \exists x \leq s_0(\vec{v})\, \varphi(\vec{v}, x) \;\wedge\; \forall \vec{v} \forall x\, \exists z \leq t_0(\vec{v}, x)\, \theta(\vec{v}, x, z).$$

and $(\mathbf{I}\Sigma_n + \Gamma_0^*)_{\Gamma_0} \vdash t_0(\vec{v}, s_0(\vec{v})) = z \leftrightarrow \psi(\vec{v}, z)$. Then

$$\mathbf{I}\Sigma_n + \Gamma_0^* \vdash \varphi(\vec{v}, x') \wedge x \leq x' \wedge \theta(\vec{v}, x, z) \rightarrow \exists z' \, (\psi(\vec{v}, z') \wedge z \leq z').$$

Since $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma_0^*) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{I}\Sigma_n + \Gamma_1)$, then

$$\mathbf{I}\Sigma_n + \Gamma_1 \vdash \varphi(\vec{v}, x') \wedge x \leq x' \wedge \theta(\vec{v}, x, z) \rightarrow \exists z' \, (\psi(\vec{v}, z') \wedge z \leq z').$$

Since $\mathbf{T} \vdash \forall \vec{v}\, \exists z\, \psi(\vec{v}, z)$, there exists $t_s(\vec{v})$ such that

$$(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall \vec{v}\, \exists z \leq t_s(\vec{v})\, \psi(\vec{v}, z).$$

So, $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash x \leq s(\vec{v}) \rightarrow t(\vec{v}, x) \leq t_s(\vec{v})$, as required.

(*(3) $\Longrightarrow$ (1)*): Let $\varphi(x, y, \vec{v}) \in \Pi_n^-$ such that $\exists y\, \varphi(x, y, \vec{v}) \in \Delta_{n+1}(\mathbf{T})$. Then there exist $\theta(x, \vec{v}), \varphi_0(x, y, \vec{v}) \in \Delta_0^\Gamma$ such that

$$(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash [\exists y\, \varphi(x, y, \vec{v}) \leftrightarrow \theta(x, \vec{v})] \;\wedge\; [\varphi(x, y, \vec{v}) \leftrightarrow \varphi_0(x, y, \vec{v})].$$

Let $\psi(x, \vec{v}, y) \in \Delta_0^\Gamma$ be $(\theta(x, \vec{v}) \wedge \varphi_0(x, y, \vec{v})) \vee (\neg\theta(x, \vec{v}) \wedge y = 0)$. Then, by 3.25, there exists $t(x, \vec{v}) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall x \forall \vec{v}\, \exists y \leq t(x, \vec{v})\, \psi(x, \vec{v}, y)$. By *(3)*, there exists $t'(u, \vec{v}) \in \mathbf{Term}(\mathcal{L}(\Gamma))$ such that $(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash x \leq u \rightarrow t(x, \vec{v}) \leq t'(u, \vec{v})$. So,

$$(\mathbf{I}\Sigma_n + \Gamma_1)_\Gamma \vdash \forall u\, \forall \vec{v}\, [\forall x \leq u\, \exists y\, \varphi(x, y, \vec{v}) \rightarrow \exists u'\, \forall x \leq u\, \exists y \leq u'\, \varphi(x, y, \vec{v})].$$

That is, $\mathbf{T} \vdash \mathbf{B}_{\varphi, x, y}$. So, $\mathbf{T}$ has Δ_{n+1}–collection. □

5. Π_n–envelopes

5.1. General properties of Π_n–envelopes. Initial segments. In this section we introduce the concept of Π_n–envelope. This generalizes the concept of envelope (see [10]) and is closely related to indicators (see [12]). Some results in this section are generalizations of results on indicators that appear in chapter 14 of [12]. However, Π_n–envelopes will provide us with Π_n–functional classes defined uniformely. This is why we include these results here. In particular, we will obtain Π_n–envelopes that will be used in section 6 to prove the hierarchy theorem.

For each formula $\varphi(u, x, y)$ let $\Gamma_\varphi = \{\varphi(k, x, y) : \; k \in \omega\}$.

Definition 5.1. *Let $\varphi(u, x, y) \in \Sigma_{n+1}^-$. We say that*

1. *$\varphi(u, x, y)$ is a Π_n–q–envelope of $\mathbf{T}$ in $\mathbf{T}_0$ if $\mathbf{T} \vdash \Gamma_\varphi^*$, and for all $k \in \omega$, $\mathbf{T}_0 \vdash \varphi(k+1, x, y) \rightarrow \exists z < y\, \varphi(k, x, z)$.*

(2) $\varphi(u,x,y)$ *satisfies* Π_n*–ENV for* $\mathbf{T}$ *and* $\mathbf{T}_0$ *if for each* $\psi(x,y) \in \Pi_n^-$ *such that* $\mathbf{T} \vdash \forall x\, \exists y\, \psi(x,y)$*, there exists* $k \in \omega$ *such that*

$$\mathbf{T}_0 \vdash \varphi(k,x,y) \to \exists z < y\, \psi(x,z).$$

(3) $\varphi(u,x,y)$ *is a* Π_n*–envelope of* $\mathbf{T}$ *in* $\mathbf{T}_0$ *if* $\varphi(u,x,y)$ *is a* Π_n*–q–envelope of* $\mathbf{T}$ *in* $\mathbf{T}_0$ *and satisfies* Π_n*–ENV for* $\mathbf{T}$ *and* $\mathbf{T}_0$.

Remark 5.2. Now we shall give some basic properties of envelopes. Let $\varphi(u,x,y) \in \Sigma_{n+1}$ a Π_n–q–envelope of $\mathbf{T}$ in $\mathbf{T}_0$. By contraction of quantifiers, part *(2)* of definition 5.1 is also true for $\psi(x,y) \in \Sigma_{n+1}^-$. We also have that

Claim 5.3. *(i) If* $\mathbf{T} \Longrightarrow \mathbf{T}_0$ *then* $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}_0 + \Gamma_\varphi^*)$.
(ii) If $\varphi \in \Pi_n$ *and* $\mathbf{T} + \mathbf{I}\Sigma_n$ *is consistent then* Γ_φ *is a* Π_n*–functional class.*

Definition 5.4. *Let* $\varphi(u,x,y) \in \Sigma_{n+1}$*. We say that* $\varphi(u,x,y)$ *satisfies* Π_n*–IND for* $\mathbf{T}$ *and* $\mathbf{T}_0$ *if for every* $\mathfrak{A} \models \mathbf{T}_0$ *countable, nonstandard and* $a, b \in \mathfrak{A}$*, the following conditions are equivalent:*

(IND-(i))*: For all* $k \in \omega$, $\mathfrak{A} \models \exists y < b\, \varphi(k,a,y)$.
(IND-(ii))*: There exists* $\mathfrak{I} \models \mathbf{T}$ *such that* $\mathfrak{I} \prec_n^e \mathfrak{A}$ *and* $a < \mathfrak{I} < b$.

Remark 5.5. Let $\varphi(u,x,y) \in \Sigma_{n+1}$ such that $\mathbf{T} \vdash \forall x\, \exists y\, \varphi(k,x,y)$, for all $k \in \omega$. Then for all theory $\mathbf{T}_0$ we have that: **IND-(ii)** $\Longrightarrow$ **IND-(i)**. So, if $\varphi(u,x,y)$ is a Π_n–q–envelope, then in order to prove that $\varphi(u,x,y)$ satisfies Π_n–IND it is enough to establish that: **IND-(i)** $\Longrightarrow$ **IND-(ii)**.

Now we shall study conditions under which it holds that Π_n–ENV is equivalent to Π_n–IND. Let us note, however, that the proof of part $\Longleftarrow$ of next theorem shows that, if $\mathbf{T}_0 \Longrightarrow \mathbf{I}\Sigma_n$, then every Π_n–q–envelope of $\mathbf{T}$ in $\mathbf{T}_0$ satisfying Π_n–IND is a Π_n–envelope.

Theorem 5.6. *($n \geq 1$) Suppose that* $\mathbf{T}_0 \Longrightarrow \mathbf{I}\Sigma_n$ *and*
(i) $\mathbf{T}$ *is recursively axiomatizable, and*
(ii) $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$.
Let $\varphi(u,x,y) \in \Sigma_{n+1}$ *be a* Π_n*–q–envelope of* $\mathbf{T}$ *in* $\mathbf{T}_0$*. Then with respect to* $\mathbf{T}$ *and* $\mathbf{T}_0$

$$\varphi(u,x,y) \text{ satisfies } \Pi_n\text{–ENV} \iff \varphi(u,x,y) \text{ satisfies } \Pi_n\text{–IND}.$$

Proof. ($\Longleftarrow$): Let $\psi(x,y) \in \Pi_n^-$ such that $\mathbf{T} \vdash \forall x\, \exists y\, \psi(x,y)$ and suppose that for all $k \in \omega$, $\mathbf{T}_0 \nvdash \varphi(k,x,y) \to \exists z < y\, \psi(x,z)$. For all $k \in \omega$ let

$$\mathbf{T}_k = \mathbf{T}_0 + \{\exists y < \mathbf{d}\, \varphi(j,\mathbf{c},y) \wedge \forall z < \mathbf{d}\, \neg\psi(\mathbf{c},z) :\ j < k\}.$$

Since, for all $k \in \omega$, $\mathbf{T}_k$ is consistent, $\mathbf{T}^* = \bigcup_{k\in\omega} \mathbf{T}_k$ is consistent. Let $\mathfrak{A}^* \models \mathbf{T}$ countable nonstandard, $\mathfrak{A} = \mathfrak{A}^*_{|\mathcal{L}}$, $a = \mathfrak{A}^*(\mathbf{c})$ and $b = \mathfrak{A}^*(\mathbf{d})$. Then $\mathfrak{A} \models \mathbf{T}_0$ and for all $k \in \omega$, $\mathfrak{A} \models \exists y < b\, \varphi(k,a,y)$. Since φ satisfies Π_n–IND for $\mathbf{T}$ and $\mathbf{T}_0$, there exists $\mathfrak{I} \models \mathbf{T}$ such that $\mathfrak{I} \prec_n^e \mathfrak{A}$ and $a < \mathfrak{I} < b$. So, there exists $e \in \mathfrak{I}$ such that $\mathfrak{I} \models \psi(a,e)$; hence, $e < b$ and $\mathfrak{A} \models \psi(a,e)$. But $\mathfrak{A}^* \models \forall z < \mathbf{d}\, \neg\psi(\mathbf{c},z)$; hence, $\mathfrak{A} \models \forall z < b\, \neg\psi(a,z)$. So, $\mathfrak{A} \models \neg\psi(a,e)$, a contradiction.

