A note on parameter free Π_1-induction and restricted exponentiation

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We characterize the sets of all Π_2 and all $\mathcal{B}(\Sigma_1)$ (= Boolean combinations of Σ_1) theorems of $I\Pi_1^-$ in terms of restricted exponentiation, and use these characterizations to prove that both sets are not deductively equivalent.

We also discuss how these results generalize to $n > 0$. As an application, we prove that a conservation theorem of Beklemishev stating that $I\Pi_{n+1}^-$ is conservative over $I\Sigma_n^-$ with respect to $\mathcal{B}(\Sigma_{n+1})$ sentences cannot be extended to Π_{n+2} sentences.

1 Introduction

A natural question in studying the strength of an arithmetic theory T, is to characterize the sets $\mathrm{Th}_\Gamma(T)$ of all theorems of T of a specifical logical complexity Γ. In this paper we shall deal with this question for the scheme of parameter free Π_1-induction $I\Pi_1^-$.

The theory $I\Pi_1^-$ is the first order theory in the language of Peano Arithmetic $\mathcal{L} = \{0, 1, +, \cdot, \leq\}$ given by finitely many algebraic axioms P^- together with the *induction scheme*

$$(I_\varphi)\ \ \varphi(0) \wedge \forall x\,(\varphi(x) \longrightarrow \varphi(x+1)) \ \longrightarrow\ \forall x\,\varphi(x)$$

for each $\varphi(x) \in \Pi_1^-$. As usual, we write $\varphi(x) \in \Gamma^-$ to mean that φ is in Γ and contains no other free variables but x (we refer the reader to [8, 6] for the fundamental notions and results of first order Arithmetic, and to [7], [9] for properties of $I\Pi_1^-$).

Our interest in $I\Pi_1^-$ is motivated by the fact that, in a sense, this theory lies in between *weak* and *strong* fragments of Arithmetic. First, observe that $I\Pi_1^-$ is axiomatized by a set of true Σ_2 sentences and hence follows from the set of all true Π_1 arithmetic sentences, $\mathrm{Th}_{\Pi_1}(\mathcal{N})$. So, $I\Pi_1^-$ cannot prove the totality of functions of more than polynomial growth and in particular exponentiation is not necessarily total in a model of $I\Pi_1^-$. From this point of view, $I\Pi_1^-$ can be considered a *weak* theory close to the scheme of induction for bounded formulas $I\Delta_0$. In fact, it is a theorem of Bigorajska that $I\Pi_1^-$ satisfies Parikh's theorem "*asymptotically.*"

Theorem 1.1 (Bigorajska [2]) *Suppose $\varphi(x, y) \in \Sigma_1$ and $I\Pi_1^- \vdash \forall x\,\exists y\,\varphi(x, y)$. Then, there is a term $t(x)$ such that $I\Pi_1^- \vdash \exists u\,\forall x > u\exists y \leq t(x)\,\varphi(x, y)$.*

Since [9] it has been known, however, that there is a tight relationship between $I\Pi_1^-$ and the *strong* system $I\Delta_0 + \exp$ (as usual $\exp$ denotes a Π_2 axiom asserting "exponentiation is total"). Although both theories are incomparable with respect to inclusion, they share the same $\mathcal{B}(\Sigma_1)$ consequences, where $\mathcal{B}(\Sigma_1)$ denotes the set of all boolean combinations of Σ_1 sentences.

Theorem 1.2 (Kaye et al. [9])

(1) $I\Delta_0 + \exp$ *is conservative over* $I\Pi_1^-$ *with respect to* Σ_2 *sentences.*

(2) $I\Pi_1^-$ *is conservative over* $I\Delta_0 + \exp$ *with respect to* Π_2 *sentences.*

(3) *So,* $\mathrm{Th}_{\Pi_1}(I\Pi_1^-) = \mathrm{Th}_{\Pi_1}(I\Delta_0 + \exp)$ *and* $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-) = \mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Delta_0 + \exp)$.

As a result $I\Pi_1^-$ proves that there are unboundedly many primes and implies the Δ_0 pigeonhole principle, for they are Π_1 consequences of $I\Delta_0 + \exp$ (whether or not these two properties can be proved from $I\Delta_0$ are important open problems). Besides, the following hierarchy of theories between $I\Delta_0$ and $I\Delta_0 + \exp$ arises:

$$I\Delta_0 \subseteq \mathrm{Th}_{\Pi_1}(I\Pi_1^-) \subseteq \mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-) \subseteq \mathrm{Th}_{\Pi_2}(I\Pi_1^-) \subset I\Delta_0 + \exp$$

In [12, 13] Wilkie and Paris studied the Π_1 theorems of $I\Pi_1^-$ under its equivalent formulation of $\mathrm{Th}_{\Pi_1}(I\Delta_0 + \exp)$. In [12] Wilkie characterized the Π_1 sentences provable in $I\Delta_0 + \exp$ as those Π_1 sentences which are interpretable in Robinson's Arithmetic Q. In [13] the authors showed that a Π_1 sentence expressing the Herbrand consistency of Q is provable in $I\Delta_0 + \exp$ but not in $I\Delta_0$ and hence $\mathrm{Th}_{\Pi_1}(I\Pi_1^-)$ is strictly stronger than $I\Delta_0$. In contrast, no similar results were known for the $\mathcal{B}(\Sigma_1)$ and the Π_2 theorems of $I\Pi_1^-$. In particular, it was not known whether $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$ follows from $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$.

The aim of this paper is to fill these gaps in our understanding of the hierarchy above by obtaining informative axiomatizations of $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ and $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$, and by proving that the inclusions are all strict. To this end, the key idea is to consider certain forms of *restricted exponentiation*. We shall show that $I\Pi_1^-$ *can* prove that x^y exists whenever x, y are restricted to be in certain initial segments determined by the Σ_1 *definable elements* of a model. We say that a is Σ_n definable in $\mathfrak{A}$ if there is $\varphi(x) \in \Sigma_n$ such that a is the unique element satisfying $\varphi(x)$. We write $\mathcal{K}_n(\mathfrak{A}, X)$ to denote the set of all Σ_n definable elements of $\mathfrak{A}$ (possibly with a parameter from X); or simply $\mathcal{K}_n(\mathfrak{A})$ if $X = \emptyset$. We write $\mathcal{I}_n(\mathfrak{A}, X)$ to denote the least initial segment of $\mathfrak{A}$ containing $\mathcal{K}_n(\mathfrak{A}, X)$. Given $\mathfrak{A} \models I\Delta_0$ we shall deal with the following hierarchy of submodels of $\mathfrak{A}$ based on iterated Σ_n definability.[1]

- $\mathcal{K}_n^0(\mathfrak{A}) = \mathcal{K}_n(\mathfrak{A})$, i.e., the submodel of $\mathfrak{A}$ whose domain consists of all Σ_n definable elements; $\mathcal{I}_n^0(\mathfrak{A}) = \mathcal{I}_n(\mathfrak{A})$, i.e., the least initial segment of $\mathfrak{A}$ containing $\mathcal{K}_n(\mathfrak{A})$.
- For each $k \geq 0$, $\mathcal{K}_n^{k+1}(\mathfrak{A}) = \mathcal{K}_n\left(\mathfrak{A}, \mathcal{I}_n^k(\mathfrak{A})\right)$, i.e., the submodel of $\mathfrak{A}$ whose domain consists of all elements which are Σ_n definable with a parameter from $\mathcal{I}_n^k(\mathfrak{A})$; $\mathcal{I}_n^{k+1}(\mathfrak{A}) = \mathcal{I}_n(\mathfrak{A}, \mathcal{I}_n^k(\mathfrak{A}))$, i.e., the least initial segment of $\mathfrak{A}$ containing $\mathcal{K}_n^{k+1}(\mathfrak{A})$.

By considering this hierarchy for $n = 1$, we can isolate the exact amount of exponentiation available in models of $I\Pi_1^-$: x^y exists whenever the base x is in $\mathcal{I}_1^1(\mathfrak{A})$ and the exponent y is in $\mathcal{I}_1(\mathfrak{A})$. In addition, the Π_2 and the $\mathcal{B}(\Sigma_1)$ theorems of $I\Pi_1^-$ can be characterized over $I\Delta_0$ by principles of the form "*For certain x, y, x^y exists.*" More precisely, next theorem collects together the main results of the paper.

Theorem 1.3

(1) $\mathrm{Th}_{\Pi_2}(I\Pi_1^-) \equiv I\Delta_0 + \forall a \in \mathcal{I}_1\, \forall b \in \mathcal{I}_1^1\, \exists y\, (y = b^a)$.

(2) $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-) \equiv I\Delta_0 + \forall a, b \in \mathcal{I}_1\, \exists y\, (y = b^a)$.

(3) $\mathrm{Th}_{\Pi_1}(I\Pi_1^-) \nvdash \mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$, *and* $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-) \nvdash \mathrm{Th}_{\Pi_2}(I\Pi_1^-)$.

Let us observe that the characterizations above are to be seen as axiomatizations in the original language $\mathcal{L} = \{0, 1, +, \cdot, \leq\}$. Indeed, one can formalize a quantifier of the form "$\forall x \in \mathcal{I}_1$" by using a *schema* of formulas (so, $\mathcal{I}_1$ is not meant to be a definable cut). Namely, a quantifier of the form "$\forall x \in \mathcal{I}_1$" in front of a formula $\Psi(x)$ is intended as a shorthand for the scheme:

$$\{\forall x, y\, (\delta(y) \wedge \forall y'(\delta(y') \longrightarrow y' = y) \wedge x \leq y \longrightarrow \Psi(x)) \;:\; \delta \in \Sigma_1\}$$

In a similar vein, a quantifier of the form "$\forall x \in \mathcal{I}_1^1$" in front of a formula $\Psi(x)$ unravels to:

$$\left\{\forall x, y, z, u \left(\begin{array}{c} \delta_1(y) \wedge \forall y'(\delta_1(y') \longrightarrow y' = y) \wedge z \leq y \\ \delta_2(u, z) \wedge \forall u'(\delta_2(u', z) \longrightarrow u' = u) \wedge x \leq u \end{array} \longrightarrow \Psi(x)\right) \;:\; \delta_1, \delta_2 \in \Sigma_1\right\}$$

[1] The use of these structures goes back to Paris and Kirby's [10], where the authors used them to construct a model of $B\Sigma_{n+1} + \neg I\Sigma_{n+1}$.

