Linear Kierst-Szpilrajn theorems

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Abstract

We prove in this paper the following result which extends in a somewhat 'linear' sense a theorem by Kierst and Szpilrajn and which holds on many 'natural' spaces of holomorphic functions in the open unit disk $\mathbb{D}$: There exist a dense linear manifold and a closed infinite-dimensional linear manifold of holomorphic functions in $\mathbb{D}$ whose domain of holomorphy is $\mathbb{D}$ except for the null function. The existence of a dense linear manifold of noncontinuable functions is also shown in any domain for its full space of holomorphic functions.

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1 Introduction and notation

The following notation will be used along this paper: $\mathbb{N}$ = the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ = the real line, $\mathbb{C}$ = the complex plane, $D(a, r)$ = the open disk with center a and radius r ($a \in \mathbb{C}$, $r > 0$), $\overline{D}(a, r)$ = the corresponding closed disk, $\mathbb{D}$ = the open unit disk $\{z \in \mathbb{C} :, |z| = 1\}$. If $A \subset \mathbb{C}$ and $z_0 \in \mathbb{C}$ then $\overline{A}$ = the closure of A, A^0 = the interior of A, ∂A = the boundary of A, and $\text{dist}(z_0, A) := \{|z_0 - a| : a \in A\}$ = the distance from z_0 to A. A domain is a nonempty open subset of G of $\mathbb{C}$, and G is said to be simply connected whenever $\mathbb{C}_\infty \setminus G$ is connected, where $\mathbb{C}_\infty$ is the one-point compactification of $\mathbb{C}$. As usual, we denote by $H(G)$ the space of all holomorphic functions on G. It is well known that $H(G)$ becomes a

Fréchet space (= completely metrizable locally convex space) when endowed with the topology of uniform convergence on compacta; in particular, it is a Baire space. By a Jordan curve we understand as usual a topological image of $\partial\mathbb{D} = \{z : |z| = 1\}$, and a Jordan domain is the bounded component of the complement of a Jordan curve. If f is a function which is holomorphic in a neighbourhood of a point $a \in \mathbb{C}$, then $\rho(f, a)$ denotes the radius of convergence of the Taylor series of f with center at a.

In 1884 Mittag-Leffler published that, given any domain G, there exists a function f having G as its domain of holomorphy, see [10, Chapter 10]. Recall that G is said to be a domain of holomorphy for f if f is holomorphy exactly on G, that is, f is holomorphic in G and f has no analytic continuation across any boundary point, in the sense that $\rho(f, a) = \mathrm{dist}(a, \partial G)$ for every point $a \in G$. Of course, if G is a domain of holomorphy then f has no holomorphic extension to any domain containing G strictly, but the converse is not true (consider, for instance, $G := \mathbb{C} \setminus (-\infty, 0]$ and $f :=$ the principal branch of $\log z$). But both properties are equivalent if G is a Jordan domain, in particular if $G = \mathbb{D}$. For any domain G, the symbol $H_e(G)$ will stand for the subclass of functions which are holomorphic exactly on G. In 1933 Kierst and Szpilrajn showed that, at least for $\mathbb{D}$, the former one is a 'generic' property; specifically, the subset $H_e(\mathbb{D})$ is not only nonempty but even residual (hence dense) in $H(\mathbb{D})$, that is, its complement in $H(\mathbb{D})$ is of first category.

Recently, Kahane [12, Theorem 3.1 and following remarks] has observed that Kierst-Szpilrajn's result can be generalized –after adapting terminology– as follows.

Theorem 1.1. *Let $G \subset \mathbb{C}$ be a domain and X be a Baire topological vector space with $X \subset H(G)$ such that the next conditions hold:*

(a) *For every $a \in G$ and every $r > \mathrm{dist}(a, \partial G)$ there exists $f \in X$ such that $\rho(f, a) < r$.*

(b) *The differentiation maps X into itself and all evaluations $f \in X \mapsto f(a) \in \mathbb{C}$ $(a \in G)$ are continuous.*

Then $X \cap H_e(G)$ is residual in X.