($\Longrightarrow$): By 5.5, it is enough to prove **IND-(i)** $\Longrightarrow$ **IND-(ii)**. We follow the proof of theorem **11.7** in [12]. Let $\mathfrak{A} \models \mathbf{T}_0$ countable, nonstandard and $a, b \in \mathfrak{A}$ such that $\mathfrak{A} \models \exists y < b\, \varphi(k,a,y)$, for all $k \in \omega$. Let

$$\mathbf{T}' = \mathbf{T} + \mathbf{B}\Sigma_{n+1} + \{\forall \vec{z}\, \psi(\mathbf{c}, \vec{z}) :\ \psi(x,\vec{z}) \in \Sigma_n,\ \mathfrak{A} \models \forall \vec{z} \leq b\, \psi(a,\vec{z})\}.$$

By *(ii)* it follows that $\mathbf{T}'$ is consistent. Since $\mathfrak{A} \models \mathbf{I}\Sigma_n$ and $n \geq 1$, the Σ_n–type of a, b in $\mathfrak{A}$ belongs to $\mathbf{SSy}(\mathfrak{A})$ (the standard system of $\mathfrak{A}$); hence, $\{\ulcorner \forall \vec{z}\, \psi(\mathbf{c},\vec{z}) \urcorner :\ \psi \in \Sigma_n,\ \mathfrak{A} \models \forall \vec{z} \leq b\, \psi(a,\vec{z})\} \in \mathbf{SSy}(\mathfrak{A})$. So, by *(i)*, $\mathbf{T}' \in \mathbf{SSy}(\mathfrak{A})$. Since $\mathbf{SSy}(\mathfrak{A})$ is a Scott system, there exists $\mathfrak{B} \models \mathbf{T}'$ countable which is $\mathbf{SSy}(\mathfrak{A})$–saturated; hence, $\mathfrak{B}$ is recursively saturated. Let $c = \mathfrak{B}(\mathbf{c})$. Then, for each $\theta(x,\vec{z}) \in \Pi_n$, if $\mathfrak{B} \models \exists \vec{z}\, \theta(c,\vec{z})$ then $\mathfrak{A} \models \exists \vec{z} \leq b\, \theta(a,\vec{z})$. So, by Friedman's theorem, there exists $H : \mathfrak{B} \precsim_n^e \mathfrak{A}$ such that $H(c) = a$ and $b \notin H(\mathfrak{B})$. Let $\mathfrak{I} = H(\mathfrak{B})$. Then $\mathfrak{I} \models \mathbf{T}$, $\mathfrak{I} \prec_n^e \mathfrak{A}$ and $a < \mathfrak{I} < b$. □

Remark 5.7. Condition *(ii)* in 5.6 cannot be deleted. We have used it there in order to prove that: **IND-(i)** $\Longrightarrow$ **IND-(ii)**. Even more, suppose that $\mathbf{T} \Longrightarrow \mathbf{T}_0 \Longrightarrow \mathbf{I}\Sigma_n$ and $\varphi(u,x,y) \in \Sigma_{n+1}$ is a Π_n–q–envelope of $\mathbf{T}$ in $\mathbf{T}_0$ that satisfies Π_n-IND for these theories. Let $\psi(x,y) \in \Pi_n$ be such that $\mathbf{T} + \mathbf{B}\Sigma_{n+1} \vdash \forall x\, \exists y\, \psi(x,y)$. Then, it holds that there is $k \in \omega$ such that $\mathbf{T}_0 \vdash \varphi(k,x,y) \to \exists z < y\, \psi(x,z)$. So, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$.

Remark 5.8. For Π_0–envelopes we have the following form of 5.6.

Claim 5.9. *Suppose that* $\mathbf{T}_0 \Longrightarrow \mathbf{I}\Delta_0 + \mathbf{exp}$, $\mathbf{T}$ *is recursively axiomatizable and* $\mathbf{Th}_{\Pi_2}(\mathbf{T}) = \mathbf{Th}_{\Pi_2}(\mathbf{T} + \mathbf{B}\Sigma_1)$. *Let* $\varphi(u,x,y) \in \Sigma_1$ *a* Π_0*–q–envelope of* $\mathbf{T}$ *in* $\mathbf{T}_0$. *Then, with respect to* $\mathbf{T}$ *and* $\mathbf{T}_0$,

$$\varphi(u,x,y) \text{ satisfies } \Pi_0\text{–ENV} \quad \Longleftrightarrow \quad \varphi(u,x,y) \text{ satisfies } \Pi_0\text{–IND}.$$

In some cases this result is also true even though $\mathbf{T}_0$ is not an extension of $\mathbf{I}\Delta_0 + \mathbf{exp}$. Using methods that appears in [1], mainly the superexponential function (see the proof of lemma **3** there), it can be proved that

Claim 5.10. *Suppose that* $\mathbf{T} \Longrightarrow \mathbf{B}\Sigma_1 + \mathbf{exp} \Longrightarrow \mathbf{T}_0 \Longrightarrow \mathbf{I}\Delta_0$, *and* $\mathbf{T}$ *is recursively axiomatizable. Let* $\varphi(u,x,y) \in \Delta_0$ *be a* Π_0*–q–envelope of* $\mathbf{T}$ *in* $\mathbf{T}_0$. *Then, with respect to* $\mathbf{T}$ *and* $\mathbf{T}_0$,

$$\varphi(u,x,y) \text{ satisfies } \Pi_0\text{–ENV} \quad \Longleftrightarrow \quad \varphi(u,x,y) \text{ satisfies } \Pi_0\text{–IND}.$$

5.2. Existence theorems of Π_n–envelopes. In this and in the next subsection we are going to use formulas in the language and in the metalanguage. In order to write expressions that are easier to read we shall use uppercase Greek letters for formulas in the metalanguage (real formulas) and lowercase Greek letters for formulas in the language (elements of a model that it thinks that are formulas). We shall use $\sigma, \tau, \ldots$ as variables (in the language of Arithmetic) for formulas, and p as variable (in the language of Arithmetic) for proofs.

Theorem 5.11. *If* $\mathbf{T}$ *is recursively axiomatizable,* Π_n*–functional and, for* $n = 0$, $\mathbf{T} \vdash \mathbf{exp}$, *then there exists a* Π_n*–envelope of* $\mathbf{T}$ *in* $\mathbf{I}\Sigma_n$.

Proof. Since $\mathbf{T}$ is Π_n–functional, $\mathbf{T}$ has Δ_{n+1}–collection and, for $n \geq 1$, $\mathbf{T} \Longrightarrow \mathbf{I}\Sigma_n \Longrightarrow \mathbf{B}\Sigma_n$. Let us consider the following cases:

Case A: $n \geq 1$. Since $\mathbf{T}$ is recursively axiomatizable, there is $\mathbf{Prf_T}(x, y) \in \Sigma_1$ that represents to $\{(\sigma, p) \in \omega^2 : \ p \text{ is a proof of } \sigma \text{ in } \mathbf{T}\}$ in $\mathbf{P}^-$. Let $\Phi'(u, x, y) \in \Pi_n$ be a formula equivalent in $\mathbf{B}\Sigma_n$ (so, also in $\mathbf{T}$), to

$$\forall p, \tau \leq u \left\{ \begin{array}{r} \mathbf{Form}_{\Pi_n^-}(\tau(v_0, v_1)) \wedge \mathbf{Prf_T}(\forall v_0 \, \exists v_1 \, \tau(v_0, v_1), p) \ \rightarrow \\ \rightarrow \forall x_0 \leq x \, \exists y_0 \leq y \, \mathbf{Sat}_{\Pi_n}(\tau(\dot{x}_0, \dot{y}_0)) \end{array} \right\}$$

Where $\mathbf{Sat}_{\Pi_n}(v)$ is a truth definition in $\mathbf{I}\Sigma_1$ for Π_n–formulas.