Our proofs are model-theoretic and are based on the proofs of Theorem 1.2 and other related results in [7] and [9]. It will be important, however, to make explicit a property that appeared only implicitly in [7] or [9]: although $I\Pi_1^-$ is much weaker than the *full* scheme of induction for Σ_1 formulas $I\Sigma_1$, $I\Pi_1^-$ does allow for induction up to $\mathcal{I}_1(\mathfrak{A})$ for Σ_1 formulas with parameters in $\mathcal{I}_1(\mathfrak{A})$. This property clarifies why a certain amount of exponentiation is available in models of $I\Pi_1^-$ and, in a sense, explains the particular behavior of $I\Pi_1^-$ mixing features of weak and strong fragments of Peano Arithmetic.

To close this section, we state some basic notions and results on definable elements that we shall use throughout the paper (for proofs we refer the reader to [6] and [8]). As usual, we write $\mathfrak{A} \prec_n \mathfrak{B}$, $\mathfrak{A} \subseteq^e \mathfrak{B}$, and $\mathfrak{A} \subseteq^{cf} \mathfrak{B}$ to mean that $\mathfrak{A}$ is an n-elementary substructure of $\mathfrak{B}$, $\mathfrak{A}$ is an initial segment of $\mathfrak{B}$, and $\mathfrak{A}$ is cofinal in $\mathfrak{B}$ (i.e., unbounded above), respectively. We say that a is Π_n *minimal* in $\mathfrak{A}$ if there is $\varphi(x) \in \Pi_n$ such that a is the least element satisfying $\varphi(x)$. We write $\mathcal{M}_n(\mathfrak{A}, X)$ to denote the set of all Π_n minimal elements of $\mathfrak{A}$ (possibly with a parameter from X). Finally, $(x)_0, (x)_1$ denote the inverse functions of Cantor's paring function.

Proposition 1.4 $(n > 0)$ *Suppose* $\mathfrak{A} \models I\Sigma_{n-1}$ *and* $X \subseteq \mathfrak{A}$.

(1) $\mathcal{K}_n(\mathfrak{A}, X) \prec_n \mathfrak{A}$; *and* $\mathcal{K}_n(\mathfrak{A}, X) \prec_n^{cf} \mathcal{I}_n(\mathfrak{A}, X) \prec_{n-1}^{e} \mathfrak{A}$. *In particular, every nonempty* Σ_n *definable set contains some* Σ_n *definable element.*

(2) $\mathcal{M}_{n-1}(\mathfrak{A}, X)$ *is cofinal in* $\mathcal{K}_n(\mathfrak{A}, X)$. *Even more, for each* a *in* $\mathcal{K}_n(\mathfrak{A}, X)$ *there is* b *in* $\mathcal{M}_{n-1}(\mathfrak{A}, X)$ *such that* $a = (b)_0$.

(3) *If* $X = \emptyset$, $\mathfrak{A} \models I\Sigma_{n-1}^-$ *suffices in* 1 *and* 2.

2 Restricted exponentiation in models of $I\Pi_1^-$

Although $I\Pi_1^-$ is known to be much weaker than the scheme of induction for Σ_1 formulas $I\Sigma_1$, our first result shows that $I\Pi_1^-$ allows for Σ_1 induction restricted to the initial segment determined by the Σ_1 definable elements (we write it for $I\Pi_{n+1}^-$ and $n \geq 0$ since we will need this general version in a subsequent section).

Proposition 2.1 *Suppose* $\mathfrak{A} \models I\Pi_{n+1}^-$. *Then, for each* $\varphi(x, v) \in \Sigma_{n+1}$ *and* $a, b \in \mathcal{I}_{n+1}(\mathfrak{A})$, *it holds that* $\mathfrak{A} \models \varphi(0, b) \wedge \forall x\, (\varphi(x, b) \longrightarrow \varphi(x+1, b)) \longrightarrow \varphi(a, b)$.

Proof. Assume $\mathfrak{A} \models \varphi(0, b) \wedge \forall x\, (\varphi(x, b) \rightarrow \varphi(x+1, b))$. We wish to prove $\mathfrak{A} \models \varphi(a, b)$. The idea is to show that $\varphi(x, b)$ is equivalent to a bounded formula for elements $x \leq a$ and then apply (parametric) Δ_0 induction. Pick $c \in \mathcal{K}_{n+1}(\mathfrak{A})$ such that $a, b < c$.

Claim 2.2 *If* $\theta(x) \in \Pi_{n+1}$ *and there is* $d \in \mathcal{K}_{n+1}(\mathfrak{A})$ *such that* $\mathfrak{A} \models \theta(d)$, *then* $\mathfrak{A} \models \exists x\, (x = (\mu t)\, (\theta(t)))$.

Towards a contradiction, assume $\mathfrak{A} \models \forall x\, (\theta(x) \rightarrow \exists x' < x\, \theta(x'))$. Put $\theta(x) \equiv \forall y\, \theta_0(x, y)$, where $\theta_0 \in \Sigma_n$.

$n = 0$: Define $\theta'(z)$ to be $\exists u\, \forall x \leq z\, \exists y \leq u\, \neg\theta_0(x, y)$.

$n > 0$: Define $\theta'(z)$ to be $\exists s\, \forall x \leq z\, \neg\theta_0(x, (s)_x)$ (observe that $\mathfrak{A} \models \exp$ and so we can make use of a standard coding for sequences in $\mathfrak{A}$).

Clearly, $\theta'(z)$ is in Σ_{n+1} and implies $\forall x \leq z\, \neg\theta(x)$. So, $\mathfrak{A} \models \theta'(0) \wedge \forall z\, (\theta'(z) \longrightarrow \theta'(z+1)) \wedge \neg\theta'(d)$. Now consider $\theta''(z, d) \equiv \neg\theta'(d - z)$. Then, $\theta'' \in \Pi_{n+1}$ and $\mathfrak{A} \models \theta''(0, d) \wedge \forall z\, (\theta''(z, d) \longrightarrow \theta''(z+1, d))$. If $\delta(v)$ is a Σ_{n+1} definition of d, we can get rid of the parameter in θ'' by considering the formula $\forall v\, (\delta(v) \longrightarrow \theta''(z, v))$. Hence $\mathfrak{A} \models \forall z\, \theta''(z, d)$ by $I\Pi_{n+1}^-$. In particular, $\mathfrak{A} \models \theta''(d, d)$ and so $\mathfrak{A} \models \theta(0)$, which is a contradiction.

Observe that 2^x exists in $\mathfrak{A}$ for each Σ_{n+1} definable exponent (for $n > 0$ it is trivial; for $n = 0$ it follows from the claim by considering the least t satisfying $\neg\exists y\, (y = 2^t)$). Hence, 2^x also exists for each exponent in $\mathcal{I}_{n+1}(\mathfrak{A})$. As a consequence, we can use a Δ_0 formula $x \in u$ expressing "*the* x*-th digit of the binary expansion of* u *is* 1" to codify finite sets included in $\mathcal{I}_{n+1}(\mathfrak{A})$. Now define $\theta(u, c)$ to be the Π_{n+1} formula:

$$\forall x, v < c\, (\varphi(x, v) \longrightarrow \langle x, v \rangle \in u)$$

Clearly, $\mathfrak{A} \models \theta\big(2^{\langle c, c \rangle} - 1, c\big)$. By the claim, there exists $e \in \mathcal{I}_{n+1}(\mathfrak{A})$ satisfying $e = (\mu u)\, (\theta(u, c))$ (again we can get rid of the parameter c in θ by using a Σ_{n+1} definition of it). Now it follows from the definition of θ that $\mathfrak{A} \models \forall x, v < c\, (\varphi(x, v) \leftrightarrow \langle x, v \rangle \in e)$. So, we can obtain $\mathfrak{A} \models \varphi(a, b)$ by applying Δ_0 induction to the formula $x < c \rightarrow \langle x, b \rangle \in e$. $\square$

It immediately follows from the previous result that $I\Pi_1^-$ proves that b^a exists whenever $a, b \in \mathcal{I}_1(\mathfrak{A})$ (it is sufficient to apply induction up to a to the Σ_1 formula $\exists y\,(y = b^x)$). However, a stronger result actually holds:

Theorem 2.3 *Suppose $\mathfrak{A} \models I\Pi_1^-$. Then, for each $a \in \mathcal{I}_1(\mathfrak{A})$ and $b \in \mathcal{I}_1^1(\mathfrak{A})$, b^a exists in $\mathfrak{A}$.*

Proof. Pick $c \in \mathcal{K}_1^1(\mathfrak{A})$ such that $b \leq c$ and let $\delta(u, d)$ be a Σ_1 definition of c, where $d \in \mathcal{I}_1(\mathfrak{A})$. By applying induction up to a to the Σ_1 formula $\exists y, u\,(\delta(u, d) \wedge y = u^x)$ (available in $\mathfrak{A}$ thanks to Proposition 2.1), we get $\mathfrak{A} \models \exists y\,(y = c^a)$. But it is straightforward to check that $\exists y\,(y = z^x) \wedge u \leq z \longrightarrow \exists y\,(y = u^x)$ is provable in $I\Delta_0$. Consequently, b^a exists in $\mathfrak{A}$. □

It is natural to ask whether Theorem 2.3 captures the amount of exponentiation available in every model of $I\Pi_1^-$. We close this section by showing that this is the case: Theorem 2.3 is best possible in terms of the hierarchy $\mathcal{I}_1^k(\mathfrak{A})$ (and hence so is Proposition 2.1). First, we need the following two lemmas.