We point out that the result for the special case $X = H(G)$ of Theorem 1.1 can be extracted from the fact that the subset of functions $f \in H(G)$ with maximal cluster set at every boundary point is residual [1]. See also

Remarks 5.2 of the present paper. Note that if G is a Jordan domain then the condition (a) of the last theorem is equivalent to

(P) For every domain Ω strictly greater than G there exists $f \in X$ which is not continuable holomorphically in Ω.

Roughly speaking, we can summarize Theorem 1.1 by saying that *in a topological sense, the set of holomorphically noncontinuable functions is large.* Our aim in this paper is to show that, under soft conditions (see Section 3) on a space X consisting of holomorphic functions in $\mathbb{D}$ (in Section 2 a number of such spaces is remembered), the set of noncontinuable functions is large not only topologically *but also algebraically.* This becomes more interesting because $H_e(\mathbb{D})$ is not a linear manifold. A positive answer will be accomplished by showing the existence of large linear manifolds of noncontinuable holomorphic functions, see Section 4. Finally, in Section 5 we deal with arbitrary domains, and the problem of functions having 'very regular' behavior on the boundary is considered.

2 Spaces of holomorphic functions

From now on X will denote a topological vector space consisting of holomorphic functions in a domain G. We devote this section to describe a collection of spaces of holomorphic functions which we are going to work with. Of course, $H(G)$ is one of them, but there will be many more.

By $H(\overline{\mathbb{D}})$ we denote the linear space of the restrictions to $\mathbb{D}$ of all holomorphic functions f on some domain $\Omega = \Omega(f)$ containing the closed unit disk $\overline{\mathbb{D}}$; equivalently, $H(\overline{\mathbb{D}})$ is the space of all complex power series centered at the origin with radius of convergence > 1, which in turn is the same as the space of holomorphic functions in $\mathbb{D}$ having no singular boundary point. The space $H(\overline{\mathbb{D}})$ has only an auxiliary interest for us. Nevertheless, it is worth mentioning that it can be endowed with a natural topology such that it becomes a complete non-metrizable locally convex space (see [2, Chapter 21]). We will not make use of this fact in the future.

For $0 < p < \infty$ the Hardy space H^p and the Bergman space B^p are defined as the set $\{f \in H(\mathbb{D}) : \|f\|_p < \infty\}$, where $\|f\|_p := \sup_{0<r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$ for $f \in H^p$ and $\|f\|_p := \left(\int \int_{\mathbb{D}} |f(z)|^p \frac{dA(z)}{\pi} \right)^{1/p}$ for $f \in B^p$ ($dA(z)$ is the normalized area measure on $\mathbb{D}$). They become F-spaces (= completely metrizable topological vector spaces) with the distance $d(f,g) = \|f-g\|_p^{\alpha(p)}$, where $\alpha(p) = 1$ if $p \ge 1$ ($= p$ if $p < 1$). If $p \ge 1$ then $\|\cdot\|_p$ is a norm on H^p or B^p, so they even are Banach spaces in this case. The set of (holomorphic) polynomials is a dense subset of every H^p and every B^p. The following inequalities can be respectively found in [7, Chapter 3], [18, page 48] and [6, page 13]:

$$|f(z)| \le 2^{1/p}\|f\|_p(1-|z|)^{-1/p} \ \ (z \in \mathbb{D},\, 0 < p < \infty,\, f \in H^p),$$

$$|f(z)| \le \|f\|_p(1-|z|^2)^{-2} \ \ (z \in \mathbb{D},\, 1 \le p < \infty,\, f \in B^p),$$

$$|f(z)| \le C\|f\|_p(1-|z|)^{-2/p} \ \ (z \in \mathbb{D},\, 0 < p < \infty,\, f \in B^p).$$

Here C is a constant depending only on p. Then the topology on H^p and on B^p is stronger than that inherited from $H(\mathbb{D})$; in other words, convergence in Hardy or Bergman spaces implies convergence on compacta in $\mathbb{D}$.