Let $\Phi(u, x, y) \in \Sigma_{n+1}$ be a formula equivalent in $\mathbf{B}\Sigma_n$ (so, also in $\mathbf{T}$), to

$$\exists y' \leq y \, [y = y' + u \wedge \Phi'(u, x, y') \wedge \forall y'' < y' \, \neg \Phi'(u, x, y'')].$$

Let $k \in \omega$. Since $\mathbf{T}$ has Δ_{n+1}–collection, $\mathbf{T} \vdash \forall x \, \exists y \, \Phi(k, x, y)$. Moreover, as $y = y' + k$, $\mathbf{I}\Sigma_n \vdash \Phi(k+1, x, y) \rightarrow \exists z < y \, \Phi(k, x, z)$ and $\mathbf{T} \vdash \text{IPF}(\Phi(k, x, y))$.

Let $\Psi(x, y) \in \Pi_n^-$ such that $\mathbf{T} \vdash \forall x \, \exists y \, \Psi(x, y)$ and $k > \ulcorner \Psi(x, y) \urcorner$. Then $\mathbf{I}\Sigma_n \vdash \Phi(k, x, y) \rightarrow \exists z < y \, \Psi(x, z)$. So, $\Phi(u, x, y)$ satisfies Π_n–ENV for $\mathbf{T}$ and $\mathbf{I}\Sigma_n$.

Case B: $n = 0$. Since $\mathbf{T}$ is recursively axiomatizable, there is $\mathbf{Prf_T}(x, y, w) \in \Delta_0$ such that $\exists w \, \mathbf{Prf_T}(x, y, w)$ represents to $\{(\sigma, p) : \ p \text{ is a proof of } \sigma \text{ in } \mathbf{T}\}$ in $\mathbf{P}^-$. Let $\Phi'(u, x, y) \in \Sigma_0$ be

$$\forall p, \rho, w \leq u \left[\begin{array}{l} \mathbf{Form}_{\Pi_0^-}(\rho(v_0, v_1)) \wedge \mathbf{Prf_T}(\forall v_0 \, \exists v_1 \, \rho(v_0, v_1), p, w) \ \rightarrow \\ \forall z, z' \leq y \left\{ \begin{array}{l} y = \langle z, z' \rangle \ \rightarrow \\ \rightarrow \left\{ \begin{array}{l} z = 2^{(x+z'+2)^{c^u}} \ \wedge \\ \forall x_0 \leq x \, \exists y_0 \leq z' \, \mathcal{V}_0(\rho, \langle x_0, y_0 \rangle, z) \end{array} \right\} \end{array} \right\} \end{array} \right]$$

Where $\mathcal{V}_0(v_1, v_2, v_3) \in \Delta_0$ is a truth definition in $\mathbf{I}\Delta_0 + \mathbf{exp}$ for Δ_0 formulas and $c \in \omega$ is a constant which depends upon the explicit definition of $\mathcal{V}_0(v_1, v_2, v_3)$ (see [10], **V.5.4**). Let $\Phi(u, x, y) \in \Sigma_0$ defined as in case A. Now, as there, it is proved that $\Phi(u, x, y)$ is a Π_0–envelope of $\mathbf{T}$ in $\mathbf{I}\Delta_0$. □

Remark 5.12. Let $\mathbf{T}$ be Π_n–functional and $\Phi'(u, x, y, w) \in \Pi_n$ such that $\exists w \, \Phi'(u, x, y, w)$ is a Π_n–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$. Let us see that there exists a Π_n formula which is a Π_n–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$.

Let $\Psi(u, x, y) \in \Pi_n$ be $\exists w, y' \leq y \, [y = \langle w, y' \rangle \wedge \Phi'(u, x, y', w)]$. For each $k \in \omega$, let $\Psi_k(x, y)$ be $\Psi(k, x, y)$. Then $\mathbf{T} \vdash \forall x \, \exists y \, \Psi_k(x, y)$. Let $\mathcal{C}_{\Psi_k}(x, y)$ be as in the proof of 3.8. The definition of $\mathcal{C}_{\Psi_k}(x, y)$ is uniform in k; so, using k as a parameter we obtain $\mathcal{C}_{\Psi}(u, x, y) \in \Pi_n$. Let $\Theta(u, x, y) \in \Pi_n$ be

$$\text{Seq}(y) \wedge \text{lg}(y) = u + 1 \wedge \forall j \leq u \, \mathcal{C}_{\Psi}(j, x, (y)_j).$$

Then $\Theta(u, x, y)$ is a Π_n–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$.

Theorem 5.13. (1) *For all $m \geq n$ ($m \geq 1$, for $n = 0$) there exists a Π_n–envelope of $\mathbf{I}\Sigma_m$ in $\mathbf{I}\Sigma_n$, $\Phi(u, x, y) \in \Pi_n$, such that*
- (a) $\mathbf{I}\Sigma_{m+1} \vdash \forall u \, \forall x \, \exists y \, \Phi(u, x, y)$.
- (b) $\mathbf{I}\Sigma_{m+1} \vdash \forall u, x, y_1, y_2 \, [\Phi(u, x, y_1) \wedge \Phi(u+1, x, y_2) \rightarrow y_1 < y_2]$.

(2) *For all $n \in \omega$ there exists a Π_n–envelope, $\Phi(u, x, y) \in \Pi_n$, of $\mathbf{PA}$ in $\mathbf{I}\Sigma_n$ such that*
- (a) $\mathbf{Th}(\mathcal{N}) \vdash \forall u \, \forall x \, \exists y \, \Phi(u, x, y)$.
- (b) $\mathbf{Th}(\mathcal{N}) \vdash \forall u, x, y_1, y_2 \, [\Phi(u, x, y_1) \wedge \Phi(u+1, x, y_2) \rightarrow y_1 < y_2]$.

Proof. Let $1 \leq n \leq m$. We will prove that the Π_n–envelope obtained in 5.12, from the one given in 5.11, satisfies the properties of *1–(a)* and *1–(b)*. Let $\Phi'(u, x, y) \in \Pi_n$ the formula

$$\forall p, \tau \leq u \left\{ \begin{array}{c} \mathbf{Form}_{\Pi_n^-}(\tau(v_0, v_1)) \wedge \mathbf{Prf}_{\mathbf{I}\Sigma_m}(\forall v_0 \, \exists v_1 \, \tau(v_0, v_1), p) \;\rightarrow \\ \rightarrow \forall x_0 \leq x \, \exists y_0 \leq y \, \mathbf{Sat}_{\Pi_n}(\tau(\dot{x}_0, \dot{y}_0)) \end{array} \right\}$$

It is enough to prove that $\mathbf{I}\Sigma_{m+1} \vdash \forall u \, \forall x \, \exists y \, \Phi'(u, x, y)$.

Let $\mathfrak{A} \models \mathbf{I}\Sigma_{m+1}$, c, p, $\tau \in \mathfrak{A}$, $p, \tau \leq c$, such that

$$\mathfrak{A} \models \mathbf{Form}_{\Pi_n^-}(\tau(v_0, v_1)) \wedge \mathbf{Prf}_{\mathbf{I}\Sigma_m}(\forall v_0 \, \exists v_1 \, \tau(v_0, v_1), p).$$

Then $\mathfrak{A} \models \forall x \, \exists p \, \mathbf{Prf}_{\mathbf{I}\Sigma_m}(\exists v_1 \, \tau(\dot{x}, v_1), p)$. By reflexion, see [10],

$$\mathbf{I}\Sigma_{m+1} \vdash \mathbf{Sent}_{\Sigma_{n+1}}(\sigma) \wedge \exists p \, \mathbf{Prf}_{\mathbf{I}\Sigma_m}(\sigma, p) \rightarrow \mathbf{Sat}_{\Sigma_{n+1}}(\sigma).$$

Hence, $\mathfrak{A} \models \forall x \, \mathbf{Sat}_{\Sigma_{n+1}}(\exists v_1 \, \tau(\dot{x}, v_1))$. So, $\mathfrak{A} \models \forall x \, \exists y \, \mathbf{Sat}_{\Pi_n}(\tau(\dot{x}, \dot{y}))$. Since $\mathfrak{A} \models \mathbf{B}\Pi_n$, for all $a \in \mathfrak{A}$ there is $b \in \mathfrak{A}$ such that

$$\mathfrak{A} \models \forall x \leq a \, \exists y \leq b \, \mathbf{Sat}_{\Pi_n}(\tau(\dot{x}, \dot{y})).$$

Then $\mathfrak{A} \models \forall x \, \exists y \, \Phi'(c, x, y)$. □

5.3. Strong Π_n–envelopes.

Definition 5.14. *Let $\Phi(u, x, y) \in \Pi_n$ a Π_n–envelope of $\mathbf{T}$ in $\mathbf{T}_0$. We say that $\Phi(u, x, y)$ is a strong Π_n–envelope if Γ_Φ is a strong Π_n–functional class.*

Lemma 5.15. *($n \geq 1$) Suppose that $\mathbf{T}$ is recursively axiomatizable and $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$. Let $\Phi(u, x, y) \in \Pi_n$ be a Π_n–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$. If $\mathbf{I}\Delta_0 + \Gamma_\Phi^* \Longrightarrow \mathbf{I}\Sigma_n$ then $\Phi(u, x, y)$ is a strong Π_n–envelope.*

