Monsters in Hardy and Bergman spaces

L. BERNAL–GONZÁLEZ and M.C. CALDERÓN–MORENO

Departamento de Análisis Matemático, Facultad de Matemáticas, Apdo. 1160,
Avda. Reina Mercedes, 41080 Sevilla, Spain.
E–mails: lbernal@us.es, mccm@us.es

Abstract

A monster in the sense of Luh is a holomorphic function on a simply connected domain in the complex plane such that it and all its derivatives and antiderivatives exhibit an extremely wild behaviour near the boundary. In this paper the Hardy spaces H^p and the Bergman spaces B^p $(1 \leq p < \infty)$ on the unit disk are considered, and it is shown that there are no Luh-monsters in them. Nevertheless, it is proved that T-monsters (as introduced by the authors in an earlier work) can be found in each of these spaces for any finite order linear differential operator T.

Key words and phrases: Luh-monster, T-monster, Hardy space, Bergman space, strongly omnipresent operator, differential operator, hypercyclic function.

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1 Introduction

As soon as the existence of a mathematical entity is established, a natural problem arises: Do such entities exist with additional (even "more perfect") properties? This has been the line of research which has motivated this paper, this time in the setting of holomorphic functions with "wild" behaviour in the boundary of the open unit disk $\mathbb{D} = \{|z| < 1\}$.

Suppose that G is a domain in the complex plane $\mathbb{C}$; by $H(G)$ we denote as usual the Fréchet space of all holomorphic functions on G, endowed with the compact-open topology. If G is simply connected then it has been proved in 1985 by W. Luh [21] the existence of a *dense* set of functions –which he called "monsters"– in $H(G)$ such that all its derivatives and antiderivatives exhibit an extremely wild behaviour near the boundary of G. Such a chaotic property can be expressed in terms of certain generalized cluster sets, introduced by Luh himself. In 1987 Grosse-Erdmann [18], see also [19, Section 4.b], showed that, in fact, there is a *residual* set of monsters. Further interesting results on this topic can be seen in [22–24].

With the aim of finding operators which are different from those of differentiation and antidifferentiation under whose action there are holomorphic functions with boundary wild behaviour, the authors have recently introduced [5] the notions of T-monsters and strongly omnipresent operators, see Definition 1.1 below.

Let us fix some terminology and notation. By ∂G we denote the boundary of a domain $G \subset \mathbb{C}$ in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}$ is the real line. An *operator* always refers to a continuous (not necessarily linear) selfmapping. We denote by $O(\partial G)$ the set of all open subsets of $\mathbb{C}_\infty$ meeting the boundary of G. If $A \subset \mathbb{C}$ then $\overline{A}$ represents the closure of A, $\|f\|_A := \sup_{z \in A} |f(z)|$, where f is a complex function defined in A, and $LT(A)$ is the set of all affine linear transformations τ, $\tau(z) = az + b$, such that $\tau(\mathbb{D}) \subset A$. In the following definition, we are allowing the point of infinity to be a boundary point of G when G is unbounded (as in [7, 9, 10]). Observe also that the domain G is allowed to be non-simply connected in (a)–(c).

Definition 1.1. (a) A function $f \in H(G)$ is a *holomorphic monster* whenever the following universality property is satisfied: For each $g \in H(\mathbb{D})$ and each $t \in \partial G$ there exists a sequence (τ_n) of affine linear transformations with

$$\tau_n(z) \to t \quad (n \to \infty) \quad \text{uniformly on } \mathbb{D} \qquad \text{and} \qquad \tau_n(\mathbb{D}) \subset G \ (n \in \mathbb{N})$$

such that

$$f(\tau_n(z)) \to g(z) \quad (n \to \infty)$$

locally uniformly in $\mathbb{D}$.

(b) Let $T : H(G) \to H(G)$ be an operator. Then a function $f \in H(G)$ is a T–*monster* if Tf is a holomorphic monster. The set of T–monsters is denoted by $\mathcal{M}(T)$.

(c) An operator $T : H(G) \to H(G)$ is *strongly omnipresent* if for all $g \in H(\mathbb{D})$, $\varepsilon > 0$, $r \in (0,1)$ and $V \in O(\partial G)$ the set

$$U(T, g, \varepsilon, r, V) := \{f \in H(G) : \text{ there exists some } \quad \tau \in LT(V \cap G)$$

$$\text{such that} \quad \|(Tf) \circ \tau - g\|_{r\overline{\mathbb{D}}} < \varepsilon\}$$

is dense in $H(G)$.