If $\beta := \{\beta(n)\}_{n=0}^\infty \subset (0,+\infty)$ is a sequence with $\liminf_{n\to\infty} \beta(n)^{1/n} \ge 1$ then its associated weighted Hardy space is the Hilbert space of all functions $f(z) = \sum_{n=0}^\infty a_n z^n$ for which the norm $\|f\| = (\sum_{n=0}^\infty |a_n|^2 \beta(n))^{1/2}$ is finite, see [16] and [5, Chapter 2]. The corresponding inner product is

$$\langle f(z) \equiv \sum_{n=0}^\infty a_n z^n, g(z) \equiv \sum_{n=0}^\infty b_n z^n \rangle = \sum_{n=0}^\infty a_n \overline{b_n} \beta(n).$$

The condition $\liminf_{n\to\infty} \beta(n)^{1/n} \ge 1$ guarantees that $H^2(\beta) \subset H(\mathbb{D})$. An easy exercise involving the Closed Graph Theorem together with the continuity of the coefficient functionals $f \in H^2(\beta) \mapsto a_n \in \mathbb{C}$ ($n \in \mathbb{N}$) (recall that $\{z^n/\beta(n)^{1/2}\}_{n=0}^\infty$ is an orthonormal basis) yield that the last inclusion is continuous or, that is the same, convergence in $H^2(\beta)$ implies convergence in $H(\mathbb{D})$. Note that for $\beta(n) \equiv \frac{1}{n+1}, 1, n+1$ the space $H^2(\beta)$ is, respectively, the classical Bergman space B^2, the unweighted Hardy space H^2, the Dirichlet space $\mathcal{D}$. By considering Taylor expansions it is easy to see that the polynomials are also dense in $H^2(\beta)$. Due to reasons that will become clear later we will impose on β the more restrictive condition

$$\liminf_{n\to\infty} \beta(n)^{1/n} = 1.$$

Let be given a bounded domain $G \subset \mathbb{C}$. We agree that $A^0(G) = A(G) := \{f \in H(G) : f \text{ has a continuous extension to } \overline{G}\}$. If $N \in \mathbb{N}$ then $A^N(G) :=$

$\{f \in H(G) : f^{(j)}$ has a continuous extension to $\overline{G}$ for all $j \in \{0,1,...,N\}\}$. It is easy to see that if $N \in \mathbb{N}_0$ then $A^N(G)$ becomes a Banach space as soon as it is endowed with the norm $\|f\| = \sum_{j=0}^N \sup_{z\in G} |f^{(j)}(z)|$. The space $A^\infty(G)$ is defined as $A^\infty(G) := \bigcap_{N\in\mathbb{N}_0} A^N(G) = \{f \in H(G) : f^{(j)}$ has a continuous extension to $\overline{G}$ for all $j \in \mathbb{N}_0\}$. The topology considered on $A^\infty(G)$ is that of the projective limit of the spaces $A^N(G)$ $(N \in \mathbb{N}_0)$. Then $A^\infty(G)$ becomes a Fréchet space. In particular, each $A^N(G)$ $(N \in \mathbb{N}_0 \cup \{\infty\})$ is a Baire space. It is evident that convergence on each of them implies uniform convergence on compacta in G. If $G = \mathbb{D}$ the Cauchy estimates together with some elementary manipulation of Taylor coefficients yield that $A^\infty(\mathbb{D}) = \{f(z) = \sum_{n=0}^\infty a_n z^n : \{n^N a_n\}_{n=0}^\infty$ is bounded for all $N \in \mathbb{N}\}$.

In this paragraph all spaces will be non-separable. The space H^∞ consists of all bounded holomorphic functions in $\mathbb{D}$. It is a Banach space when endowed with the supremum norm, so H^∞ is a Baire space. Its topology is clearly finer than that of uniform convergence on compacta. The Korenblum space $A^{-\infty}$ is defined as the inductive limit of the weighted Banach spaces $A^{-q} := \{f \in H(\mathbb{D}) : \|f\|_q < \infty\}$ $(q > 0)$, where $\|f\|_q := \sup_{z\in\mathbb{D}}(1-|z|)^q|f(z)|$. Anew after using Cauchy's estimates and some manipulation of Taylor coefficients we obtain

$$A^{-\infty} = \bigcup_{q>0} A^{-q} = \{f(z) = \sum_{n=0}^\infty a_n z^n : \text{there is } N = N(f) \in \mathbb{N}$$

$$\text{such that } \{n^N a_n\}_{n=1}^\infty \text{ is bounded}\}.$$

The topology of each A^{-q} (so that of $A^{-\infty}$) is finer than that of uniform convergence on compacta. But $A^{-\infty}$ is neither Baire nor metrizable, see [9, Section 4.3].