Proof. By 5.3, Γ_Φ is a Π_n–functional class; hence, $\mathbf{I}\Sigma_n + \Gamma_\Phi^*$ is consistent. Let $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma_\Phi^*$ and $\mathfrak{I} \prec_0^e \mathfrak{A}$ such that for all $k \in \omega$ and $a \in \mathfrak{I}$ there exists $b \in \mathfrak{I}$ such that $\mathfrak{A} \models \Phi(k, a, b)$. Let us see, using Tarski–Vaught test's, that $\mathfrak{I} \prec_n \mathfrak{A}$. Let $\Theta(x, y) \in \Pi_{n-1}$, $a \in \mathfrak{I}$ such that $\mathfrak{A} \models \exists y \, \Theta(a, y)$ and $c \in \mathfrak{A}$ such that $\mathfrak{I} < c$. Then for all $k \in \omega$, $\mathfrak{A} \models \exists y < c \, \Phi(k, a, y)$. So, by 5.6, there exists $\mathfrak{I}_1 \models \mathbf{T}$ such that $\mathfrak{I}_1 \prec_n^e \mathfrak{A}$ and $a < \mathfrak{I}_1 < c$. Hence, $\mathfrak{I}_1 \models \exists y \, \Theta(a, y)$ and $\mathfrak{A} \models \exists y < c \, \Theta(a, y)$. Then, by underspill ($\mathfrak{A} \models \mathbf{I}\Sigma_n$), there exists $d \in \mathfrak{I}$ such that $\mathfrak{A} \models \exists y < d \, \Theta(a, y)$. So, there is $b \in \mathfrak{I}$ such that $\mathfrak{A} \models \Theta(a, b)$. □

Theorem 5.16. *($n \geq 1$) There exists a formula $\mathbb{E}_n(u, x, y) \in \Pi_n$ such that*

1. *For every $k \in \omega$, $\mathbf{I}\Sigma_{n-1} \vdash$ IPF$(\mathbb{E}_n(k, x, y))$.*
2. *$\mathbb{E}_n(u, x, y)$ is a Π_n–q–envelope of $\mathbf{I}\Sigma_n$ in $\mathbf{I}\Sigma_n$.*
3. *$\mathbf{I}\Sigma_n \Longleftrightarrow \mathbf{I}\Delta_0 + \Gamma_{\mathbb{E}_n}^*$.*
4. *$\Gamma_{\mathbb{E}_n}$ is a strong Π_n–functional class.*
5. *Let $\mathbb{K}_n(x) = y$ be the formula $\mathbb{E}_n(x, x, y)$ and $\Gamma_n = \{\mathbb{K}_n(x) = y\}$. Then*
   (a) *$\mathbf{I}\Sigma_n \vdash$ IPF$(\mathbb{K}_n)$.*
   (b) *$\mathbf{I}\Sigma_n \vdash \forall x \, \exists y \, [\mathbb{K}_n(x) = y]$.*
   (c) *Γ_n is a strong Π_n–functional class.*

Proof. The idea is to define $\mathbb{E}_n(u, x, y)$ as the strong Π_n–functional class of 4.17, $\mathbb{H}_n$, in an uniform way. A Σ_{n+1} formula similar to $\mathbb{K}_n(x) = y$ has been considered by R. Kaye in [11] and [13]. However, we need a Π_n formula.

The proof of the theorem is by induction on $n \geq 1$.

<u>$(n = 1)$</u>: Let $\Theta_1'(\sigma, x, w, z) \in \Pi_1$ be the following formula

$$\left\{\begin{array}{l} \neg\exists v \leq x\,\exists y\,\exists z'\,[2^{(x+v+y+2)^{c^\sigma}} \leq z' \wedge \mathcal{V}_0(\sigma,\langle v,y\rangle,z') \wedge w=0] \vee \\ \exists w_1,w_2 \leq w \left\{\begin{array}{l} w=\langle w_1,w_2\rangle \wedge w_1 \leq x \wedge \mathcal{V}_0(\sigma_{\mu,w_2},\langle w_1,w_2\rangle,z) \wedge \\ \forall v \leq x \left\{\begin{array}{l} \forall y\,\forall z' \left\{\begin{array}{c} 2^{(x+v+y+2)^{c^\sigma}} \leq z' \wedge \mathcal{V}_0(\sigma,\langle v,y\rangle,z') \;\rightarrow \\ \rightarrow \exists y \leq w_2\,\mathcal{V}_0(\sigma,\langle v,y\rangle,z) \end{array}\right\} \\ \wedge\; (\mathcal{V}_0(\sigma_{\mu,w_2},\langle v,w_2\rangle,z) \rightarrow v \leq w_1) \end{array}\right. \end{array}\right. \end{array}\right.$$

where $\mathcal{V}_0(\sigma_{\mu,w_2},\langle w_1,w_2\rangle,z)$ is the formula

$$\mathcal{V}_0(\sigma,\langle w_1,w_2\rangle,z) \wedge \forall w < w_2 \neg\mathcal{V}_0(\sigma,\langle w_1,w\rangle,z).$$

Let $\Theta_1(\sigma,x,y)$ and $\mathbb{E}_1(u,x,y)$ be the following Π_1–formulas

$$\exists z,w \leq y \left\{\begin{array}{l} y=\langle w,z\rangle \wedge z = 2^{(x+w+2)^{c^\sigma}} \;\wedge \\ \left\{\begin{array}{l} [\mathbf{Form}_{\Pi_0^-}(\sigma(v_0,v_1)) \wedge \Theta_1'(\sigma,x,w,z)] \vee \\ [\neg\mathbf{Form}_{\Pi_0^-}(\sigma(v_0,v_1)) \wedge w=0] \end{array}\right. \end{array}\right.$$

$$\mathrm{Seq}(y) \wedge \mathrm{lg}(y) = u+1 \wedge \forall j \leq u\,\Theta_1(j,x,(y)_j).$$

The proof of *(1)* is as in 4.17, and *(2)* follows easily from the definition of $\mathbb{E}_1(u,x,y)$. We also have that

Claim 5.17. $\mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_1} \Longrightarrow \mathbf{S}\Pi_0^-$.

Proof of Claim. Let $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_1}$, $\Psi(x,y) \in \Pi_0^-$ and $a \in \mathfrak{A}$. Let us see that

$$\mathfrak{A} \models \exists w\,\forall x \leq a\,[\exists y\,\Psi(x,y) \rightarrow \exists y \leq w\,\Psi(x,y)].$$

Let $\psi = \ulcorner\Psi\urcorner$. Since $\mathfrak{A} \models \Gamma^*_{\mathbb{E}_1}$, there exists $b \in \mathfrak{A}$ such that $\mathfrak{A} \models \mathbb{E}_1(\psi,a,b)$. Then $\mathfrak{A} \models \Theta(\psi,a,(b)_\psi)$; so, $\mathfrak{A} \models \Theta_1'(\psi,a,b',b'')$, where $(b)_\psi = \langle b',b''\rangle$. Let $d \in \mathfrak{A}$ such that $d \leq a$ and $\mathfrak{A} \models \exists y\,\Psi(d,y)$. Then

$$\mathfrak{A} \models \exists y\,\exists z'\,[2^{(a+d+y+2)^{c^\psi}} \leq z' \wedge \mathcal{V}_0(\psi,\langle d,y\rangle,z')].$$

So, $\mathfrak{A} \models \exists y \leq (b')_2\,\mathcal{V}_0(\psi,\langle d,y\rangle,b'')$. That is, $\mathfrak{A} \models \exists y \leq b\,\Psi(d,y)$. □

Since $\mathbf{S}\Pi_0^- \Longleftrightarrow \mathbf{S}\Pi_0 \Longleftrightarrow \mathbf{I}\Sigma_1$, then $\mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_1} \Longrightarrow \mathbf{I}\Sigma_1$. This proves *(3)*. To prove *(4)*, follow the proof of 4.17. Now we prove *(5)*. From the definition of $\mathbb{K}_1(x)=y$, $\mathbf{I}\Sigma_1 \vdash \mathrm{IPF}(\mathbb{K}_1(x)=y)$. This gives *5–(a)*. We have that

Claim 5.18. $\mathbf{I}\Sigma_1 \vdash \forall x\,\exists y\,[\mathbb{K}_1(x)=y]$.

Proof of Claim. Since $\mathbf{I}\Sigma_1 \Longleftrightarrow \mathbf{S}\Pi_0$, then

$$\mathbf{I}\Sigma_1 \vdash \exists z_1\,\forall v \leq x \left[\begin{array}{l} \exists y,z\,(\mathcal{V}_0(\sigma,\langle v,y\rangle,z) \wedge 2^{(v+y+2)^{c^\sigma}} \leq z) \;\rightarrow \\ \quad \rightarrow \exists y,z \leq z_1(\mathcal{V}_0(\sigma,\langle v,y\rangle,z) \wedge 2^{(v+y+2)^{c^\sigma}} \leq z) \end{array}\right]$$

By properties of $\mathcal{V}_0(v_1,v_2,v_3)$, (see theorem **V.5.4** of [10]), we have that

$$\mathbf{I}\Sigma_1 \vdash 2^{(v+y+2)^{c^\sigma}} \leq z_1,z_2 \rightarrow [\mathcal{V}_0(\sigma,\langle v,y\rangle,z_1) \leftrightarrow \mathcal{V}_0(\sigma,\langle v,y\rangle,z_2)].$$

Then, as in 4.17, we get that $\mathbf{I}\Sigma_1 \vdash \forall\sigma,x\,\exists w,z\,\Theta_1'(\sigma,x,w,z)$. □

This completes the proof of *5–(b)*. Now we prove *5–(c)*.