(d) If G is simply connected, then a function $f \in H(G)$ is a *Luh–monster* whenever every derivative $f^{(n)}$ $(n \in \mathbb{N}_0)$ and every antiderivative $f^{(-n)}$ $(n \in \mathbb{N})$ is a holomorphic monster.

Observe that f is a holomorphic monster if and only if it is an I-monster ($I :=$ the identity operator), and that f is a Luh-monster if and only if f is simultaneously a D^N-monster (for all $N \in \mathbb{N}_0$) and a D_a^{-N}-monster (for all $N \in \mathbb{N}$ and all $a \in G$). Here D^N $(N \in \mathbb{N}_0)$ is the differentiation operator $D^N f = f^{(N)}$, $D^0 = I$, and D_a^{-N} is the antidifferentiation operator given by $D_a^{-N} f :=$ the unique antiderivative F of order N of f such that $F(a) = \cdots = F^{(N-1)}(a) = 0$.

It happens that an operator T on $H(G)$ is strongly omnipresent if and only if the set $\mathcal{M}(T)$ is residual (see [5, Theorem 2.2]). Hence Grosse-Erdmann [18, Kapitel 3] had showed in fact that every D^N and every D_a^{-N} is strongly omnipresent. He and the authors have identified several kinds of strongly omnipresent operators, including infinite order differential and antidifferential operators, integral operators, composition and multiplication operators [5, 9, 10].

For other kinds of operators –also introduced by the authors– under whose action certain functions have some type of boundary chaotic behaviour, the reader is referred to [1] (omnipresent operators), [2, 6, 14] (DI-operators) and [7] (totally omnipresent operators), see also [8]. It happens that every totally omnipresent operator is strongly omnipresent and DI, and that if an operator is either strongly omnipresent or DI then it is omnipresent. By using totally omnipresent operators, the authors have recently proved (see [7]) that there is a dense linear manifold in $H(G)$ all of whose nonzero functions are Luh-monsters.

We will need some terminology about universality (see [19] for an excellent survey, updated till 1999). If X and Y are (Hausdorff) topological vector spaces over the same field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$) and $T_n : X \to Y$ ($n \in \mathbb{N}$) is a sequence of continuous linear mappings, then (T_n) is said to be *hypercyclic* (or *universal*) if and only if there is a vector $x \in X$, called also *hypercyclic* for (T_n), such that the orbit $\{T_n x : \ n \in \mathbb{N}\}$ is dense in Y. The sequence (T_n) is called *densely hypercyclic* whenever the set $HC((T_n))$ of hypercyclic vectors for (T_n) is dense. If $X = Y$ and T is a linear operator on X then T is called hypercyclic if and only if the sequence (T^n) of iterates is hypercyclic. It is easy to see that in such a case (T^n) is indeed densely hypercyclic. The existence of T-invariant dense linear manifolds of hypercyclic vectors for each hypercyclic linear operator T on a (real or complex) locally convex space was shown by Herrero, Bourdon and Bès [11, 12, 20] (see also [4] for the additional property of maximal algebraical cardinality to such manifolds). In 1999 the first author extended this result to hypercyclic sequences of mappings, see [3] and Theorem 2.6.

For the sake of convenience, we will keep in this work the notion of hypercyclicity even when the spaces X, Y and the mappings T, T_n are *not* linear.

The aim of this paper is to study the existence of Luh-monsters and in general of T-monsters in (not necessarily closed) subspaces of $H(G)$, mainly in the maybe most emblematic spaces of analytic functions on $G = \mathbb{D}$, namely, the Hardy spaces H^p and the Bergman spaces B^p ($1 \le p < \infty$). Recall that H^p is the class of all functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_p := \sup_{0<r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty,$$

while B^p is the class of all $f \in H(\mathbb{D})$ satisfying

$$\|f\|_p := \left(\int_{\mathbb{D}} |f(z)|^p \, dA(z) \right)^{1/p} < \infty.$$

Each one becomes a Banach space under the corresponding norm $\| \cdot \|_p$. Recall also that $H^p \subset B^p$ with continuous inclusion. We have denoted by $dA(z)$ the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is 1. The existence of dense linear manifolds of holomorphic monsters in these spaces is also considered.