Lemma 2.4 *Suppose $\mathfrak{A} \models I\Pi_1^-$, $\mathfrak{B} \prec_0 \mathfrak{A}$ and $\mathcal{K}_1^1(\mathfrak{A}) \subseteq \mathfrak{B}$. Then, $\mathfrak{B}$ also satisfies $I\Pi_1^-$.*

Proof. Let $\varphi(x) \in \Pi_1$. We need to prove that $\mathfrak{B} \models I_{\varphi(x)}$. Put $\varphi(x) \equiv \forall y\, \varphi_0(x, y)$, where $\varphi_0 \in \Delta_0$. The induction axiom for φ can be reexpressed as

$$\exists x, y, z\,[\neg\varphi_0(0, y) \vee (\varphi(x) \wedge \neg\varphi_0(x + 1, z)) \vee \forall x\, \varphi(x)]$$

Let $\theta(x, y, z)$ denote the Π_1 formula in brackets above. It suffices to show that θ can be 'witnessed' in $\mathcal{K}_1^1(\mathfrak{A})$, for it follows from $\mathfrak{B} \prec_0 \mathfrak{A}$ and $\mathcal{K}_1^1(\mathfrak{A}) \subseteq \mathfrak{B}$ that $\mathfrak{B} \models \exists x, y, z\, \theta(x, y, z)$.

Case 1: $\mathfrak{A} \models \forall x\, \varphi(x)$. Then, $\mathfrak{A} \models \theta(0, 0, 0)$.

Case 2: $\mathfrak{A} \models \neg\varphi(0)$. There is $a \in \mathcal{K}_1(\mathfrak{A})$ such that $\mathfrak{A} \models \neg\varphi_0(0, a)$ and then $\mathfrak{A} \models \theta(0, a, 0)$.

Case 3: $\mathfrak{A} \models \varphi(0) \wedge \exists x\, \neg\varphi(x)$. Pick $a \in \mathcal{K}_1(\mathfrak{A})$ such that $\mathfrak{A} \models \neg\varphi(a)$. Then, there is $b \leq a$ such that $\mathfrak{A} \models \varphi(b) \wedge \neg\varphi(b+1)$ (otherwise, it would follow from the induction axiom for $x \leq a \to \varphi(x)$ that $\mathfrak{A} \models \varphi(a)$). Since $b \in \mathcal{I}_1(\mathfrak{A})$, there is $c \in \mathcal{K}_1^1(\mathfrak{A})$ satisfying $\mathfrak{A} \models \neg\varphi_0(b + 1, c)$. Thus, $\mathfrak{A} \models \theta(b, 0, c)$. □

Lemma 2.5 *Suppose $\mathfrak{A} \models I\Sigma_1$ with nonstandard Σ_1 definable elements. Then, $\{\mathcal{I}_1^k(\mathfrak{A}) : k \in \omega\}$ forms a proper hierarchy, i.e., $\mathcal{K}_1^{k+1}(\mathfrak{A}) \neq \mathcal{I}_1^k(\mathfrak{A})$ for each $k \in \omega$.*

Proof. Assume that there is $k \in \omega$ such that $\mathcal{K}_1^{k+1}(\mathfrak{A}) = \mathcal{I}_1^k(\mathfrak{A})$ (we may also assume that $\mathcal{K}_1^{m+1}(\mathfrak{A}) \neq \mathcal{I}_1^m(\mathfrak{A})$ for all $m < k$). Then, $\mathcal{I}_1^k(\mathfrak{A})$ is a nonstandard model of $I\Sigma_1$. To see this, observe that if $\mathcal{I}_1^k(\mathfrak{A}) = \mathfrak{A}$, then the result is immediate; otherwise, it follows from $\mathcal{I}_1^k(\mathfrak{A}) \prec_1^e \mathfrak{A} \models I\Delta_0$ and $\mathcal{I}_1^k(\mathfrak{A}) \neq \mathfrak{A}$ that $\mathcal{I}_1^k(\mathfrak{A})$ satisfies the Σ_2 collection scheme $B\Sigma_2$ which is well-known to imply $I\Sigma_1$. In addition, there is $a \in \mathcal{I}_1^k(\mathfrak{A})$ satisfying that $\mathcal{K}_1(\mathcal{I}_1^k(\mathfrak{A}), (\leq a))$ is cofinal in $\mathcal{I}_1^k(\mathfrak{A})$ (if $k = 0$, consider $a = 0$; otherwise, pick $a \in \mathcal{K}_1^k(\mathfrak{A}) - \mathcal{I}_1^{k-1}(\mathfrak{A})$). But this is a contradiction, for the Σ_1 definable elements (even with parameters from a proper cut) of a nonstandard model of $I\Sigma_1$ are not cofinal in it (see, e.g., [6, Chapter IV, Lemma 1.37]). □

Proposition 2.6

(1) *There are $\mathfrak{A} \models I\Pi_1^-$ and $a \in \mathcal{K}_1^1(\mathfrak{A})$ such that 2^a does not exist in $\mathfrak{A}$.*

(2) *There are $\mathfrak{A} \models I\Pi_1^-$, $a \in \mathcal{I}_1(\mathfrak{A})$ and $b \in \mathcal{K}_1^2(\mathfrak{A})$ such that b^a does not exist in $\mathfrak{A}$.*

Proof. (1): Let $\mathfrak{B}$ be a model of $I\Pi_1^- + \neg\mathrm{con}(I\Delta_0)$, where $\mathrm{con}(I\Delta_0)$ is a Π_1 sentence expressing the consistency of $I\Delta_0$. Consider $\mathfrak{A} = \mathcal{K}_1^1(\mathfrak{B})$. Then, $\mathcal{K}_1^1(\mathfrak{A}) = \mathfrak{A}$, $\mathfrak{A} \models \neg\mathrm{con}(I\Delta_0)$, and $\mathfrak{A} \models I\Pi_1^-$ by Lemma 2.4.

Claim 2.7 $\mathfrak{A} \not\models I\Delta_0 + \exp$.

Towards a contradiction, assume $\mathfrak{A} \models I\Delta_0 + \exp$. Let $z = \mathrm{supexp}(x, y)$ be a Δ_0 formula defining the graph of the superexponential function and satisfying the usual recursion laws of superexponentiation in $I\Delta_0 + \exp$. Pick $a, b \in \mathcal{K}_1(\mathfrak{A})$. By applying induction up to a to the Σ_1 formula $\exists z\,(z = \mathrm{supexp}(b, y))$ (available in $\mathfrak{A}$ thanks to Proposition 2.1), we get $\mathfrak{A} \models \exists z\,(z = \mathrm{supexp}(b, a))$. Hence, $\mathcal{K}_1(\mathfrak{A})$ is a model of $I\Delta_0 + \mathrm{supexp} \vdash \mathrm{con}(I\Delta_0)$. So, $\mathfrak{A} \models \mathrm{con}(I\Delta_0)$, which is impossible.

Consequently, $\mathfrak{A}$ satisfies $I\Pi_1^-$ and there is $a \in \mathcal{K}_1^1(\mathfrak{A})$ such that 2^a does not exist in $\mathfrak{A}$.

(2): Let $\mathfrak{B}$ be a model of $I\Sigma_1$ containing nonstandard Σ_1 definable elements. Pick $b \in \mathcal{M}_0(\mathfrak{B}, \mathcal{I}_1^1(\mathfrak{B})) - \mathcal{I}_1^1(\mathfrak{B})$ (such an element exists by Lemma 2.5) and define $\mathfrak{A}$ to be the initial segment of $\mathfrak{B}$ determined by the standard powers of b, i.e., $\mathfrak{A} = \{a \in \mathfrak{B} : \exists k \in \omega,\ a \leq b^k\}$. Then, $\mathfrak{A} \models I\Pi_1^-$ by Lemma 2.4, b is in $\mathcal{K}_1^2(\mathfrak{A})$, and b^a does not exist in $\mathfrak{A}$ for any nonstandard exponent a. □

3 The Π_2 and the $\mathcal{B}(\Sigma_1)$ Theorems of $I\Pi_1^-$

We have shown that $I\Pi_1^-$ proves "$\forall a \in \mathcal{I}_1\, \forall b \in \mathcal{I}_1^1\, b^a$ exists." This principle can be reexpressed as a set of Π_2 sentences of $\mathcal{L}$ and hence follows from $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$. Interestingly, the converse is also true. Our first characterization theorem states that, over $I\Delta_0$ declaring "$\forall a \in \mathcal{I}_1\, \forall b \in \mathcal{I}_1^1\, b^a$ exists" allows us to recover all Π_2 theorems of $I\Pi_1^-$. As a by-product, we also obtain a *Parsons-like* characterization of $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$. A well-known result of C. Parsons [11] states that the Π_2 theorems of $I\Sigma_1$ can be characterized as the closure of $I\Delta_0$ under the Σ_1 induction rule $I\Delta_0 + \Sigma_1$–IR. Likewise, the Π_2 theorems of $I\Pi_1^-$ can be captured by using a variant of that inference rule: the Σ_1 induction rule up to $\mathcal{I}_1$ with parameters in $\mathcal{I}_1^1$, denoted by $\left(\Sigma_1, \mathcal{I}_1, \mathcal{I}_1^1\right)$–IR. This inference rule is given by

$$\frac{\varphi(0,v) \wedge \forall x\,(\varphi(x,v) \longrightarrow \varphi(x+1,v))}{\forall v \in \mathcal{I}_1^1\, \forall x \in \mathcal{I}_1\, \varphi(x,v)},$$

where $\varphi(x,v)$ runs over Σ_1. Then, $I\Delta_0 + \left(\Sigma_1, \mathcal{I}_1, \mathcal{I}_1^1\right)$–IR denotes the closure of $I\Delta_0$ under $\left(\Sigma_1, \mathcal{I}_1, \mathcal{I}_1^1\right)$–IR; whereas $\left[I\Delta_0, \left(\Sigma_1, \mathcal{I}_1, \mathcal{I}_1^1\right)\text{–IR}\right]$ denotes the closure of $I\Delta_0$ under *unnested* applications of the rule.