Let $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_1}$ and $\mathfrak{I} \subset^e \mathfrak{A}$ such that for all $a \in \mathfrak{I}$ there is $b \in \mathfrak{I}$ such that $\mathfrak{A} \models \mathbb{K}_1(a) = b$. Let $k \in \omega$, $a \in \mathfrak{I}$ and $c = \max(k,a)$. Then $c \in \mathfrak{I}$; hence, there exists

$d \in \mathfrak{I}$ such that $\mathfrak{A} \models \mathbb{K}_1(c) = d$; that is, $\mathfrak{A} \models \mathbb{E}_1(c, c, d)$. Since $\mathbf{I}\Delta_0 \vdash \mathbb{E}_1(u, x, y) \to \forall v < u\, \exists z < y\, \mathbb{E}_1(v, x, z)$, there exists $b \in \mathfrak{I}$ such that $\mathfrak{A} \models \mathbb{E}_1(k, a, b)$. Since $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_1}$, by *(4)*, $\mathfrak{I} \prec_1 \mathfrak{A}$.

This proves 5–*(c)* and completes the proof of the theorem for $n = 1$.

<u>$(\leq n \to n+1)$</u>: For each m, $1 \leq m \leq n$, let $\mathbb{E}_m(u, x, y)$ be a Π_m formula that satisfies *(1)–(5)*. Let $\Theta'_{n+1}(\sigma, x, w) \in \Pi_{n+1}(\mathbf{I}\Sigma_n)$ the formula

$$\left\{\begin{array}{l} [\neg\exists v \leq x\, \exists y\, \mathbf{Sat}_{\Pi_n}(\sigma(\dot{v}, \dot{y})) \wedge w = 0] \vee \\ \exists w_1, w_2 \leq w \left\{\begin{array}{l} w = \langle w_1, w_2\rangle \wedge w_1 \leq x \wedge \mathbf{Sat}_{\Pi_n}(\sigma_{\mu, w_2}(\dot{w}_1, \dot{w}_2)) \wedge \\ \forall v \leq x \left\{\begin{array}{l} \exists y\, \mathbf{Sat}_{\Pi_n}(\sigma(\dot{v}, \dot{y})) \to \exists y \leq w_2\, \mathbf{Sat}_{\Pi_n}(\sigma(\dot{v}, \dot{y})) \wedge \\ \mathbf{Sat}_{\Pi_n}(\sigma_{\mu, w_2}(\dot{v}, \dot{w}_2)) \to v \leq w_1 \end{array}\right. \end{array}\right. \end{array}\right.$$

(where $\mathbf{Sat}_{\Pi_n}(\sigma_{\mu, v_2}(\dot{v}_1, \dot{v}_2))$ is $\mathbf{Sat}_{\Pi_n}(\sigma(\dot{v}_1, \dot{v}_2)) \wedge \forall y < v_2\, \neg\mathbf{Sat}_{\Pi_n}(\sigma(\dot{v}_1, \dot{y}))$). Let $\Theta_{n+1}(u, x, y)$ and $\mathbb{E}_{n+1}(u, x, y)$ be the following Π_{n+1}–formulas

$$\mathrm{Seq}(y) \wedge \mathrm{lg}(y) = u + 1 \wedge \forall j \leq u\, \Theta'_{n+1}(j, x, (y)_j),$$
$$\mathrm{Seq}(y) \wedge \mathrm{lg}(y) = n + 1 \wedge [\textstyle\bigwedge_{1 \leq m \leq n} \mathbb{E}_m(u, x, (y)_{m-1})] \wedge \Theta_{n+1}(u, x, (y)_n).$$

It is clear that for all $k \in \omega$, $\mathbf{I}\Sigma_{n+1} \vdash \forall x\, \exists y\, \Theta'_{n+1}(k, x, y)$. So, for all $k \in \omega$, $\mathbf{I}\Sigma_n \vdash \mathrm{IPF}(\mathbb{E}_{n+1}(k, x, y))$ and $\mathbb{E}_{n+1}(u, x, y)$ is a Π_{n+1}–q–envelope of $\mathbf{I}\Sigma_{n+1}$ in $\mathbf{I}\Sigma_{n+1}$. This proves *(1)* and *(2)* for $\mathbb{E}_{n+1}(u, x, y)$. We also have that

Claim 5.19. $\mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_{n+1}} \Longrightarrow \mathbf{I}\Sigma_n$.

Proof of Claim. Let $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_{n+1}}$. By induction on m, $1 \leq m \leq n$, let us see that $\mathfrak{A} \models \mathbf{I}\Sigma_m$.
<u>$(m = 1)$</u>: By *(1)*, for $n = 1$, $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_1}$. So, by *(3)* (for $n = 1$), $\mathfrak{A} \models \mathbf{I}\Sigma_1$.
<u>$(m \to m + 1 \leq n)$</u>: By induction hypothesis (on n, using *(1)* for $m + 1$), for all $k \in \omega$, $\mathbf{I}\Sigma_m \vdash \mathrm{IPF}(\mathbb{E}_{m+1}(k, x, y))$. By induction hypothesis (on m) $\mathfrak{A} \models \mathbf{I}\Sigma_m$, so $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_{m+1}}$. Then, by induction hypothesis (on n, using *(3)* for $m + 1 \leq n$), $\mathfrak{A} \models \mathbf{I}\Sigma_{m+1}$. □

Claim 5.20. $\mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_{n+1}} \Longrightarrow \mathbf{I}\Sigma_{n+1}$.

Proof of Claim. Let $\mathfrak{A} \models \mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_{n+1}}$. By 5.19, $\mathfrak{A} \models \mathbf{I}\Sigma_n$; so, $\Theta'_{n+1}(u, x, y)$ is Π_{n+1} in $\mathfrak{A}$. Now, as in 5.17, we get that $\mathfrak{A} \models \mathbf{S}\Pi^-_n$; so, $\mathfrak{A} \models \mathbf{I}\Sigma_{n+1}$. □

This proves *(3)*. The proofs of *(4)* and *(5)* are as for $n = 1$. □

Theorem 5.21. *$(n \geq 1)$ If $\mathbf{T}$ is recursively axiomatizable and Π_n–functional then there is a Π_n–formula which is a strong Π_n–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$.*

Proof. By 5.11 and 5.12, there exists $\Theta(u, x, y) \in \Pi_n$ which is a Π_n–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$. Using 5.16 we get $\mathbb{E}_n(u, x, y) \in \Pi_n$ which is a Π_n–q–envelope of $\mathbf{I}\Sigma_n$ in $\mathbf{I}\Sigma_n$. Let $\Phi(u, x, y) \in \Pi_n$ the following formula

$$\mathrm{Seq}(y) \wedge \mathrm{lg}(y) = 2 \cdot (u + 1) \wedge \forall j \leq u\, [\mathbb{E}_n(j, x, (y)_{2j}) \wedge \Theta(j, x, (y)_{2j+1})].$$

Since $\Theta(u, x, y)$ is a Π_n–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$, $\Phi(u, x, y)$ is a Π_n–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$. By 5.16–*(3)*, $\mathbf{I}\Delta_0 + \Gamma^*_{\mathbb{E}_n} \Longleftrightarrow \mathbf{I}\Sigma_n$; hence, $\mathbf{I}\Delta_0 + \Gamma^*_{\Phi} \Longrightarrow \mathbf{I}\Sigma_n$. So, by 5.15, $\Phi(u, x, y)$ is a strong Π_n–envelope of $\mathbf{T}$ in $\mathbf{I}\Sigma_n$. □

Corollary 5.22. (1) *($n \geq 1$) For all $m \geq n$ there exists a strong Π_n–envelope of $\mathbf{I\Sigma}_m$ in $\mathbf{I\Sigma}_n$, $\Phi(u,x,y) \in \Pi_n$, such that*
  (a) $\mathbf{I\Sigma}_{m+1} \vdash \forall u \, \forall x \, \exists y \, \Phi(u,x,y)$.
  (b) $\mathbf{I\Sigma}_{m+1} \vdash \forall u, x, y_1, y_2 \, [\Phi(u,x,y_1) \wedge \Phi(u+1,x,y_2) \rightarrow y_1 < y_2]$.

(2) *For all $n \in \omega$ there is a strong Π_n–envelope, $\Phi(u,x,y) \in \Pi_n$, of $\mathbf{PA}$ in $\mathbf{I\Sigma}_n$ such that*
  (a) $\mathbf{Th}(\mathcal{N}) \vdash \forall u \, \forall x \, \exists y \, \Phi(u,x,y)$.
  (b) $\mathbf{Th}(\mathcal{N}) \vdash \forall u, x, y_1, y_2 \, [\Phi(u,x,y_1) \wedge \Phi(u+1,x,y_2) \rightarrow y_1 < y_2]$.

6. The hierarchy $\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_m)$, $m \geq n$

For each formula $\psi(x,\vec{y})$ and term $t(x)$ of $\mathcal{L}(\Gamma)$, let $[\psi,t](z;\vec{y})$ denotes

$$z \leq t(\max(\vec{y})) \wedge \psi(z,\vec{y}) \wedge \forall x \, (\psi(x,\vec{y}) \rightarrow x = z).$$

Definition 6.1. *Let Γ be a Π_n–functional class, $\mathfrak{A} \models (\mathbf{I\Delta}_0 + \Gamma^*)_\Gamma$ and $\emptyset \neq X \subseteq \mathfrak{A}$. Let $\mathcal{K}_0^\Gamma(\mathfrak{A};X)$ be the substructure of $\mathfrak{A}$ whose universe is*

$$\{a \in \mathfrak{A} : \ \mathfrak{A} \models [\psi,t](a;\vec{b}), \ \psi \in \Delta_0^\Gamma, \ t \in \mathbf{Term}(\mathcal{L}(\Gamma)), \ \vec{b} \in X\},$$

and let $\mathcal{I}_0^\Gamma(\mathfrak{A};X)$ be the initial segment of $\mathfrak{A}$ given by $\mathcal{K}_0^\Gamma(\mathfrak{A};X)$.