2 Monsters in subspaces of $H(\mathbb{D})$

Up to date, all results about monsters have been addressed to state their existence, with success except for very special examples. However, in the current setting, we are on the point to obtain a general statement of non-existence of classical monsters in function spaces, see Theorem 2.2 below. Before this, we establish an auxiliary lemma whose content is probably well known. Since we have not been able to find a reference for it, we provide with an elementary proof.

Lemma 2.1. *If $f \in H(\mathbb{D})$ and $f' \in B^p$ for some $p \in [1, \infty)$ then $f \in H^p$.*

Proof. By hypothesis, $\int_{\mathbb{D}} |f'(z)|^p \, dA(z) < \infty$. Passing to polar coordinates, we get $\int_0^1 \int_0^{2\pi} |f'(se^{i\theta})|^p s \, ds d\theta < \infty$. Since $|f'|^p$ is Lebesgue-integrable on a neighbourhood of the origin, we can drop the factor s, that is,

$$\int_0^1 \int_0^{2\pi} |f'(se^{i\theta})|^p \, ds d\theta < \infty.$$

It is evident that we can suppose $f(0) = 0$. Then $f(re^{i\theta}) = \int_0^r f'(se^{i\theta}) e^{i\theta} \, ds$ for all $r \in [0, 1)$ and all $\theta \in [0, 2\pi]$. Hence, by the Hölder inequality,

$$|f(re^{i\theta})|^p \le \left(\int_0^r |f'(se^{i\theta})| \, ds \right)^p$$

$$\le r^{p-1} \int_0^r |f'(se^{i\theta})|^p \, ds \le \int_0^1 |f'(se^{i\theta})|^p \, ds.$$

Discarding the terms in the middle, an integration over $[0, 2\pi]$ yields the desired result. □

Theorem 2.2. *There are no Luh-monsters in any Bergman space B^p $(1 \le p < \infty)$, so in any Hardy space H^p $(1 \le p < \infty)$.*

Proof. Assume that $f \in B^p$ and that F is a holomorphic function on $\mathbb{D}$ such that $F'' = f$. Then F' is in H^p by Lemma 2.1, so F is in the disk algebra, that is, it can be extended continuously on $\overline{\mathbb{D}}$: this is asserted, for instance, in [16, Chapter 5, Exercise

9] for $p > 1$, but in the case $p = 1$ a well known result of Privalov establishes that for $h \in H(\mathbb{D})$ the function $h' \in H^1$ if and only if h has a continuous extension to $\overline{\mathbb{D}}$ that is absolutely continuous on $\partial\mathbb{D}$ [16, Theorem 3.11]. If f were a Luh-monster then for some sequence $(\tau_n) \subset LT(\mathbb{D})$ with $\tau_n \to 1$ $(n \to \infty)$ uniformly on $\mathbb{D}$ we would have

$$F(\tau_n(z)) \to g(z) \qquad (n \to \infty)$$

in $H(\mathbb{D})$ for the constant function $g \in H(\mathbb{D})$ with $g(z) := 1 + \max_{\overline{\mathbb{D}}} |F|$ $(z \in \mathbb{D})$, which is clearly impossible. Thus, f cannot be a Luh-monster, as required. $\square$

Observe that according to the last proof the antiderivatives are to blame for the nonexistence of Luh-monsters. Nevertheless, we will be able to deal with the existence of D^N-monsters in H^p and B^p for any *nonnegative* integer N. In fact, much more will be obtained, see Theorem 2.7.