Theorem 3.1 *The following theories are equivalent.*

(1) $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$.
(2) $I\Delta_0 + \left(\Sigma_1, \mathcal{I}_1, \mathcal{I}_1^1\right)$–*IR*.
(3) $\left[I\Delta_0, \left(\Sigma_1, \mathcal{I}_1, \mathcal{I}_1^1\right)\text{–}\mathit{IR}\right]$.
(4) $I\Delta_0 + \forall a \in \mathcal{I}_1\ \forall b \in \mathcal{I}_1^1\ \exists y\,(y = b^a)$.

Proof. (1⇒2): Since $I\Delta_0 + \left(\Sigma_1, \mathcal{I}_1, \mathcal{I}_1^1\right)\text{–IR} \subseteq \Pi_2$, it suffices to show that it can be proved from $I\Pi_1^-$. Indeed, we show the stronger result that $I\Pi_1^-$ is closed under $\left(\Sigma_1, \mathcal{I}_1, \mathcal{I}_1^1\right)$–IR. To this end, suppose $\mathfrak{A} \models I\Pi_1^-$ and $I\Pi_1^- \vdash \forall v\, \varphi(0,v) \wedge \forall v, x\,(\varphi(x,v) \longrightarrow \varphi(x+1,v))$, with $\varphi(x,v) \in \Sigma_1$. Let $a \in \mathcal{I}_1(\mathfrak{A})$ and $b \in \mathcal{I}_1^1(\mathfrak{A})$. Pick $c \in \mathcal{K}_1^1(\mathfrak{A})$ such that $b \leq c$ and let $\delta(v,d)$ be a Σ_1 definition of c, where $d \in \mathcal{I}_1(\mathfrak{A})$. Write $\varphi(x,v) \equiv \exists y\, \varphi_0(x,y,v)$, with $\varphi_0 \in \Delta_0$, and define $\theta(x,d)$ to be the Σ_1 formula

$$\exists v\,(\delta(v,d) \wedge \exists u\, \forall w \leq v\, \exists y \leq u\, \varphi_0(x,y,w))$$

Clearly, $\mathfrak{A} \models \forall x\,(\theta(x,d) \rightarrow \theta(x+1,d))$. Since $I\Pi_1^-$ proves $\forall v\, \exists y\, \varphi_0(0,y,v)$ and $I\Pi_1^-$ is closed under Σ_1 collection rule, $I\Pi_1^-$ also proves $\forall v\, \exists u\, \forall w \leq v\, \exists y \leq u\, \varphi_0(0,y,w)$ and so $\mathfrak{A} \models \theta(0,d)$. Thus, it follows from Proposition 2.1 that $\mathfrak{A} \models \theta(a,d)$ and hence $\mathfrak{A} \models \varphi(a,b)$.
(2⇒3), (3⇒4): Immediate.
(4⇒1): Suppose that $\mathfrak{A}$ is a (countable) model of $I\Delta_0$ and for each $a \in \mathcal{I}_1(\mathfrak{A})$, $b \in \mathcal{I}_1^1(\mathfrak{A})$, b^a exists in $\mathfrak{A}$. We need to show that $\mathfrak{A} \models \mathrm{Th}_{\Pi_2}(I\Pi_1^-)$.

Case 1: $\mathcal{I}_1^1(\mathfrak{A}) \neq \mathfrak{A}$.

Then, $\mathfrak{A}$ satisfies the least number axiom scheme for parameter free Σ_1 formulas $L\Sigma_1^-$, which is known to be equivalent to $I\Pi_1^-$. To see that, let $\varphi(x) \in \Sigma_1$ and $c \in \mathcal{K}_1(\mathfrak{A})$ such that $\mathfrak{A} \models \varphi(c)$. Put $\varphi(x) \equiv \exists y\, \varphi_0(x,y)$, where $\varphi_0 \in \Delta_0$, and pick $d \in \mathfrak{A}$ satisfying $\mathcal{I}_1^1(\mathfrak{A}) < d$. Then, $\mathfrak{A} \models \forall x \leq c\,(\exists y\, \varphi_0(x,y) \rightarrow \exists y \leq d\, \varphi_0(x,y))$ and so $\varphi(x)$ is equivalent to the Δ_0 formula $\exists y \leq d\, \varphi_0(x,y)$ for elements $x \leq c$. By applying Δ_0-minimization in $\mathfrak{A}$, we get that there is a least element satisfying $\varphi(x)$.

Case 2: $\mathcal{I}_1^1(\mathfrak{A}) = \mathfrak{A}$.

Then, we can assume that b^a exists in $\mathfrak{A}$ for $a \in \mathcal{I}_1(\mathfrak{A})$ and all $b \in \mathfrak{A}$. Now we essentially follow the proof of the Π_2 conservativity of $I\Pi_1^-$ over $I\Delta_0 + \exp$ included in Kaye's Ph.D. thesis [7, Theorem 10.8]. Towards a contradiction, assume that there is a Π_2 sentence θ such that $I\Pi_1^- \vdash \theta$ and $\mathfrak{A} \models \neg\theta$. By compactness, there are $\varphi_1(x), \ldots, \varphi_n(x) \in \Pi_1$ such that $I\Delta_0 + I_{\varphi_1} + \cdots + I_{\varphi_n} \vdash \theta$. For the sake of simplicity, we first observe that

Claim 3.2 *There is a single* Π_1 *formula* $\varphi(x)$ *such that* $\mathrm{Th}_{\Sigma_2}(\mathfrak{A}) + I_\varphi \vdash \theta$.

By reordering the formulas $\varphi_i(x)$ if necessary, we can assume that there is $m \leq n$ such that: (a) none of $I_{\varphi_1}, \dots, I_{\varphi_m}$ is true in $\mathfrak{A}$ (hence $\exists x\, \neg\varphi_i(x)$ is in $\mathrm{Th}_{\Sigma_2}(\mathfrak{A})$ for $1 \leq i \leq m$); and (b) $I_{\varphi_{m+1}}, \dots, I_{\varphi_n}$ are all true in $\mathfrak{A}$ (and hence are included in $\mathrm{Th}_{\Sigma_2}(\mathfrak{A})$). It is well-known that the least number axiom for a formula $\neg\delta(x)$, $L_{\neg\delta(x)}$, is equivalent to the induction axiom for $\forall y \leq x\, \delta(y)$. In addition, it is easy to check that $\mathrm{Th}_{\Sigma_2}(\mathfrak{A}) + L_{\neg\varphi_1((x)_1) \wedge \dots \wedge \neg\varphi_m((x)_m)} \vdash L_{\neg\varphi_1(x)} \wedge \dots \wedge L_{\neg\varphi_m(x)}$, where $(x)_1, \dots, (x)_m$ denote the inverse functions for a coding for m-tuples. Thus, it is sufficient to define $\varphi(x)$ to be the Π_1 formula $\forall y \leq x\, (\varphi_1((y)_1) \vee \dots \vee \varphi_m((y)_m))$.

Let $\mathfrak{B}$ be a recursively saturated elementary extension of $\mathfrak{A}$. It follows from the saturation of $\mathfrak{B}$ that there exists $c \in \mathfrak{B}$ such that, for each $\delta(x) \in \Pi_1$, $\mathfrak{B} \models \exists x\, \delta(x) \to \exists x \leq c\, \delta(x)$. As a result, $\mathcal{I}_1(\mathfrak{B}) < c$ and every initial segment of $\mathfrak{B}$ containing c is a model of $\mathrm{Th}_{\Sigma_2}(\mathfrak{A})$. Since $\mathfrak{A}$ and $\mathfrak{B}$ are elementary equivalent, b^a also exists in $\mathfrak{B}$ for $a \in \mathcal{I}_1(\mathfrak{B})$ and all $b \in \mathfrak{B}$. In addition, I_φ must fail in $\mathfrak{B}$ and so:

(i) $\mathfrak{B} \models \varphi(0) \wedge \forall x\, (\varphi(x) \to \varphi(x+1))$, and

(ii) there is $a \in \mathcal{K}_1(\mathfrak{B})$ such that $\mathfrak{B} \models \neg\varphi(a)$.

Put $\varphi(x) \equiv \forall y\, \varphi_0(x, y)$, with $\varphi_0 \in \Delta_0$. Then, we have:

Claim 3.3 $\exists a' \leq a\, \exists b\, \forall k \in \omega :\ \mathfrak{B} \models \forall z \leq (b + c)^k\, \varphi_0(a', z) \wedge \neg\varphi_0(a' + 1, b)$.

By the saturation of $\mathfrak{B}$, it is sufficient to show that $\forall k \in \omega\, \exists a' \leq a\, \exists b :\ \mathfrak{B} \models \forall z \leq (b + c)^k\, \varphi_0(a', z) \wedge \neg\varphi_0(a' + 1, b)$. Assume not. Then, there is some $k \in \omega$ such that:

$$\mathfrak{B} \models \forall x \leq a\, \forall y\, (\forall z \leq (c + y)^k\, \varphi_0(x, z) \to \varphi_0(x + 1, y))$$

Since $\mathfrak{B} \models \neg\varphi(a)$, there is $d \in \mathcal{K}_1(\mathfrak{B})$ such that $\mathfrak{B} \models \neg\varphi_0(a, d)$. Define $\psi(x)$ to be the Δ_0 formula

$$\forall z \leq (c + d)^{k^{((2a+1)-2x)}}\, \varphi_0(x, z)$$

Observe that $k^{((2a+1)-2x)}$ is in $\mathcal{I}_1(\mathfrak{B})$ and hence $(c + d)^{k^{((2a+1)-2x)}}$ exists. Clearly, $\mathfrak{B} \models \psi(0)$ and it is easy to check that $\mathfrak{B} \models \forall x < a\, (\psi(x) \to \psi(x + 1))$. Thus, by applying $I\Delta_0$ in $\mathfrak{B}$, we get that $\mathfrak{B} \models \psi(a)$ and so $\mathfrak{B} \models \varphi_0(a, d)$, which is a contradiction.