Remark 6.2. Let Γ be a Π_n–functional class, $\mathfrak{A} \models \mathbf{I\Delta}_0^{\Gamma^*}$ and $\emptyset \neq X \subseteq \mathfrak{A}$. The structures $\mathcal{K}_0^\Gamma(\mathfrak{A};X)$ and $\mathcal{I}_0^\Gamma(\mathfrak{A};X)$ have similar properties, with respect to $\mathcal{L}(\Gamma)$, that $\mathcal{K}_1(\mathfrak{A};X)$ and $\mathcal{I}_1(\mathfrak{A};X)$. In particular, $\mathcal{K}_0^\Gamma(\mathfrak{A};X) \prec_0 \mathfrak{A}$ and $\mathcal{K}_0^\Gamma(\mathfrak{A};X) \prec_1^e \mathcal{I}_0^\Gamma(\mathfrak{A};X) \prec_0^e \mathfrak{A}$ as $\mathcal{L}(\Gamma)$–structures.

Remark 6.3. Let Γ be a strong Π_n–functional class, $\mathfrak{A} \models (\mathbf{I\Delta}_0 + \Gamma^*)_\Gamma$ and $\emptyset \neq X \subseteq \mathfrak{A}$. Here we prove some basic facts on $\mathcal{K}_0^\Gamma(\mathfrak{A};X)$ and $\mathcal{I}_0^\Gamma(\mathfrak{A};X)$. Let us first observe that since $\mathfrak{A} \models (\mathbf{I\Delta}_0 + \Gamma^*)_\Gamma$, by 4.12, $\mathfrak{A} \models \mathbf{I\Delta}_0^{\Gamma^*}$.

Claim 6.4. *$\mathcal{I}_0^\Gamma(\mathfrak{A},X) \prec_n^e \mathfrak{A}$, as $\mathcal{L}$–structures, and $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \models \mathbf{I\Delta}_0^{\Gamma^*}$.*

Proof of Claim. The first part follows from 6.2. As a consequence, since $\Gamma \subseteq \Pi_n$, $\mathcal{I}_0^\Gamma(\mathfrak{A},X)$ is a model of $\mathbf{I\Delta}_0$ and for all $\varphi \in \Gamma$ it satisfies IPF(φ) and the definition axiom of G_φ. So, $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \models (\mathbf{I\Delta}_0 + \Gamma^*)_\Gamma$. □

Claim 6.5. *$\mathcal{K}_0^\Gamma(\mathfrak{A},X) \prec_{n+1} \mathcal{I}_0^\Gamma(\mathfrak{A},X)$ and $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \prec_n \mathfrak{A}$, as $\mathcal{L}$–structures. Also $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \models \mathbf{I\Delta}_0^{\Gamma^*}$.*

Proof of Claim. Let $\psi(x,\vec{w}) \in \Pi_n$ and $\vec{b} \in \mathcal{K}_0^\Gamma(\mathfrak{A},X)$ such that $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \models \exists x \, \psi(x,\vec{b})$. Let $\theta(x,\vec{w}) \in \Delta_0^\Gamma$ such that $\mathbf{I\Delta}_0^{\Gamma^*} \vdash \psi(x,\vec{w}) \leftrightarrow \theta(x,\vec{w})$. Then, by 6.4, $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \models \exists x \, \theta(x,\vec{b})$. So, by 6.2, $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \models \exists x \, \theta(x,\vec{b})$. Let $a \in \mathcal{K}_0^\Gamma(\mathfrak{A},X)$, such that $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \models \theta(a,\vec{b})$. Then $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \models \theta(a,\vec{b})$. Hence, by 6.4, $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \models \psi(a,\vec{b})$. By Tarski–Vaught's test, $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \prec_{n+1} \mathcal{I}_0^\Gamma(\mathfrak{A},X)$. From this and 6.4 we get that $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \prec_n \mathfrak{A}$. So, as in the second part of 6.4, $\mathcal{K}_0^\Gamma(\mathfrak{A},X) \models \mathbf{I\Delta}_0^{\Gamma^*}$. □

Claim 6.6. *If $\mathcal{K}_0^\Gamma(\mathfrak{A};X)$ is not cofinal in $\mathfrak{A}$ then $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \models (\mathbf{B}\Sigma_{n+1} + \Gamma^*)_\Gamma$.*

Proof of Claim. Since Γ is a strong Π_n–functional class, $\mathfrak{A} \models \mathbf{I}\Sigma_n$. We also have that $\mathcal{I}_0^\Gamma(\mathfrak{A},X) \prec_n^e \mathfrak{A}$ and is proper; hence, $\mathcal{I}_0^\Gamma(\mathfrak{A};X) \models (\mathbf{B}\Sigma_{n+1} + \Gamma^*)_\Gamma$. □

Remark 6.7. Let $\varphi(u,x,y) \in \Pi_n$ be a strong Π_n–envelope of $\mathbf{T}$ in $\mathbf{T}$, where $\mathbf{T} \Longrightarrow \mathbf{I}\Sigma_n$. We shall denote by G_k the function symbol of $\mathcal{L}(\Gamma_\varphi)$ associated with $\varphi(k,x,y)$ and by $[\psi,k](x;\vec{y})$ the formula $[\psi,G_k](x;\vec{y})$. Let $\mathfrak{A} \models \mathbf{T}_{\Gamma_\varphi}$ and $a \in \mathfrak{A}$ nonstandard. We have that

Claim 6.8. *$\{G_k(a) : k \in \omega\}$ is cofinal in $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$.*

Proof of Claim. Let $b \in \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$. Then there are $\psi(x,y) \in \Delta_0^{\Gamma_\varphi}$ and a term $t(x)$ of $\mathcal{L}(\Gamma_\varphi)$ such that $\mathfrak{A} \models [\psi,t](b;a)$. By 4.14 there exists $\theta(x,z) \in \Sigma_{n+1}$ such that $(\mathbf{I}\Delta_0 + \Gamma_\varphi^*)_\varphi \vdash t(x) = z \leftrightarrow \theta(x,z)$. Since $\mathbf{T} \Longrightarrow \mathbf{I}\Delta_0 + \Gamma_\varphi^*$, then $\mathbf{T} \vdash \forall x\, \exists z\, \theta(x,z)$; so, there is $k \in \omega$ such that $\mathbf{T} \vdash \varphi(k,x,y) \to \exists z \le y\, \theta(x,z)$. Hence, $(\mathbf{T} + \Gamma_\varphi^*)_{\Gamma_\varphi} \vdash t(x) \le G_k(x)$. So, $\mathfrak{A} \models b \le t(a) \le G_k(a)$. □

Claim 6.9. *Suppose that $\mathfrak{A} \models \forall u,x,y_1,y_2\, [\varphi(u,x,y_1) \wedge \varphi(u+1,x,y_2) \to y_1 < y_2]$ and $\mathfrak{A} \models \forall u\, \forall x\, \exists y\, \varphi(u,x,y)$. Then ω is definable in $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ by the formula $\exists y\, \varphi(u,a,y)$; that is, by a Σ_{n+1} formula with parameters.*

Proof of Claim. Let us see that $\{c \in \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) : \mathcal{K}_0^{\Gamma_\varphi} \models \exists y\, \varphi(c,a,y)\} \subseteq \omega$. Let $c,d \in \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ such that $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \varphi(c,a,d)$. Since $\varphi \in \Pi_n$, by 6.5, $\mathfrak{A} \models \varphi(c,a,d)$. By 6.8, there exists $k \in \omega$ such that $\mathfrak{A} \models d \le G_k(a)$. Then, $c \le k$; hence, $c \in \omega$. □

Theorem 6.10. *Let $\mathbf{T}$ be a Π_n–functional theory (if $n = 0$ we assume that $\mathbf{T} \vdash \mathbf{exp}$), $\varphi(u,x,y) \in \Pi_n$ a strong Π_n–envelope of $\mathbf{T}$ in $\mathbf{T}$, $\mathfrak{A} \models \mathbf{T}_{\Gamma_\varphi}$ and $a \in \mathfrak{A}$ nonstandard. Then*

1. *$\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \prec_1^c \mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a) \prec_0^e \mathfrak{A}$, as $\mathcal{L}(\Gamma_\varphi)$–structures.*
2. *$\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \prec_{n+1}^c \mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a) \prec_n^e \mathfrak{A}$ and $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \prec_n \mathfrak{A}$, as $\mathcal{L}$–structures.*
3. *$\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \mathbf{I}\Delta_0^{\Gamma_\varphi}$ and $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \not\models \mathbf{B}\Sigma_{n+1}$.*
4. *$\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a) \not\models \mathbf{I}\Sigma_{n+1}$.*
5. *If $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ is not cofinal in $\mathfrak{A}$ then $\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \mathbf{B}\Sigma_{n+1}$.*
6. *$\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T})$.*
7. *$\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \mathbf{I}\Delta_{n+1}(\mathbf{T})$.*