In view of the negative result provided by Theorem 2.2, we now focus our attention on the search of some suitable condition on an operator T defined on $H(G)$ and on a subspace $X \subset H(G)$ in order that T-monsters can exist in X, i.e., $\mathcal{M}(T) \cap X \neq \emptyset$. We even get that $\mathcal{M}(T) \cap X$ is residual in X under suitable conditions. This will be made in Lemma 2.3. Afterwards, with the help of a strong theorem due to Bourdon and Shapiro, this lemma is applied in the proof of Theorem 2.5, in which the existence of many holomorphic monsters (this is the case $T = I$) in Hardy and Bergman spaces is obtained. We also show how having holomorphic monsters plus a (purely set-theoretic) soft condition on a general operator T is sufficient to have T-monsters in X, see Theorem 2.6. Before establishing all these results, recall that if $G \subset \mathbb{C}$ is a domain and $\varphi \in H(\mathbb{D})$ satifies $\varphi(\mathbb{D}) \subset G$ then the composition mapping $C_\varphi : f \in H(G) \mapsto f \circ \varphi \in H(\mathbb{D})$ is well defined and continuous. In particular, φ can be any member of $LT(G)$. Sometimes, in the case $G = \mathbb{D}$, the composition operator C_φ maps continuously an F-space (= a complete linear metric space) $X \subset H(\mathbb{D})$ into itself for every holomorphic selfmapping φ on $\mathbb{D}$. For instance, this holds for each Hardy space H^p and each Bergman space B^p $(p > 0)$ due to Littlewood's subordination theorem, see [26, Chapter 10]. See also [15] for a collection of such spaces X.

The following auxiliar statement gives us a positive answer to the problem of existence of monsters on subspaces in terms of the existence of some kind of hypercyclic sequences.

Lemma 2.3. *Assume that X is an F-space with $X \subset H(G)$ and that T is an operator on $H(G)$ satisfying the following two conditions:*

(a) *Convergence in X implies locally uniform convergence.*

(b) *For every boundary point $t \in \partial G$ there is a sequence $(\tau_n) \subset LT(G)$ tending to t uniformly on $\mathbb{D}$ such that the sequence of mappings $C_{\tau_n} T : X \to H(\mathbb{D})$ $(n \in \mathbb{N})$ is densely hypercyclic.*

Then the set $\{f \in X : \ f \text{ is a } T\text{-monster}\}$ *is residual in* X.

Proof. Fix two sets $U(T, g, \varepsilon, r, V)$ and $B_X(h, \alpha)$, where $g \in H(\mathbb{D})$, $h \in X$, $\varepsilon > 0$, $\alpha > 0$, $0 < r < 1$, $V \in O(\partial G)$ and $B_X(h, \alpha) := \{f \in X : \ d(f, h) < \alpha\}$ is an open ball for a translation-invariant distance d compatible with the topology of X. Note that each set $U(T, g, \varepsilon, r, V)$ is open in $H(G)$ due to the continuity of T, hence $U(T, g, \varepsilon, r, V) \cap X$ is open in X by condition (a). On the other hand, there are countable many sets U_n $(n \in \mathbb{N})$ of the type $U(T, g, \varepsilon, r, V)$ such that $\mathcal{M}(T) = \bigcap_{n\in\mathbb{N}} U_n$, see [5]. Then $\mathcal{M}(T) \cap X = \bigcap_{n\in\mathbb{N}} U_n \cap X$, so $\mathcal{M}(T) \cap X$ is a G_δ-subset of X. Since X is a Baire space, it suffices to show that every intersection $U(T, g, \varepsilon, r, V) \cap X$ is dense in X or, equivalently, that

$$U(T, g, \varepsilon, r, V) \cap B_X(h, \alpha) \neq \emptyset. \tag{1}$$

Choose any point $t \in V \cap \partial G$ and consider the sequence $(\tau_n) \subset LT(G)$ given by hypothesis (b). By dense hypercyclicity, there is an $f \in X$ with $d(f, h) < \alpha$ and a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ in $\mathbb{N}$ such that

$$(Tf) \circ \tau_{n_k} \to g \ (k \to \infty) \ \text{ uniformly on } r\overline{\mathbb{D}}.$$

Since $\tau_n(z) \to t$ $(n \to \infty)$ uniformly on $\mathbb{D}$, there is $k_0 \in \mathbb{N}$ such that $\tau_{n_{k_0}}(\mathbb{D}) \subset V \cap G$ and $\|(Tf) \circ \tau_{n_{k_0}} - g\|_{r\overline{\mathbb{D}}} < \varepsilon$. Hence $f \in U(T, g, \varepsilon, r, V) \cap B_X(h, \alpha)$ and (1) is fulfilled. $\square$

For instance, in the case $X = H(G)$ condition (a) is trivially satisfied and, for $T = I$, (b) is even fulfilled for every $t \in \partial G$ by *any* sequence $(\tau_n) \subset LT(G)$ tending to t uniformly on $\mathbb{D}$, see [7].