Let $a' \leq a$ and b as in the Claim above and consider $\mathfrak{C} = \{d \in \mathfrak{B} : \exists k \in \omega,\ d \leq (c+b)^k\}$. Since $\mathfrak{C}$ is an initial segment of $\mathfrak{B}$ containing c, $\mathfrak{C}$ satisfies $\mathrm{Th}_{\Sigma_2}(\mathfrak{A})$ and so $\mathfrak{C} \models \neg\theta$. However, by the Claim, $\mathfrak{C} \models \varphi(a') \wedge \neg\varphi(a'+1)$ and hence $\mathfrak{C}$ satisfies $\mathrm{Th}_{\Sigma_2}(\mathfrak{A}) + I_\varphi \vdash \theta$, which gives the desired contradiction. □

As a first application of Theorem 3.1, we obtain that

Corollary 3.4 $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$ *is not finitely axiomatizable.*

Proof. Let T_0 be a finite part of $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$. Since $\mathrm{Th}_{\Pi_1}(\mathcal{N}) \vdash \mathrm{Th}_{\Pi_2}(I\Pi_1^-)$, there is a true Π_1 sentence φ satisfying $I\Delta_0 + \varphi \vdash T_0$. Let $\mathfrak{A}$ be a model of $I\Sigma_1 + \varphi$ with nonstandard Σ_1 definable elements. Since $\mathcal{I}_1^1(\mathfrak{A}) \neq \mathcal{I}_1(\mathfrak{A})$ by Lemma 2.5, there is $b \in \mathcal{M}_0(\mathfrak{A}, \mathcal{I}_1(\mathfrak{A})) - \mathcal{I}_1(\mathfrak{A})$. Define $\mathfrak{B}$ to be the initial segment of $\mathfrak{A}$ determined by the standard powers of b. Clearly, $\mathfrak{B} \models I\Delta_0 + \varphi \vdash T_0$. In addition, $\mathfrak{B} \not\models \mathrm{Th}_{\Pi_2}(I\Pi_1^-)$, for b is in $\mathcal{I}_1^1(\mathfrak{B})$ and b^a does not exist in $\mathfrak{B}$ for any nonstandard exponent a. □

From Theorem 3.1, we can also derive a characterization of the $\mathcal{B}(\Sigma_1)$ consequences of $I\Pi_1^-$. First, we need the following simple but useful observation (recall that $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *1–elementary equivalent*, written $\mathfrak{A} \equiv_1 \mathfrak{B}$, if for each Σ_1 sentence φ, $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$).

Lemma 3.5 *Suppose* $\mathfrak{A} \models I\Delta_0$. *The following are equivalent*:

(1) $\mathfrak{A} \models \mathrm{Th}_{\mathcal{B}(\Sigma_1)}(T)$.

(2) *There exists* $\mathfrak{B} \models T$ *such that* $\mathfrak{A} \equiv_1 \mathfrak{B}$.

(3) $\mathcal{K}_1(\mathfrak{A}) \models \mathrm{Th}_{\Pi_2}(T)$.

Proof. (1⇒2): It suffices to show that $T + \mathrm{Th}_{\Pi_1}(\mathfrak{A}) + \mathrm{Th}_{\Sigma_1}(\mathfrak{A})$ is a consistent theory. Assume not, then there are $\varphi_1 \in \Pi_1$ and $\varphi_2 \in \Sigma_1$ satisfying that $T + \varphi_1 + \varphi_2$ is inconsistent and $\mathfrak{A} \models \varphi_1 \wedge \varphi_2$. Define θ to be $\neg\varphi_1 \vee \neg\varphi_2$. Then, θ is a $\mathcal{B}(\Sigma_1)$ sentence provable in T and $\mathfrak{A} \models \neg\theta$, which is impossible since $\mathfrak{A} \models \mathrm{Th}_{\mathcal{B}(\Sigma_1)}(T)$.

(2⇒3): Assume $\mathfrak{B}$ is a model of T satisfying that $\mathfrak{A} \equiv_1 \mathfrak{B}$. Then, $\mathcal{K}_1(\mathfrak{B})$ satisfies $\mathrm{Th}_{\Pi_2}(T)$ since $\mathcal{K}_1(\mathfrak{B}) \prec_1 \mathfrak{B}$. It follows from $\mathfrak{A}, \mathfrak{B} \models I\Delta_0$ and $\mathfrak{A} \equiv_1 \mathfrak{B}$ that there exists a canonical isomorphism between $\mathcal{K}_1(\mathfrak{A})$ and $\mathcal{K}_1(\mathfrak{B})$ (if $\varphi(x) \in \Sigma_1$ defines $a \in \mathfrak{A}$, map a to the unique $b \in \mathfrak{B}$ satisfying $\mathfrak{B} \models \varphi(b)$). So, $\mathcal{K}_1(\mathfrak{A}) \models \mathrm{Th}_{\Pi_2}(T)$.

(3⇒1): Since $\mathcal{K}_1(\mathfrak{A}) \prec_1 \mathfrak{A}$, $\mathfrak{A} \models \mathrm{Th}_{\Sigma_2}(\mathrm{Th}_{\Pi_2}(T))$. But clearly $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(T) \subseteq \mathrm{Th}_{\Sigma_2}(\mathrm{Th}_{\Pi_2}(T))$. □

Corollary 3.6

(1) *If T implies $I\Delta_0$ then* $\mathrm{Th}_{\Sigma_2}(\mathrm{Th}_{\Pi_2}(T)) \equiv \mathrm{Th}_{\mathcal{B}(\Sigma_1)}(T)$.

(2) $\mathrm{Th}_{\Sigma_2}(I\Delta_0 + \exp) \equiv \mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Delta_0 + \exp) \equiv \mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$.

We are now ready to give our characterization of the $\mathcal{B}(\Sigma_1)$ consequences of $I\Pi_1^-$ (and hence also of the Σ_2 consequences of $I\Delta_0 + \exp$) in terms of restricted exponentiation. As in Theorem 3.1, we also obtain a Parsons–like characterization of this class. To this end, we introduce the parameter free Σ_1 induction rule up to $\mathcal{I}_1$, denoted by $(\Sigma_1^-, \mathcal{I}_1)$–IR. This inference rule is given by

$$\frac{\varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1))}{\forall x \in \mathcal{I}_1\, \varphi(x)},$$

where $\varphi(x) \in \Sigma_1^-$.

Theorem 3.7 *The following theories are equivalent.*

(1) $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$.

(2) $I\Delta_0 + (\Sigma_1^-, \mathcal{I}_1)$*–IR.*

(3) $[I\Delta_0, (\Sigma_1^-, \mathcal{I}_1)$*–IR*$]$.

(4) $I\Delta_0 + \forall a, b \in \mathcal{I}_1\, \exists y\,(y = b^a)$.

Proof. (1⇒2): We prove the stronger fact that $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ is closed under $(\Sigma_1^-, \mathcal{I}_1)$–IR. Assume that $\mathfrak{A}$ is a model of $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ and $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-) \vdash \varphi(0) \wedge \forall x\,(\varphi(x) \to \varphi(x+1))$, with $\varphi(x) \in \Sigma_1^-$. Let $a \in \mathcal{I}_1(\mathfrak{A})$ and let $b \in \mathcal{K}_1(\mathfrak{A})$ such that $a \leq b$. By Lemma 3.5, there exists $\mathfrak{B} \models I\Pi_1^-$ such that $\mathfrak{A} \equiv_1 \mathfrak{B}$ (and so $\mathcal{K}_1(\mathfrak{A}) = \mathcal{K}_1(\mathfrak{B})$). Write $\varphi(x) \equiv \exists y\, \varphi_0(x, y)$, where $\varphi_0 \in \Delta_0$, and define $\theta(z)$ to be the Σ_1 formula $\exists u\, \forall x \leq z\, \exists y \leq u\, \varphi_0(x, y)$. By applying Σ_1 induction up to b (available in $\mathfrak{B}$ thanks to Proposition 2.1), we get $\mathfrak{B} \models \theta(b)$ and so $\mathfrak{A} \models \theta(b)$ since $\mathfrak{A} \equiv_1 \mathfrak{B}$. Thus, $\mathfrak{A} \models \varphi(a)$.

(2⇒3): Immediate.

(3⇒4): Assume $\mathfrak{A} \models [I\Delta_0, (\Sigma_1^-, \mathcal{I}_1)$–IR$]$. It follows that 2^a exists for each exponent $a \in \mathcal{I}_1(\mathfrak{A})$. So, $\mathcal{I}_1(\mathfrak{A}) \models I\Delta_0 + \exp$ and hence (4) follows.

(4⇒1): Assume $\mathfrak{A} \models I\Delta_0$ and b^a exists for all $a, b \in \mathcal{I}_1(\mathfrak{A})$. Clearly, $\mathcal{K}_1(\mathfrak{A}) \models I\Delta_0 + \exp$ and so $\mathcal{K}_1(\mathfrak{A}) \models \mathrm{Th}_{\Pi_2}(I\Pi_1^-)$ by Theorem 3.1. Thus, $\mathfrak{A} \models \mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ by Lemma 3.5. □

Reasoning as in Corollary 3.4, we also get that

Corollary 3.8 $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ *is not finitely axiomatizable.*

4 Separation results

We start by proving that $\mathrm{Th}_{\Pi_1}(I\Pi_1^-)$ is strictly weaker than $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$. Roughly speaking, given a nonstandard Δ_0 minimal element a, we consider the function

$$f_a(x) = \begin{cases} 2^x & \text{if } x = a \\ 0 & \text{if } x \neq a \end{cases}$$

Then, $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ can prove that f_a is a total function since $a \in \mathcal{I}_1(\mathfrak{A})$ but $\mathrm{Th}_{\Pi_1}(I\Pi_1^-)$ cannot; for otherwise 2^a would be bounded by a polynomial in a by Parikh's theorem. More formally:

Definition 4.1 Let $\delta(x)$ be a Δ_0 formula. Define $y = f_\delta(x)$ to be the following Δ_0 formula:

$$(x = (\mu z)\,(\delta(z)) \,\wedge\, y = 2^x) \,\vee\, (x \neq (\mu z)\,(\delta(z)) \,\wedge\, y = 0)$$

Theorem 4.2 *Let* $\delta(x)$ *be a* Δ_0*-formula such that* $\mathcal{N} \models \forall x\,\neg\delta(x)$ *and* $I\Pi_1^- \nvdash \forall x\,\neg\delta(x)$ $(\mathrm{con(PA)} \equiv \forall x\,\neg\delta(x)$, *say). Then*:

(1) $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-) \vdash \forall x\,\exists! y\,(y = f_\delta(x))$.