Proof. Part *(1)* follows from 6.2 and part *(2)* from 6.3.
(*(3)*): By 6.5 it is only necessary to prove that $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \not\models \mathbf{B}\Sigma_{n+1}$. Let $b \in \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$. Then there exist $\psi(x,y) \in \Delta_0^{\Gamma_\varphi}$ and $k \in \omega$ such that $\mathfrak{A} \models [\psi,k](b;a)$. By 4.14, there exist $\theta(x,y,z) \in \Pi_n$ and a term $t(x,y)$ of $\mathcal{L}(\Gamma_\varphi)$ such that $\mathbf{I}\Delta_0^{\Gamma_\varphi} \vdash \forall z \ge t(x,y)\, [\psi(x,y) \leftrightarrow \theta(x,y,z)]$. Let $c_1,c_2 \in \mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ such that $c_1 \ge b$ and $c_2 \ge t(c_1,a)$. Then $\mathfrak{A} \models \psi(x,a) \leftrightarrow [\theta(x,a,c_2) \wedge x \le c_1]$ and $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},X) \models \psi(x,a) \leftrightarrow [\theta(x,a,c_2) \wedge x \le c_1]$. Let $\theta'(x,y,z_1,z_2)$ be the formula $\theta(x,y,z_2) \wedge x \le z_1$. We consider the following cases.

Case A: $n=0$. For all $\delta(y_1,\dots,y_m)\in\Delta_0$ there exists $r\in\omega$ such that

$$\mathbf{I}\Delta_0 \vdash 2^{(\max(y_1,\dots,y_m)+2)^r} \le u \to [\delta(y_1,\dots,y_m) \leftrightarrow \mathcal{V}_0(\ulcorner\delta\urcorner, \langle y_1,\dots,y_m\rangle, u)].$$

Since $\mathbf{T}\vdash \mathbf{exp}$, $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \mathcal{V}_0(\ulcorner\theta'\urcorner, \langle b,a,c_1,c_2\rangle, 2^{(\max(b,a,c_1,c_2)+2)^r})$. Let $d\in\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ nonstandard. Then $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ satisfies the following formula

$$\exists\sigma\le d \begin{cases} \mathbf{Form}_{\Delta_0}(\sigma)\ \wedge \\ \exists z_1\,\exists z_2 \begin{cases} \mathcal{V}_0(\sigma,\langle b,a,z_1,z_2\rangle, 2^{(\max(b,a,z_1,z_2)+2)^a})\ \wedge \\ \forall x<b\,\neg\mathcal{V}_0(\sigma,\langle x,a,z_1,z_2\rangle, 2^{(\max(x,a,z_1,z_2)+2)^a}) \end{cases} \end{cases}$$

Since b is an arbitrary element of $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$, then $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ satisfies

$$\forall u\le d+1\,\exists\sigma\le d \begin{cases} \mathbf{Form}_{\Delta_0}(\sigma)\ \wedge \\ \exists z_1\,\exists z_2 \begin{cases} \mathcal{V}_0(\sigma,\langle u,a,z_1,z_2\rangle, 2^{(\max(u,a,z_1,z_2)+2)^a})\ \wedge \\ \forall x<u\,\neg\mathcal{V}_0(\sigma,\langle x,a,z_1,z_2\rangle, 2^{(\max(x,a,z_1,z_2)+2)^a}) \end{cases} \end{cases}$$

Let $\gamma(d,a)$ denotes this formula. Then $\gamma(d,a)\in\Sigma_1$ (in $\mathbf{B}\Sigma_1$). Assume that $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ is a model of $\mathbf{B}\Sigma_1$. Since $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)\prec_0\mathfrak{A}$, $\mathfrak{A}\models\gamma(d,a)$. So, there is in $\mathfrak{A}$ an one–one Σ_1–mapping from $(\le d+1)$ to $(\le d)$, a contradiction.

Case B: $n\ge 1$. We proceed as in case A but now we use $\mathbf{Sat}_{\Pi_n}(x)$. Let $\gamma(d,a)$ be the following formula

$$\forall u\le d+1\,\exists\sigma\le d \begin{cases} \mathbf{Form}_{\Pi_n}(\sigma)\ \wedge \\ \exists z_1\,\exists z_2 \begin{cases} \mathbf{Sat}_{\Pi_n}(\sigma(\dot u,\dot a,\dot z_1,\dot z_2))\ \wedge \\ \forall x<u\,\neg\mathbf{Sat}_{\Pi_n}(\sigma(\dot x,\dot a,\dot z_1,\dot z_2)) \end{cases} \end{cases}$$

We have that $\gamma\in\Sigma_{n+1}(\mathbf{B}\Sigma_{n+1})$. Assume that $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)\models\mathbf{B}\Sigma_{n+1}$. Since $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)\prec_n\mathfrak{A}$, $\mathfrak{A}\models\gamma(d,a)$. So, there exists in $\mathfrak{A}$ an one–one Σ_{n+1}–mapping from $(\le d+1)$ to $(\le d)$, a contradiction.

(*(4)*): Suppose that $\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a)\models\mathbf{I}\Sigma_{n+1}$. By *(2)* we have that for all $k\in\omega$, $\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a)\models\exists y\,[\varphi(k,a,y)\wedge\forall u<k\,\exists z<y\,\varphi(u,a,z)]$. Then by Σ_{n+1}–overspill there exists $c\in\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ nonstandard such that

$$\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a)\models\exists y\,[\varphi(c,a,y)\wedge\forall u<c\,\exists z<y\,\varphi(u,a,z)].$$

Let $b\in\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ such that $\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a)\models\varphi(c,a,b)\wedge\forall u<c\,\exists z<y\,\varphi(u,a,z)$. Then for all $k\in\omega$, $\mathcal{I}_0^{\Gamma_\varphi}(\mathfrak{A},a)\models G_k(a)<b$, a contradiction (see 6.8).

(*(5)*): This follows from 6.6.

(*(6)*): Let $\psi(x,z)\in\Pi_n$ and $k\in\omega$ such that $\mathbf{T}\vdash\forall x\,\exists z\,\psi(x,z)$ and

$$\mathbf{T}\vdash\forall x\,\forall y\,[\varphi(k,x,y)\to\exists z\le y\,\psi(x,z)].$$

Let $\psi'(x,z)\in\Delta_{n+1}(\mathbf{B}\Sigma_n)$ be $\psi(x,z)\wedge\forall w<z\,\neg\psi(x,w)$. For all $b\in\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$ there is $c\in\mathfrak{A}$ such that $\mathfrak{A}\models\psi'(b,c)$. Since Γ_φ is a strong Π_n–envelope, by 4.12, $\psi'(x,z)\in\Delta_{n+1}(\mathbf{I}\Delta_0+\Gamma_\varphi^*)$; hence, there exists $\theta(x,z)\in\Delta_0^{\Gamma_\varphi}$ such that $\mathbf{I}\Delta_0^{\Gamma_\varphi}\vdash\psi'(x,z)\leftrightarrow\theta(x,z)$. Since $\mathfrak{A}\models\mathbf{I}\Delta_0^{\Gamma_\varphi}$, $\mathfrak{A}\models[\theta,k](c;b)$ and $c\in\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$. Since $\theta\in\Delta_0^{\Gamma_\varphi}$, by *(1)*, $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)\models\theta(b,c)$. So, by *(3)*, $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)\models\psi'(b,c)$; hence, $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)\models\psi(b,c)$. So, $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)\models\forall x\,\exists z\,\psi(x,z)$. This proves *(6)*.

(*(7)*): This follows from 2.13 ($\mathbf{T}$ is Π_n–functional and 3.8) and *(6)*. □

Theorem 6.11 *(The Hierarchy Theorem). Let* $\mathbf{T}$ *be a* Π_n*–functional theory (if* $n = 0$ *we assume that* $\mathbf{T} \vdash \mathbf{exp}$*),* $\varphi(u,x,y)$ *a strong* Π_n*–envelope of* $\mathbf{T}$ *in* $\mathbf{T}$ *and* $\mathbf{T}'$ *an extension of* $\mathbf{T}$ *such that* $\mathbf{T}' \vdash \forall u\, \forall x\, \exists y\, \varphi(u,x,y)$*, and* $\mathbf{T}' \vdash \varphi(u,x,y_1) \wedge \varphi(u+1,x,y_2) \rightarrow y_1 < y_2$*. Then*

(1) *For each* $\mathfrak{A} \models \mathbf{T}'_{\Gamma_\varphi}$ *and* $a \in \mathfrak{A}$ *nonstandard,* $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \mathbf{I}\Delta_{n+1}(\mathbf{T})$ *and* $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \not\models \mathbf{I}\Delta_{n+1}(\mathbf{T}')$.

(2) $\mathbf{I}\Delta_{n+1}(\mathbf{T}') \models\!\!\!\Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{T})$.

Proof. Part *(2)* follows from *(1)*. By 6.10–*(7)*, $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \models \mathbf{I}\Delta_{n+1}(\mathbf{T})$. Since $\exists y\, \varphi(u,x,y) \in \Delta_{n+1}(\mathbf{T}')$ and, by 6.9, $\exists y\, \varphi(u,a,y)$ defines ω in $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a)$, then $\mathcal{K}_0^{\Gamma_\varphi}(\mathfrak{A},a) \not\models \mathbf{I}\Delta_{n+1}(\mathbf{T}')$. □

Theorem 6.12. (1) *For all* $m \leq n$, $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_m) \Longleftrightarrow \mathbf{I}\Sigma_n$.