The next assertion is a version for sequences of the Hypercyclicity Comparison Principle, see [25, p. 111]. Its proof is trivial, so it is dropped. The lemma will be used in the proof of the second part of Theorem 2.6.

Lemma 2.4. *Suppose that X_1, X_2, X_3 are topological spaces in such a way that $X_3 \subset X_1$, X_3 is dense in X_1 and the topology of X_3 is stronger than that of X_1. Assume also that $S_n : X_1 \to X_2$ $(n \in \mathbb{N})$ is a sequence of continuous mappings with the property that the sequence $S_n|_{X_3} : X_3 \to X_2$ $(n \in \mathbb{N})$ is densely hypercyclic. Then (S_n) is densely hypercyclic.*

Theorem 2.5. *Assume that $p \in [1, +\infty)$. We have:*

(1) The set $\{f \in H^p : \ f \text{ is a holomorphic monster}\}$ *is residual in H^p.*

(2) The set $\{f \in B^p : \ f \text{ is a holomorphic monster}\}$ *is residual in B^p.*

Proof. (1) Conditions (a)–(b) in Lemma 2.3 should be checked for $G = \mathbb{D}$, $X = H^p$, $T = I$. Property (a) follows from the well known estimate

$$|f(z)| \le 2^{1/p} \|f\|_p (1 - |z|)^{-1/p} \qquad (z \in \mathbb{D}),$$

which holds even for $0 < p < \infty$, see for instance [16, Chapter 3]. Property (b) is more delicate. In order to check it, fix a point $t \in \partial\mathbb{D}$ and consider the function

$$\varphi(z) = \frac{z + t}{2}.$$

Trivially, $\varphi \in LT(\mathbb{D})$ and φ is not an automorphism of $\mathbb{D}$. Moreover, its fixed points are t $(\in \partial\mathbb{D})$ and ∞ $(\notin \overline{\mathbb{D}})$, therefore φ is a non-parabolic non-automorphism without fixed points in $\mathbb{D}$. Hence, the Linear Fractional Hypercyclicity Theorem due to Bourdon and Shapiro (see [25, Chapter 7] and [13]; the result is obtained for $p = 2$ but the proof equally works for $1 \le p < \infty$ because it is ultimately based on the fact that for every $\alpha \in \partial\mathbb{D}$ the collection of polynomials vanishing at α is dense in H^p, which in turn is a consequence of Beurling's approximation theorem, see [16, pp. 113–114]) tells us that the operator $C_\varphi : H^p \to H^p$ is hypercyclic, so (C_φ^n) is *densely* hypercyclic (see Section 1). But $C_\varphi^n = C_{\tau_n}$, where $\tau_n := \varphi \circ \cdots \circ \varphi$ (n-fold), i. e.,

$$\tau_n(z) = \frac{z + (2^n - 1)t}{2^n} \qquad (n \in \mathbb{N}).$$

Finally, observe that $\tau_n(z) \to t$ ($n \to \infty$) uniformly on $\mathbb{D}$ and that $C_{\tau_n} : H^p \to H(\mathbb{D})$ ($n \in \mathbb{N}$) is also densely hypercyclic, because H^p is dense in $H(\mathbb{D})$ and its norm-topology is stronger than the compact-open one.

(2) Choose again $G = \mathbb{D}$, $T = I$ in Lemma 2.3, with $X = B^p$ this time. Property (a) is derived from the inequality

$$(1 - |z|)^2 |f(z)| \leq ||f||_p \quad (z \in \mathbb{D},\, p \geq 1),$$

see [26, p. 48]. As for property (b), it is enough to consider the fact that $C_{\tau_n} : H^p \to H(\mathbb{D})$ ($n \in \mathbb{D}$) is densely hypercyclic (where C_{τ_n} is as in the proof of the first part) together with Lemma 2.4 as applied on $X_1 = B^p$, $X_2 = H(\mathbb{D})$, $X_3 = H^p$, $S_n = C_{\tau_n} : B^p \to H(\mathbb{D})$ ($n \in \mathbb{N}$). Note that H^p is dense in B^p because the polynomials are dense in B^p and H^p contains each polynomial. This finishes the proof. □