Let us consider a final, very small space. Fix $\alpha \in (0,1)$ and define

$$X_\alpha = \{f(z) = \sum_{n=0}^\infty a_n z^n : \{a_n n^{n^\alpha}\}_{n=1}^\infty \text{ is bounded}\}.$$

With no difficulty one can see that X_α is a Banach space when endowed with the norm $\|f\| := |a_0| + \sup_{n\in\mathbb{N}} |n^{n^\alpha} a_n|$, that $X_\alpha \subset A^\infty(\mathbb{D})$ (use $\alpha > 0$), that $\bigcup_{0<\alpha<1} X_\alpha \neq A^\infty(\mathbb{D})$ (take $f(z) = \sum_{n=1}^\infty n^{-n^{\log n}} z^n$), and that the polynomials are dense in X_α. The inequality $|f(z)| \le \|f\|[1 + \sum_{n=1}^\infty r^n/n^{n^\alpha}]$ $(|z| = r < 1)$ shows that the topology in X_α is finer than that of uniform convergence on compacta in $\mathbb{D}$.

3 Conditions on our spaces

It appears to be convenient to list the properties on our spaces X which will be used repeatedly along this paper, see (A)–(E) below. But let us first recall that if $f(z) := \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ then the *support* of f (or of the sequence $\{a_n\}_{n=0}^{\infty}$) is the set $\mathrm{supp}\,(f) = \{n \in \mathbb{N}_0 : a_n \neq 0\}$. If $Q \subset \mathbb{N}_0$ then we denote by $H_Q(\mathbb{D})$ the space of all $f \in H(\mathbb{D})$ with gaps outside Q, that is, such that $\mathrm{supp}\,(f) \subset Q$. The symbol P_Q will stand for the natural projection $P_Q : \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) \mapsto \sum_{n \in Q} a_n z^n \in H_Q(\mathbb{D})$.

In the following enumeration, it is assumed that $G = \mathbb{D}$ in (A), (B) and (E).

(A) X is *stable under projections*, that is, $P_Q(X) \subset X$ for every $Q \subset \mathbb{N}_0$.

(B) Some denumerable subset of $H(\overline{\mathbb{D}})$ is a dense subset of X.

(C) All evaluation functionals $f \in X \mapsto f^{(k)}(a) \in \mathbb{C}$ $(a \in G;\ k \in \mathbb{N}_0)$ are continuous.

(D) For every $a \in G$ and every $r > \mathrm{dist}(a, \partial G)$ there exists $f \in X$ such that $\rho(f, a) < r$.

(E) $X \not\subset H(\overline{\mathbb{D}})$.

Observe that properties (A), (D) and (E) do not require any topological or algebraic structure on X. Note that (D) is the condition (a) in Theorem 1.1, so (again) it is equivalent to (P) if G is a Jordan domain, specially if $G = \mathbb{D}$. We also point out that condition (b) in Theorem 1.1 can be considerably weakened. Indeed, the same proof of [12, Theorem 3.1] taken word-for-word works if we replace (b) to (C). Thus, within our conventions, Theorem 1.1 can be reinforced as follows.

Theorem 3.1. *Let $G \subset \mathbb{C}$ be a domain and X be a Baire topological vector space with $X \subset H(G)$ satisfying* (C) *and* (D). *Then $X \cap H_e(G)$ is residual in X.*

This reformulation allows, for instance, each Hardy space H^p and each Bergman space B^p $(0 < p < \infty)$ –which are not stable under differentiation– to be one of the 'lucky' spaces X. Theorem 3.1 will be employed several times in the subsequent sections.