(2) $\mathrm{Th}_{\Pi_1}(I\Pi_1^-) \nvdash \forall x\,\exists y\,(y = f_\delta(x))$.

(3) *Hence,* $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ *does not follow from* $\mathrm{Th}_{\Pi_1}(I\Pi_1^-)$.

Proof. (1) Let $\mathfrak{A}$ be a model of $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ and $a \in \mathfrak{A}$. If a is not the least element satisfying $\delta(x)$ then $\mathfrak{A} \models 0 = f_\delta(a)$. Otherwise, a is in $\mathcal{K}_1(\mathfrak{A})$ and hence $\mathfrak{A} \models \exists y\,(y = 2^a)$ by Theorem 3.7.

(2) Towards a contradiction, assume that $\mathrm{Th}_{\Pi_1}(I\Pi_1^-)$ proves $\forall x\,\exists y\,(y = f_\delta(x))$. By Parikh's theorem there is a term $t(x)$ such that $\mathrm{Th}_{\Pi_1}(I\Pi_1^-) \vdash \forall x\,\exists y \leq t(x)\,(y = f_\delta(x))$. Let $\mathfrak{A}$ be a model of $I\Pi_1^- + \exists x\,\delta(x)$ and $a = (\mu x)\,(\delta(x))$. Then, $y = f_\delta(x)$ defines a total function in $\mathfrak{A}$ and $2^a = f_\delta(a)$. From $\mathfrak{A} \models \exists y \leq t(a)\,(y = f_\delta(a))$ it follows that $2^a \leq t(a)$, which is a contradiction since a is nonstandard. □

Let us observe that the proof of the previous result actually shows the following more general fact:

Corollary 4.3 $\mathrm{Th}_{\Pi_1}(\mathcal{N})$ *is the only consistent* Π_1 *theory to imply* $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$.

In what follows we show that $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ is strictly weaker than $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$. Following the ideas in Theorem 4.2, given two Δ_0 minimal elements a, c and a Δ_0 formula $\delta(z, v)$, we consider the function:

$$g(x) = \begin{cases} (x)_3^{(x)_0} & \text{if } (x)_0 = a \wedge (x)_1 = c \wedge (x)_2 \leq (x)_1 \wedge (x)_3 = (\mu z)\,(\delta(z, (x)_2)) \\ 0 & \text{otherwise} \end{cases}$$

Then, $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$ proves that g defines a total function $\big($observe that if an element x satisfies the first condition in the definition of g then $(x)_3$ is in $\mathcal{I}_1^1(\mathfrak{A})$ and $(x)_0$ is in $\mathcal{I}_1(\mathfrak{A})\big)$. However, since $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ is not Π_1 axiomatizable, now we cannot make use of Parikh's theorem to obtain that the totality of g is not provable in $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$. Instead, we need to show that a (finer) version of Bigorajska's theorem holds for $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$. First of all, it is worth noticing that Bigorajska's theorem can be derived from Parikh's theorem as follows (Bigorajska's proof was based on an ultrapower construction).

Proposition 4.4 *Suppose* $\varphi(x, y) \in \Sigma_1$ *and* $I\Delta_0 + \exists u\,\theta(u) \vdash \forall x\,\exists y\,\varphi(x, y)$, *where* $\theta(u) \in \Pi_1$.

(1) *There is a term* $t(x)$ *such that* $I\Delta_0 \vdash \forall u\,(\theta(u) \rightarrow \forall x > u\,\exists y \leq t(x)\,\varphi(x, y))$.

(2) *So,* $I\Delta_0 + \exists u\,\theta(u) \vdash \exists u\,\forall x > u\,\exists y \leq t(x)\,\varphi(x, y)$.

Proof. Clearly $I\Delta_0 \vdash \forall x, u\,\exists y\,(\theta(u) \rightarrow \varphi(x, y))$. By Parikh's theorem there is a term $t'(x, u)$ such that $I\Delta_0 \vdash \forall x, u\,\exists y \leq t'(x, u)\,(\theta(u) \rightarrow \varphi(x, y))$. By the monotonicity of terms, $I\Delta_0 \vdash x > u \rightarrow t'(x, u) \leq t'(x, x)$ and so $I\Delta_0 \vdash \forall u\,(\theta(u) \rightarrow \forall x > u\,\exists y \leq t'(x, x)\,\varphi(x, y))$. Now it is sufficient to put $t(x) = t'(x, x)$. □

Our proof makes clear that Bigorajska's theorem actually holds for every Σ_2 extension of $I\Delta_0$. Besides, it gives us a useful piece of information: if a Σ_2 theory T proves the totality of a Σ_1 function f and the axioms of T can be "witnessed" in a certain subset $\mathcal{A}$, then there is an element u *already in* $\mathcal{A}$ such that f is bounded by a polynomial for all $x > u$. Thus, we have:

Proposition 4.5 *Suppose* $\varphi(x, y) \in \Sigma_1$ *and* $T \vdash \forall x\,\exists y\,\varphi(x, y)$. *Let* $\mathfrak{A}$ *be a model of* T.

(1) *If* $T \subseteq \mathcal{B}(\Sigma_1)$, *there are a term* $t(x)$ *and* $a \in \mathcal{I}_1(\mathfrak{A})$ *such that* $\mathfrak{A} \models \forall x > a\,\exists y \leq t(x)\,\varphi(x, y)$.

(2) *If* $T = I\Pi_1^-$, *there are a term* $t(x)$ *and* $a \in \mathcal{I}_1^1(\mathfrak{A})$ *such that* $\mathfrak{A} \models \forall x > a\,\exists y \leq t(x)\,\varphi(x, y)$.

Proof. If $T \subseteq \mathcal{B}(\Sigma_1)$, every axiom of T can be "witnessed" in $\mathcal{K}_1(\mathfrak{A})$ since $\mathcal{K}_1(\mathfrak{A}) \prec_1 \mathfrak{A}$. If $T = I\Pi_1^-$, we know from the proof of Lemma 2.4 that every axiom of $I\Pi_1^-$ can be "witnessed" in $\mathcal{K}_1^1(\mathfrak{A}) \subseteq \mathcal{I}_1^1(\mathfrak{A})$. □

We are now ready to obtain the desired non collapse result. First, we write down a formal definition of the function g described above.

Definition 4.6 Let $D = (\delta_1(x), \delta_2(x), \delta_3(x, v))$ be a triple of Δ_0 formulas. Consider $\theta_D(x)$ to be

$$(x)_0 = (\mu z)\,(\delta_1(z)) \;\wedge\; (x)_1 = (\mu z)\,(\delta_2(z)) \;\wedge\; (x)_2 \leq (x)_1 \;\wedge\; (x)_3 = (\mu z)\,(\delta_3(z, (x)_2))$$

and define $y = g_D(x)$ to be the Δ_0 formula: $\big(\theta_D(x) \wedge y = (x)_3^{(x)_0}\big) \vee (\neg\theta_D(x) \wedge y = 0)$.

Theorem 4.7 *There is a triple of* Δ_0 *formulas* $D = (\delta_1(x), \delta_2(x), \delta_3(x, v))$ *satisfying that:*

(1) $\mathrm{Th}_{\Pi_2}(I\Pi_1^-) \vdash \forall x\, \exists! y\, (y = g_D(x))$.

(2) $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-) \nvdash \forall x\, \exists y\, (y = g_D(x))$.

(3) *Hence,* $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$ *does not follow from* $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$.

Proof. (1) Assume $\mathfrak{A} \models \mathrm{Th}_{\Pi_2}(I\Pi_1^-)$ and $a \in \mathfrak{A}$. If $\mathfrak{A} \models \neg\theta_D(a)$ then $\mathfrak{A} \models 0 = g_D(a)$. Otherwise, $(a)_0$ is in $\mathcal{I}_1(\mathfrak{A})$ and $(a)_3$ is in $\mathcal{I}_1^1(\mathfrak{A})$ and hence $(a)_3^{(a)_0}$ exists by Theorem 3.1.

(2) Let $\mathfrak{A}$ be a model of $I\Sigma_1$ with nonstandard Σ_1 definable elements. By Lemma 2.5, $\mathcal{K}_1^1(\mathfrak{A}) \neq \mathcal{I}_1(\mathfrak{A})$ and so there are $c \in \mathcal{M}_0(\mathfrak{A})$ and $d \leq c$ such that $\mathcal{M}_0(\mathfrak{A}, d) - \mathcal{I}_1(\mathfrak{A}) \neq \emptyset$. Pick $a \in \mathcal{M}_0(\mathfrak{A})$ nonstandard and $b \in \mathcal{M}_0(\mathfrak{A}, d) - \mathcal{I}_1(\mathfrak{A})$. Let $\delta_1(x)$, $\delta_2(x)$ and $\delta_3(x, v)$ be Δ_0 formulas such that a is the least element satisfying $\delta_1(x)$, c is the least element satisfying $\delta_2(x)$, and b is the least element satisfying $\delta_3(x, d)$. Finally, let $D = (\delta_1, \delta_2, \delta_3)$ and let $y = g_D(x)$ be as in Definition 4.6. Towards a contradiction, assume that $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ proves $\forall x\, \exists y\, (y = g_D(x))$. Then, it follows from Proposition 4.5 that there are a term $t(x)$ and $a' \in \mathcal{I}_1(\mathfrak{A})$ such that $\mathfrak{A} \models \forall x > a'\, \exists y \leq t(x)\, (y = g_D(x))$. Since $e = \langle a, c, d, b\rangle \geq a'$, $\mathfrak{A} \models \exists y \leq t(e)\, (y = g_D(e))$ and so $\mathfrak{A} \models \exists y \leq t(e)\, (y = b^a)$. Since $a, c, d \leq b$, $t(e) \leq b^k$ for some $k \in \omega$. Hence, $b^a \leq b^k$ for some $k \in \omega$, which is a contradiction since a is nonstandard. □

Combining Theorem 4.7 and Proposition 3.5 we obtain

Corollary 4.8 $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$ *is not* Σ_2*-axiomatizable.*

We close this section with some remarks.