An inmediate consequence of Theorem 2.5 is that the set $\{f \in H^p : \ f \text{ is a } C_\varphi\text{-monster}\}$ is residual in H^p for every automorphism φ of $\mathbb{D}$. Indeed, the operator $T := C_\varphi|_{H^p}$ maps homeomorphically H^p onto itself due to Littlewood's subordination theorem. Now, the latter set is $\mathcal{M}(C_\varphi) \cap H^p = C_\varphi^{-1}(\mathcal{M}(I)) \cap H^p = T^{-1}(\mathcal{M}(I) \cap H^p)$, which is residual in H^p because $\mathcal{M}(I) \cap H^p$ is. Of course, the same holds if H^p is replaced to B^p.

For future references, we point out that the proof of the Linear Fractional Hypercyclicity Theorem [25] also works for any subsequence $(C_\varphi^{n_k})$ $(n_1 < n_2 < n_3 < \cdots)$ of (C_φ^n).

Theorem 2.6. *Assume that X is an F-space with $X \subset H(G)$ such that there is some holomorphic monster in X. Suppose that T is an operator on $H(G)$ satisfying $T(X) \supset X$. Then there is some T-monster in X.*

Proof. The proof is, like the hypothesis on T itself, purely set-theoretic. By hypothesis, $\mathcal{M}(I) \cap X \neq \emptyset$, and it should be shown that $\mathcal{M}(T) \cap X \neq \emptyset$. But $\mathcal{M}(T) \cap X = T^{-1}(\mathcal{M}(I)) \cap X$, which is nonempty if and only if $\mathcal{M}(I) \cap T(X) \neq \emptyset$, which is true because $T(X) \supset X$. $\square$

As an application of the last two theorems, we get in the next statement the existence in Hardy spaces of monsters with respect to each nontrivial finite order linear differential operator with constant coefficients. Every of these operators has the form $P(D) = a_0 I + a_1 D + \cdots + a_N D^N$, where $P(z) = a_0 + \cdots + a_N z^N$ is a complex nonzero polynomial.

Theorem 2.7. *Let $1 \leq p < \infty$ and let P be a complex nonzero polynomial. Then there is some $P(D)$-monster in H^p, hence in B^p.*

Proof. In view of Theorems 2.5–2.6, we would be done if we were able to show that the range of the mapping $P(D) : H^p \to H(\mathbb{D})$ contains H^p. Since $P(D)$ can be written as a finite composition of operators aI, $D - \lambda I$ $(a, \lambda \in \mathbb{C},\ a \neq 0)$, it is sufficient to demonstrate that $aI(H^p) \supset H^p$ (this is trivial) and that $(D - \lambda I)(H^p) \supset H^p$. For this, fix $f \in H^p$ and consider the function

$$F(z) = e^{\lambda z} \int_0^z f(t) e^{-\lambda t} dt.$$

Since $e^{-\lambda z}$ is entire, the function $f(z)e^{-\lambda z}$ is again in H^p, so the integral in the display is (extendable to) a continuous function on $\overline{\mathbb{D}}$ by Privalov's theorem. Hence F has also this property because $e^{\lambda z}$ is entire. Thus $F \in H^p$ and, by a simple calculation, $(D - \lambda I)F = f$, which finishes the proof. $\square$

Finally, we prove the existence of a dense linear submanifold of H^p all of whose nonzero members are holomorphic monsters (Theorem 2.9), so proving that not only topologically but also algebraically the size of $\mathcal{M}(I) \cap H^p$ is huge. We will make use of the following result –whose proof can be seen in [7]– which in turn is an improvement of a statement due to the first author [3].

Theorem 2.8. *Let X and Y be two metrizable topological vector spaces such that X is Baire and separable. Assume that, for each $k \in \mathbb{N}$ and each sequence $\{n_1 < n_2 < n_3 < \cdots\} \subset \mathbb{N}$, $T_{n_j}^{(k)} : X \to Y$ $(j \in \mathbb{N})$ is a densely hypercyclic sequence of continuous linear mappings. Then there is a dense linear submanifold $M \subset X$ such that*

$$M \setminus \{0\} \subset \bigcap_{k \in \mathbb{N}} HC((T_n^{(k)})).$$

Theorem 2.9. *For each p with $1 \leq p < \infty$, there exists a dense linear submanifold of the Hardy space H^p whose nonzero members are holomorphic monsters. Consequently, the same holds for the Bergman space B^p.*