Remarks 3.2. 1. There are plenty of natural spaces enjoying property (A), apart from $H(\mathbb{D})$ itself. They include many spaces given by inequalities or by convergence of series involving the Taylor coefficients. For instance, the spaces $H^2(\beta)$, $A^\infty(\mathbb{D})$, $A^{-\infty}$ and X_α $(0 < \alpha < 1)$ are stable under projections. On the negative side, there exist rather natural spaces $X \subset H(\mathbb{D})$ which have not this kind of stability. In fact, every Hardy space H^p with $p \neq 2$ does not satisfy (A). To see this, fix $p < 2$ and select a function $f(z) = \sum_{n=0}^\infty a_n z^n \in H^p \setminus H^2$. Then f cannot be bounded, so $\limsup_{n\to\infty} |a_n|^{1/n} = 1$. In addition, due to a celebrated theorem of Littlewood (see [7, Appendix A]) there is a sequence of signs $\{\varepsilon_n : n \in \mathbb{N}_0\} \subset \{-1, 1\}$ such that $g(z) := \sum_{n=0}^\infty \varepsilon_n a_n z^n$ has radial limit almost nowhere $e^{i\theta} \in \partial\mathbb{D}$. Hence $g \notin H^p$ by Fatou's Theorem. Define $Q := \{n \in \mathbb{N}_0 : \varepsilon_n = 1\}$. Then it is clear that $g = P_Q(f) - P_{\mathbb{N}_0\setminus Q}(f)$, so at least one of the functions $P_Q(f)$, $P_{\mathbb{N}_0\setminus Q}(f)$ must be out of H^p, which shows the non-stability of this space. Let us fix now a real number $p > 2$ and select this time a function $f(z) = \sum_{n=0}^\infty a_n z^n \in H^2 \setminus H^p$. As before, $\limsup_{n\to\infty} |a_n|^{1/n} = 1$. By the aforementioned theorem of Littlewood there is a sequence of signs $\{\varepsilon_n : n \in \mathbb{N}_0\} \subset \{-1, 1\}$ such that $g(z) := \sum_{n=0}^\infty \varepsilon_n a_n z^n$ is in H^q for all $q \in (0, +\infty)$; in particular $g \in H^p$. Define again $Q := \{n \in \mathbb{N}_0 : \varepsilon_n = 1\}$. Then it is clear that $f = P_Q(f) + P_{\mathbb{N}_0\setminus Q}(f)$, so at least one of the functions $P_Q(f)$, $P_{\mathbb{N}_0\setminus Q}(f)$ is out of H^p. But $P_Q(f) = P_Q(g)$ and $P_{\mathbb{N}_0\setminus Q}(f) = -P_{\mathbb{N}_0\setminus Q}(g)$. Therefore at least one of the functions $P_Q(g)$, $P_{\mathbb{N}_0\setminus Q}(g)$ is not in H^p, hence H^p is not projection-stable either.

2. Property (B) holds if, for instance, the set of polynomials is a dense subset of X (the continuity of the sum and of the multiplication by scalars on a topological vector space makes the denumerable set of polynomials with rational real and imaginary parts another dense subset of X). Hence the spaces $H(\mathbb{D})$, $H^2(\beta)$, $A^N(\mathbb{D})$ $(N \in \mathbb{N}_0 \cup \{\infty\})$, H^p, B^p $(0 < p < \infty)$, X_α $(0 < \alpha < 1)$ enjoy property (B). But the spaces H^∞, A^{-q} $(0 < q < 1)$, $A^{-\infty}$ do not satisfy it since they are not separable.

3. The Weierstrass theorem about convergence of sequences of holomorphic functions yields that if convergence in X implies uniform convergence on compacta in G then (C) holds. Therefore all the spaces $H(\mathbb{D})$, $H^2(\beta)$, $A^N(\mathbb{D})$, H^p, B^p, X_α, H^∞, A^{-q}, $A^{-\infty}$ satisfy property (C).

4. It is clear that property (D) holds if $H_e(G) \cap X \neq \emptyset$. Then $H(G)$ enjoys (D) due to the Mittag-Leffler theorem mentioned in the Introduction. If $G = \mathbb{D}$ then both properties (D) and (E) are satisfied whenever $H_e(\mathbb{D}) \cap X \neq \emptyset$. Therefore the space $X = H(\mathbb{D})$ satisfies (D)–(E). This