(2) *For all* $m \geq n$, $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{m+1}) \models\!\!\!\Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_m)$.

(3) $\mathbf{I}\Delta_{n+1}(\mathcal{N}) \models\!\!\!\Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{PA})$.

Proof. *(1)* follows from 2.18. Let us see *(2)*. By 5.22–*(1)*, for every $m \geq n$ there exists a strong Π_n–envelope that satisfies the hypothesis of 6.11 for $\mathbf{T} = \mathbf{I}\Sigma_m$ and $\mathbf{T}' = \mathbf{I}\Sigma_{m+1}$; hence, *(2)* follows from 6.11–*(1)*. Part *(3)* is proved in a similar way using 5.22–*(2)*. □

Lemma 6.13. *For every* $m \geq n$, $\mathbf{B}\Sigma_{n+1} \not\Longrightarrow \mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{m+1})$.

Proof. Since $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1})$ is a Π_{n+2}–axiomatizable theory (see [8], theorem **1.1**, or [7], [15]) and, by 6.12, $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{n+1}) \models\!\!\!\Longrightarrow \mathbf{I}\Sigma_n$, the result follows from 1.3. □

Theorem 6.14. (1) *For all* $m \geq n$, $\mathbf{I}\Delta_{n+1}(\mathbf{I}\Sigma_{m+1}) \models\!\!\!\Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{I}\Sigma_{m+1})$.

(2) $\mathbf{I}\Delta_{n+1}(\mathbf{PA}) \models\!\!\!\Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{PA})$.

(3) $\mathbf{I}\Delta_{n+1}(\mathcal{N}) \models\!\!\!\Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathcal{N})$.

Proof. First observe that for every theory $\mathbf{T}$, $\mathbf{B}\Sigma_{n+1} \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$, and if $\mathbf{T}$ has Δ_{n+1}–collection then, by 2.10, $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{T})$. Since $\mathbf{I}\Sigma_{m+1}$, $m \geq n$, $\mathbf{PA}$ and $\mathbf{Th}(\mathcal{N})$ have Δ_{n+1}–collection, then *(1)*, *(2)* and *(3)* follow from 6.13. □

7. Remarks and open questions

The main problem we have considered in this work is the Paris–Friedman's Conjecture in three versions

(1) Paris–Friedman's Conjecture: $\mathbf{I}\Delta_{n+1} \Longleftrightarrow \mathbf{L}\Delta_{n+1}$.

(2) Uniform Paris–Friedman's Conjecture: $\mathbf{UI}\Delta_{n+1} \Longleftrightarrow \mathbf{UL}\Delta_{n+1}$.

(3) Parameter Free Paris–Friedman's Conjecture: $\mathbf{I}\Delta^-_{n+1} \Longleftrightarrow \mathbf{L}\Delta^-_{n+1}$.

From Slaman's result, it holds $\mathbf{I}\Delta_{n+1} \Longleftrightarrow \mathbf{L}\Delta_{n+1}$, for $n \geq 1$. We have studied here the relativization of these problems to Δ_{n+1} formulas in a theory $\mathbf{T}$. This gives a new version of the Conjecture.

4. Relativized Paris–Friedman's Conjecture: $\mathbf{I}\Delta_{n+1}(\mathbf{T}) \Longleftrightarrow \mathbf{L}\Delta_{n+1}(\mathbf{T})$.

We have proved that if $\mathbf{T}$ satisfies some conditions then the relativized Paris–Friedman's Conjecture for $\mathbf{T}$ holds. So, we consider the following strong forms of these Conjectures.

Problem 1. Does it hold that for all $\mathbf{T}$ extension of $\mathbf{I}\Sigma_n$

(1) If $\mathbf{T}$ is Δ_{n+1}–closed then $\mathbf{T}$ has Δ_{n+1}–collection?
(2) If $\mathbf{T}$ has Δ_{n+1}–induction then $\mathbf{T}$ has Δ_{n+1}–collection?
(3) If $\mathbf{T}$ is Δ_{n+1}–PF then $\mathbf{T}$ has Δ_{n+1}–collection?

Let us observe that if every (complete) extension of $\mathbf{I}\Sigma_n$ satisfies 1–*(2)* then the Uniform Paris–Friedman's Conjecture holds.

Condition 3.10–*(2)* is related with the Uniform Paris-Friedman's Conjecture. Let $\mathfrak{A} \models \mathbf{UI}\Delta_{n+1}$ and $\varphi(x,y) \in \Pi_n^-$ such that $\mathfrak{A} \models \forall x\,\exists y\,\varphi(x,y)$. Let $F_\varphi : \mathfrak{A} \longrightarrow \mathfrak{A}$ be defined by: $F_\varphi(a) = (\mu y)[\varphi(a,y)]$. Let F_φ^* be, the bounding map of F_φ, defined by

$$F_\varphi^*(a) = (\mu x)_{\leq a}[\forall u \leq a\,(F_\varphi(u) \leq F_\varphi(x))]$$

Claim. Let $\mathfrak{A} \models \mathbf{UI}\Delta_{n+1}$. If for each $\varphi(x,y) \in \Pi_n^-$ such that $\mathfrak{A} \models \forall x\,\exists! y\,\varphi(x,y)$, it holds that F_φ^* is a total function on $\mathfrak{A}$ then $\mathfrak{A} \models \mathbf{UL}\Delta_{n+1}$.

Let us consider the following question.

Problem 2. In the above conditions. Is F_φ^* a total function?

In 3.12 we have obtained a conservativeness property, $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$, under which $\mathbf{T}$ is Π_n–functional and, hence, satisfies the Relativized Paris–Friedman's Conjecture. We have also extended this result in 3.13 for Σ_{n+2} extensions of Π_{n+2} axiomatizable theories. Let us consider the following problems.

Problem 3. (1) Let $\mathbf{T}$ be a theory such that $\mathbf{T}+\mathbf{B}\Sigma_{n+1}$ is consistent. Are the following conditions equivalent?
   (a) $\mathbf{T}$ is Π_n–functional.
   (b) $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1})$.
   (c) $\mathbf{Th}_{\Pi_{n+2}}(\mathbf{T}) = \mathbf{Th}_{\Pi_{n+2}}(\mathbf{T} + \mathbf{B}\Sigma_{n+1}^-)$.

(2) Let $\mathbf{T}$ be a Π_{n+2}–axiomatizable extension of $\mathbf{I}\Sigma_n$ and let $\mathbf{T}'$ be Σ_{n+2}–axiomatizable such that $\mathbf{T} + \mathbf{T}'$ is consistent. Does it hold that

$$\mathbf{T} \text{ is } \Pi_n\text{–functional} \iff \mathbf{T} + \mathbf{T}' \text{ is } \Pi_n\text{–functional?}$$

In 5.11 it is proved that if $\mathbf{T}$ is Π_n–functional and recursively axiomatizable then $\mathbf{T}$ has a Π_n–envelope in $\mathbf{I}\Sigma_n$ (for $n = 0$ we add that $\mathbf{T} \vdash \mathbf{exp}$). For all $n \in \omega$, $\mathbf{Th}_{\Pi_{n+2}}(\mathcal{N})$ is Π_n–functional and proves $\mathbf{exp}$. Nevertheless, $\mathbf{Th}_{\Pi_{n+2}}(\mathcal{N})$ does not have a Π_n–envelope in $\mathbf{I}\Sigma_n$. So, it cannot be omitted that $\mathbf{T}$ is recursively axiomatizable.

Now, we will consider if $\mathbf{T} \vdash \mathbf{exp}$ could be eliminated for $n = 0$. The theory $\mathbf{I}\Pi_1^-$ has Π_0–collection, is recursively axiomatized and $\mathbf{I}\Pi_1^- \nvdash \mathbf{exp}$. It holds that if $\varphi(x,y) \in \Delta_0^-$ and $\mathbf{I}\Pi_1^- \vdash \forall x\,\exists y\,\varphi(x,y)$ then there exists $k \in \omega$ such that $\mathbf{I}\Pi_1^- \vdash \exists z\,\forall x\,[z < x \rightarrow \exists y < x^k\,\varphi(x,y)]$ (see [5]).

From this it follows that $\varphi(u,x,y) \equiv x^u + u = y$ is a Π_0–envelope of $\mathbf{I}\Pi_1^-$ in $\mathbf{Th}_{\Pi_1}(\mathcal{N})$. Let us consider the following problem.

Problem 4. Is there a Π_0–envelope of $\mathbf{I\Pi}_1^-$ in $\mathbf{I\Delta}_0$?

In section 6 the models $\mathcal{K}_0^\Gamma(\mathfrak{A}, a)$ have been used to separate the fragments $\mathbf{I\Delta}_{n+1}(\mathbf{I\Sigma}_m)$, $m \geq n$. Theorem 1.1 sums up results obtained using these models. Let us consider the following problem.

Problem 5. Is strict the following chain of theories?

$$\mathbf{B}^*\Delta_{n+1}(\mathcal{N}) \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{PA}) \Longrightarrow \ldots \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{I\Sigma}_{n+1}) \Longrightarrow \mathbf{B}^*\Delta_{n+1}(\mathbf{I\Sigma}_n)$$

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