(a) Our results give us a description of what total functions of $I\Pi_1^-$ look like. Suppose $I\Pi_1^- \vdash \forall x\, \exists! y\, (y = f(x))$, where $y = f(x)$ is a Σ_1 formula. Let $\mathfrak{A}$ be a nonstandard model of $I\Pi_1^-$. By Proposition 4.5 there are $b \in \mathcal{I}_1^1(\mathfrak{A})$ and a polynomial $p_1(x)$ such that $f(x) \leq p_1(x)$ for all $x > b$. If we define $\mathfrak{B}$ to be the submodel given by $\{c \in \mathfrak{A} : \forall z \in \mathcal{M}_0(\mathfrak{A}) - \omega,\ c \leq z\}$, then it is easy to check that $\mathfrak{B}$ is the largest initial segment of $\mathfrak{A}$ satisfying $\mathrm{Th}_{\Pi_1}(\mathcal{N})$. By applying Parikh's theorem for $\mathrm{Th}_{\Pi_1}(\mathcal{N})$, we get that there is a polynomial $p_2(x)$ such that $f(x) \leq p_2(x)$ for all $x \in \mathfrak{B}$. By Δ_0-overspill there is $a \in \mathcal{M}_0(\mathfrak{A})$ such that $f(x) \leq p_2(x)$ for all $x \leq a$. Put $p(x) = p_1(x) + p_2(x)$. Thus, f is bounded by p for all x in $[0, a] \cup [b, +\infty)$; whereas f may present an "exponential interference" in the interval $(a, b) \subseteq \mathcal{I}_1^1(\mathfrak{A})$.

(b) Our separation results also inform us of the limits of Parikh's and Bigorajska's theorems. Let $y = f_\delta(x)$ and $y = g_D(x)$ be as in Theorem 4.2 and Theorem 4.7, respectively. Firstly, by Theorem 4.2, $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ proves $\forall x\, \exists! y\, (y = f_\delta(x))$ but does not prove $\forall x\, \exists y \leq t(x)\, (y = f_\delta(x))$ for any term $t(x)$. Secondly, it follows from Theorem 4.7 and Corollary 3.6 that $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$ proves $\forall x\, \exists! y\, (y = g_D(x))$ but does not prove $\exists u\, \forall x > u\, \exists y \leq t(x)\, (y = g_D(x))$ for any term $t(x)$. Thus, we have:

Proposition 4.9

(1) $\mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ *satisfies Bigorajska's theorem and does not satisfy Parikh's theorem.*

(2) $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$ *does not satisfy Bigorajska's theorem.*

This result solves in the negative a question posed in [3], where the authors studied the quantifier complexity of the induction schema for the class of Δ_{n+1} formulas in an arithmetic theory T, $I\Delta_{n+1}(T)$. Whereas for $n > 0$ this scheme is not Σ_{n+2} axiomatizable for any theory T, for $n = 0$ the authors gave a number of examples for which $I\Delta_1(T)$ is Σ_2 axiomatizable (e.g., $T = I\Delta_0$, or $B\Sigma_1$). Indeed, for all of those examples $I\Delta_1(T)$ is also a Π_1 axiomatizable theory. This fact motivated the following question.

Question 4.10 (Problem 7.2 of [3]) *Suppose $I\Delta_0 \subseteq T \subseteq I\Sigma_1$ and T is closed under Σ_1 collection rule. Are the following conditions equivalent*?

(1) *$I\Delta_1(T)$ is Π_1 axiomatizable.*

(2) *$I\Delta_1(T)$ satisfies Parikh's theorem.*

(3) *$I\Delta_1(T)$ is Σ_2 axiomatizable.*

(4) *$I\Delta_1(T)$ satisfies Bigorajska's theorem.*

It follows from Theorem 2.4 of [3] that, under the assumptions of Question 4.10 $I\Delta_1(T) \equiv \mathrm{Th}_{\Pi_2}(T)$. So, it is clear that (1) and (2) are equivalent; and it is easy to check that (3) and (4) are equivalent too. However, if we put $T = \mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ then $I\Delta_1(T) \equiv \mathrm{Th}_{\mathcal{B}(\Sigma_1)}(I\Pi_1^-)$ and so $I\Delta_1(T)$ is Σ_2 axiomatizable and not Π_1, for it does not satisfy Parikh's theorem. Consequently, (1) and (3) are not equivalent.

Let us note, in passing, that in the statement of [3, Question 4.10] the assumption $T \subseteq I\Sigma_1$ is missing; as well as in the remarks following that problem it is erroneously claimed that $\mathrm{Th}_{\Pi_2}(I\Pi_1^-)$ satisfies Bigorajska's theorem.

5 Concluding remarks

It is natural to ask how the results in the previous sections generalize to the induction scheme for parameter free Π_{n+1} formulas $I\Pi_{n+1}^-$ for $n > 0$. Whereas for $n = 0$ we have the anomalous situation $I\Pi_1^- \vdash I\Delta_0^- \equiv I\Delta_0$, for $n > 0$, $I\Sigma_n^-$ and $I\Sigma_n$ cease to be equivalent and $I\Sigma_n + I\Pi_{n+1}^-$ and $I\Pi_{n+1}^-$ differ significantly in strength (in fact, in [1] it is shown that $I\Sigma_n + I\Pi_{n+1}^-$ has a larger class of provably total computable functions than that of $I\Pi_{n+1}^-$). It turns out that the ideas in the previous sections apply equally well to theories $I\Sigma_n + I\Pi_{n+1}^-$ for $n > 0$. The key point is to think of the exponential function as the *iteration* of a polynomial. Indeed, building on a well–known construction of Kaye, in [5] it is shown that, for each $n \geq 0$, there is a Π_n formula $y = \mathbb{K}_n(x)$ satisfying:

(a) $I\Sigma_n \vdash \forall x\, \exists! y\, (y = \mathbb{K}_n(x))$; and

(b) initial segments of $\mathfrak{B} \models I\Sigma_n$ closed under function $y = \mathbb{K}_n(x)$ are Σ_n-elementary substructures of $\mathfrak{B}$.

Observe that for $n = 0$ it suffices to put $\mathbb{K}_0(x) = (x+2)^2$; and indeed for $n > 0$ functions $y = \mathbb{K}_n^k(x)$, with $k \in \omega$, will play the role of polynomials. Moreover, in Section 3 of [3] a Π_n formula $y = \mathbb{K}_n^z(x)$ that expresses the *iteration* of the function $y = \mathbb{K}_n(x)$ is presented. It is not hard to check that our characterization and non collapse results remain true for $n > 1$ and for $I\Sigma_{n-1} + I\Pi_n^-$ when restricted exponentiation is replaced with restricted iterations of functions $y = \mathbb{K}_n(x)$. Namely,

Theorem 5.1

(1) $\mathrm{Th}_{\Pi_{n+2}}(I\Sigma_n + I\Pi_{n+1}^-) \equiv I\Sigma_n + \forall a \in \mathcal{I}_{n+1}\, \forall b \in \mathcal{I}_{n+1}^1\, \exists y\, \big(y = \mathbb{K}_n^a(b)\big)$.

(2) $\mathrm{Th}_{\mathcal{B}(\Sigma_{n+1})}(I\Sigma_n + I\Pi_{n+1}^-) \equiv I\Sigma_n^- + \forall a, b \in \mathcal{I}_{n+1}\, \exists y\, \big(y = \mathbb{K}_n^a(b)\big)$.

(3) $\mathrm{Th}_{\Pi_{n+2}}(I\Sigma_n + I\Pi_{n+1}^-)$ *does not follow from* $\mathrm{Th}_{\mathcal{B}(\Sigma_{n+1})}(I\Sigma_n + I\Pi_{n+1}^-)$*, which, in turn, does not follow from* $\mathrm{Th}_{\Pi_{n+1}}(I\Sigma_n + I\Pi_{n+1}^-)$.

With respect to theories $I\Pi_{n+1}^-$ for $n > 0$, the following questions are in order.

Question 5.2 *Does* $\mathrm{Th}_{\mathcal{B}(\Sigma_{n+1})}(I\Pi_{n+1}^-)$ *follow from* $\mathrm{Th}_{\Pi_{n+1}}(I\Pi_{n+1}^-)$?

Question 5.3 *Does* $\mathrm{Th}_{\Pi_{n+2}}(I\Pi_{n+1}^-)$ *follow from* $\mathrm{Th}_{\mathcal{B}(\Sigma_{n+1})}(I\Pi_{n+1}^-)$?

The negative answer to Question 5.2 can be derived from axiomatization properties of $I\Sigma_n^-$. In [4] it is proved that, for $n > 0$, $\mathrm{Th}_{\Pi_{n+1}}(\mathcal{N})$ is the only Π_{n+1} theory to imply $I\Sigma_n^-$. Thus, $\mathrm{Th}_{\mathcal{B}(\Sigma_{n+1})}(I\Pi_{n+1}^-)$ does not follow from $\mathrm{Th}_{\Pi_{n+1}}(I\Pi_{n+1}^-)$ since $\mathrm{Th}_{\mathcal{B}(\Sigma_{n+1})}(I\Pi_{n+1}^-)$ implies $I\Sigma_n^-$. Question 5.3 is related to a conservation theorem of Beklemishev:

Theorem 5.4 (Beklemishev [1]) $(n > 0)$ *$I\Pi_{n+1}^-$ is conservative over $I\Sigma_n^-$ w.r.t. $\mathcal{B}(\Sigma_{n+1})$ sentences.*

As a result, $\mathrm{Th}_{\mathcal{B}(\Sigma_{n+1})}(I\Pi_{n+1}^-) \equiv I\Sigma_n^-$ and Question 5.3 is equivalent to asking whether Beklemishev's conservation theorem can be extended to Π_{n+2} sentences. We conclude by proving that Question 5.3 also has the negative answer and hence Beklemishev's conservation theorem is best possible (observe that it cannot be extended to Σ_{n+2} sentences either, for the least number principle for parameter free Δ_{n+1} formulas $L\Delta^-_{n+1}$ is a Σ_{n+2} theory that can be proved from $I\Pi^-_{n+1}$ but not from $I\Sigma^-_n$.)