Proof. Fix a dense countable subset $\{t_k : \ k \in \mathbb{N}\}$ of ∂G. Assume that, for each $k \in \mathbb{N}$, $\varphi = \varphi_k$ is the affine linear transformation given in the proof of Theorem 2.5, that is, $\varphi_k(z) = \frac{z+t_k}{2}$. By the remark given after that theorem, each subsequence $(C_{\varphi_k}^{n_j})_j$ is densely hypercyclic on H^p, whence $T_{n_j}^{(k)} : H^p \to H(\mathbb{D})$ $(j \in \mathbb{N})$ is also densely hypercyclic for each k, since H^p is dense in $H(\mathbb{D})$. We have denoted $T_n^{(k)} := C_{\varphi_k}^n$. Choose $X = H^p$, $Y = H(\mathbb{D})$. It is evident that

$$\bigcap_{k \in \mathbb{N}} HC((T_n^{(k)})) \subset \mathcal{M}(I) \cap X.$$

Thus, an application of Theorem 2.8 yields the desired result. $\square$

3 Concluding remarks and questions

1. We do *not* know whether $\mathcal{M}(P(D)) \cap H^p$ is *residual* (or, equivalently, dense) in H^p for every nonzero polynomial P, compare Theorem 2.7. If this were true then each set $\mathcal{M}(D^N) \cap H^p$ $(N \in \mathbb{N}_0)$ would be residual, hence its intersection $H^p \cap \bigcap_{N=0}^{\infty} \mathcal{M}(D^N)$ would also be residual. That is, we would obtain the existence of (many) "Luh-semimonsters" in H^p (Luh-monsters are prohibited by Theorem 2.2). Of course, the same question makes sense for B^p.

2. More generally, Theorem 2.7 still holds if one replaces $P(D)$ by a finite composition of operators with each one either of the form C_φ (φ =an automorphism of $\mathbb{D}$) or of the form $D - a(z)I$, where $a(z) \in H^\infty := \{f \in H(\mathbb{D}) :\ f \text{ is bounded}\}$. Indeed, $C_\varphi(H^p) = H^p \supset H^p$, and $(D - a(z)I)(H^p) \supset (D - a(z)I)(H^\infty) \supset H^p$, which is achieved just by changing the function F in the proof of Theorem 2.5 by
$$F(z) = e^{\int_0^z a(t)dt} \cdot \int_0^z f(t)e^{-\int_0^t a(u)du}dt.$$
But we do *not* know whether there is some $\Phi(D)$-monster in H^p, where $\Phi(D)$ is the *infinite* order differential operator associated to an entire function $\Phi(z) = \sum_0^\infty a_n z^n$ of subexponential type. Nevertheless, it is known that every nonzero operator $\Phi(D)$ is strongly omnipresent, see [5].

3. Recently, Gallardo and Montes [17] have characterized the hypercyclicity of the composition operator C_φ ($\varphi :=$ a Möbius transformation with $\varphi(\mathbb{D}) \subset \mathbb{D}$) in terms of φ on certain weighted Hardy spaces, so completing some results of Zorboska [27] which in turn extended Bourdon-Shapiro's theorems. Specifically, let S_ν ($\nu \in \mathbb{R}$) be the Hilbert space of all functions $f(z) = \sum_{n=0}^\infty a_n z^n$ for which the norm $\|f\| = (\sum_{n=0}^\infty |a_n|^2 (n+1)^{2\nu})^{1/2}$ is finite (observe that for $\nu = -\frac{1}{2}, 0, \frac{1}{2}$ the space S_ν is, respectively, the classical Bergman space B^2, the Hardy space H^2, the Dirichlet space $\mathcal{D}$). As for what we are concerned, they prove that if φ is a Möbius selfmap of $\mathbb{D}$ which is a hyperbolic non-automorphism (like that used in the proof of Theorems 2.5 and 2.9) then C_φ is hypercyclic if and only if $\nu < 1/2$ (so including the Bergman space and the Hardy space, but not the Dirichlet space). Consequently, the statements of Theorems 2.5 and 2.9 hold if one replaces H^p to any S_ν ($\nu < 1/2$).

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