The idea to separate $I\Sigma^-_n$ and $\mathrm{Th}_{\Pi_{n+2}}(I\Pi^-_{n+1})$ is to consider the following model-theoretic property:

$$(A) \;\equiv\; \forall a \in \mathcal{K}^1_{n+1}(\mathfrak{A}) \text{ “}\mathcal{K}_n(\mathfrak{A}, a) \text{ is not cofinal in } \mathfrak{A}\text{”}$$

Recall that in models of $I\Sigma_{n-1}$ the set of the Π_{n-1} minimal elements $\mathcal{M}_{n-1}(\mathfrak{A}, a)$ is cofinal in $\mathcal{K}_n(\mathfrak{A}, a)$. Hence, (A) can be reexpressed as: $(A') \equiv \forall a \in \mathcal{K}^1_{n+1}(\mathfrak{A})$ “$\mathcal{M}_{n-1}(\mathfrak{A}, a)$ is not cofinal in $\mathfrak{A}$.” Let $\mathrm{Sat}_{n-1}(x, y, v)$ denote a truth predicate for Π_{n-1} formulas (which is available in $I\Sigma_{n-1} + \exp$) expressing “*x satisfies the Π_{n-1} formula y with a parameter v*.” Then, it is straightforward to write down a Δ_n formula $\psi(x, y, v)$ formalizing the property “*x is the least element satisfying the Π_{n-1} formula y with a parameter v*.” We can now formalize over $I\Sigma_{n-1} + \exp$, (a stronger version of) property (A') using the following set of Π_{n+2} sentences of $\mathcal{L}$:

$$(B) \;\equiv\; \forall a \in \mathcal{I}^1_{n+1}\, \forall b \in \mathcal{I}_{n+1}\, \exists u\, \forall x\, \forall y \leq b\, (\psi(x, y, a) \longrightarrow x \leq u)$$

Theorem 5.5 ($n > 0$)

(1) $\mathrm{Th}_{\Pi_{n+2}}(I\Pi^-_{n+1}) \vdash (B)$.

(2) $\mathrm{Th}_{\mathcal{B}(\Sigma_{n+1})}(I\Pi^-_{n+1}) \nvdash (B)$.

(3) *Hence,* $\mathrm{Th}_{\Pi_{n+2}}(I\Pi^-_{n+1})$ *does not follow from* $\mathrm{Th}_{\mathcal{B}(\Sigma_{n+1})}(I\Pi^-_{n+1}) \equiv I\Sigma^-_n$.

Proof. (1) Since $(B) \subseteq \Pi_{n+2}$, it suffices to show that $I\Pi^-_{n+1} \vdash (B)$. Suppose $\mathfrak{A} \models I\Pi^-_{n+1}$, $a \in \mathcal{K}^1_{n+1}(\mathfrak{A})$ and $b \in \mathcal{K}_{n+1}(\mathfrak{A})$. There exist $\delta(v, w) \in \Sigma_{n+1}$ and $c \in \mathcal{I}_{n+1}(\mathfrak{A})$ such that $\delta(v, c)$ defines a. Let $\varphi(z, c)$ be the Σ_{n+1} formula $\exists v\, [\delta(v, c) \wedge \exists u\, \forall y \leq z\, (\exists x\, \mathrm{Sat}_{n-1}(x, y, v) \rightarrow \exists x \leq u\, \mathrm{Sat}_{n-1}(x, y, v))]$. Clearly, $\mathfrak{A} \models \varphi(0, c)$ and $\mathfrak{A} \models \forall z\, (\varphi(z, c) \rightarrow \varphi(z+1, c))$. So, $\mathfrak{A} \models \varphi(b, c)$ by Proposition 2.1. Hence, property (B) holds in $\mathfrak{A}$, for it follows from the definition of ψ that $I\Sigma_{n-1} + \exp \vdash \psi(x, y, v) \wedge \mathrm{Sat}_{n-1}(x', y, v) \rightarrow x \leq x'$.

(2) Let $\mathfrak{A}$ be a model of $I\Sigma_{n+1}$ with nonstandard Σ_{n+1} definable elements. Reasoning as in the proof of Lemma 2.5, we get that $\mathcal{K}^1_{n+1}(\mathfrak{A}) \neq \mathcal{I}_{n+1}(\mathfrak{A})$ and so there is $a \in \mathcal{M}_n(\mathfrak{A}, b) - \mathcal{I}_{n+1}(\mathfrak{A})$, with $b \in \mathcal{I}_{n+1}(\mathfrak{A})$. Pick $c \in \mathcal{M}_n(\mathfrak{A})$ such that $b \leq c$. Put $d = \langle a, b, c\rangle$ and define $\mathfrak{C}$ to be $\mathcal{I}_n(\mathfrak{A}, d)$, i.e., the least initial segment of $\mathfrak{A}$ containing $\mathcal{K}_n(\mathfrak{A}, d)$. Then, $\mathcal{I}_{n+1}(\mathfrak{A}) \subset \mathfrak{C}$ and $\mathfrak{C} \prec_{n-1} \mathfrak{A}$ by Proposition 1.4. In addition, it follows from theorem 1.33, Chapter IV of [6] that $\mathfrak{C} \models B\Sigma_n$ and $\mathfrak{C}$ satisfies $\mathrm{Th}_{\Pi_{n+1}}(\mathfrak{A}; d)$, i.e., the set of all Π_{n+1} sentences (possibly containing the parameter d) which are true in $\mathfrak{A}$. Then, we have:

(i) $\mathfrak{C} \models \mathrm{Th}_{\mathcal{B}(\Sigma_{n+1})}(I\Pi^-_{n+1})$.
Since $\mathfrak{A} \models I\Pi^-_{n+1}$, it is sufficient to prove that $\mathfrak{C} \models \mathrm{Th}_{\Pi_{n+1}}(\mathfrak{A}) \cup \mathrm{Th}_{\Sigma_{n+1}}(\mathfrak{A})$. Clearly, $\mathfrak{C} \models \mathrm{Th}_{\Pi_{n+1}}(\mathfrak{A})$. To prove that $\mathfrak{C} \models \mathrm{Th}_{\Sigma_{n+1}}(\mathfrak{A})$, suppose $\mathfrak{A} \models \exists x\, \theta(x)$, where $\theta(x)$ is in Π_n. It follows from $\mathcal{I}_{n+1}(\mathfrak{A}) < a$ that $\mathfrak{A} \models \exists x \leq a\, \theta(x)$ and so $\mathfrak{C} \models \exists x \leq a\, \theta(x)$ since $\mathfrak{C} \prec_{n-1} \mathfrak{A}$ and $[0, a] \subseteq \mathfrak{C}$.

(ii) $\mathfrak{C} \not\models (B)$.
By Proposition 1.4, $\mathcal{K}_n(\mathfrak{A}, d) \prec_n \mathfrak{C}$ and so $\mathcal{K}_n(\mathcal{K}_n(\mathfrak{A}, d), d) = \mathcal{K}_n(\mathfrak{C}, d)$. It follows from $\mathcal{K}_n(\mathfrak{A}, d) \prec_n \mathfrak{A}$ that $\mathcal{K}_n(\mathcal{K}_n(\mathfrak{A}, d), d) = \mathcal{K}_n(\mathfrak{A}, d)$. So, $\mathcal{K}_n(\mathfrak{C}, d) = \mathcal{K}_n(\mathfrak{A}, d)$ and hence $\mathcal{K}_n(\mathfrak{C}, d)$ is cofinal in $\mathfrak{C}$. Thus, in order to prove that (B) fails in $\mathfrak{C}$ we only need to show that d is in $\mathcal{K}^1_{n+1}(\mathfrak{C})$. Indeed, it is sufficient to show that $d = \langle a, b, c\rangle$ is in $\mathcal{M}_n(\mathfrak{C}, \mathcal{I}_{n+1}(\mathfrak{C}))$, for it follows from $\mathfrak{C} \models B\Sigma_n$ that $\mathcal{M}_n(\mathfrak{C}, \mathcal{I}_{n+1}(\mathfrak{C})) \subseteq \mathcal{K}^1_{n+1}(\mathfrak{C})$. First, observe that $c \in \mathcal{M}_n(\mathfrak{C})$ since $c \in \mathcal{M}_n(\mathfrak{A})$, $\mathfrak{C} \models \mathrm{Th}_{\Pi_{n+1}}(\mathfrak{A}; d)$ and $c = (d)_2$. Second, it follows from $\mathfrak{C} \models B\Sigma_n$ that $\mathcal{M}_n(\mathfrak{C}) \subseteq \mathcal{K}_{n+1}(\mathfrak{C})$ and so $b \in \mathcal{I}_{n+1}(\mathfrak{C})$. Finally, $a \in \mathcal{M}_n(\mathfrak{C}, \mathcal{I}_{n+1}(\mathfrak{C}))$ since $a \in \mathcal{M}_n(\mathfrak{A}, b)$, $\mathfrak{C} \models \mathrm{Th}_{\Pi_{n+1}}(\mathfrak{A}; d)$ and $a = (d)_0$, $b = (d)_1$. Hence, a, b, c are in $\mathcal{M}_n(\mathfrak{C}, \mathcal{I}_{n+1}(\mathfrak{C}))$ and so is d. □

Let us observe that $I\Sigma_n$ does prove property (B) for $n > 0$. Indeed, the following question is open.

Question 5.6 *For $n > 0$, does* $\mathrm{Th}_{\Pi_{n+2}}(I\Pi^-_{n+1})$ *follow from* $I\Sigma_n$?

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