is by the special case $G = \mathbb{D}$ in Mittag-Leffler's result, but we have another, more direct approach: Take the function $f(z) = \sum_{j=0}^{\infty} z^{2^j}$, which has radius of convergence 1 and Hadamard gaps, so it is in $H_e(\mathbb{D})$ by the Hadamard lacunary theorem (see [15, Chapter 16]). A similar fact happens with the much smaller space $A^\infty(\mathbb{D})$: Consider this time the function $f(z) = \sum_{j=0}^{\infty} a_j \eta_j z^j$, where $a_j = 1$ if j is a power of 2, $a_j = 0$ otherwise, and $\eta_j = \exp(-\sqrt{j})$, see again [15, Chapter 16]. This together with the fact that $A^\infty(\mathbb{D})$ is included in each of the spaces H^p, B^p, $A^N(\mathbb{D})$, H^∞, A^{-q}, $A^{-\infty}$ ($0 < p < \infty$, $N \in \mathbb{N}_0 \cup \{\infty\}$, $q > 0$) shows that all these spaces satisfy (D)–(E) too. Also each weighted Hardy space $H^2(\beta)$ enjoys (D)–(E) by the former reason: The function $f(z) := \displaystyle\sum_{j=1}^{\infty} \frac{z^{m_j}}{m_j \beta(m_j)^{1/2}}$ is in $H^2(\beta)$, where $\{m_j\}_{j=1}^{\infty}$ is a sequence of positive integers satisfying $m_{j+1} > 2m_j$ ($j \in \mathbb{N}$) and $\lim_{j\to\infty} \beta(m_j)^{1/m_j} = 1$ (recall that $\liminf_{n\to\infty} \beta(n)^{1/n} = 1$), so the radius of convergence of the power series of f is 1 and the mentioned Hadamard theorem can anew be applied; hence $f \in H_e(\mathbb{D}) \cap H^2(\beta)$. As for the small space X_α, the function $f(z) := \sum_{n=1}^{\infty} n^{-n^\alpha} z^n$ belongs to X_α but not to $H(\overline{\mathbb{D}})$ (use the fact $\alpha < 1$), so (E) holds for this space. In fact, (D) also holds: Make sufficiently many gaps in the last series. On the other hand, the space $H(\overline{\mathbb{D}})$ trivially does not satisfy (E), but it satisfies (D). Indeed, fix a domain Ω containing $\mathbb{D}$ strictly and choose any $z_0 \in \Omega \setminus \overline{\mathbb{D}}$. Then the function $f(z) = \sum_{j=0}^{\infty} (z/|z_0|)^{2^j}$ has radius of convergence $|z_0|$ and Hadamard gaps, so $D(0, |z_0|)$ is its domain of holomorphy; therefore it belongs to $H(\overline{\mathbb{D}})$ but it cannot be holomorphically continued to Ω. Finally, if we fix any domain Ω as before with $\partial\Omega \cap \partial\mathbb{D} \neq \emptyset$ and choose any function in $H_e(\Omega)$ then it is immediately shown that $X :=$ {the restrictions to $\mathbb{D}$ of the functions of $H(\Omega)$} satisfies (E) but not (D).

4 Large linear manifolds of noncontinuable holomorphic functions

We are going to see how large linear manifolds of holomorphic functions having $\mathbb{D}$ as its domain of holomorphy can be constructed. This will be done in a twofold way, namely, with dense linear manifolds (Theorem 4.2) and with closed infinite-dimensional linear manifolds (Theorem 4.3). For this, the natural mild assumptions (A)–(E) given in Section 2 are to be applied timely.

In the statements of Theorems 4.2–4.3, it is understood that conditions (C), and (D) are referred to the domain $G = \mathbb{D}$.

We now present the following auxiliary result, which might be interesting in itself. It will reveal useful in the proof of our main results in this section.

Lemma 4.1. *Suppose that X is a topological vector space X with $X \subset H(\mathbb{D})$ satisfying* (A) *and that $F \in X \setminus H(\overline{\mathbb{D}})$. Then there exists an infinite-dimensional linear manifold $L(F) \subset H_{\mathrm{supp}\,(F)}(\mathbb{D}) \cap X$ such that $L(F) \setminus \{0\} \subset H_e(\mathbb{D})$.*

Proof. Assume that $F \in X \setminus H(\overline{\mathbb{D}})$. Then we can write $F(z) := \sum_{n=0}^{\infty} a_n z^n$, where the radius of convergence of the power series is 1. By the Cauchy-Hadamard formula, we have

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1.$$

Therefore there exists a strictly increasing sequence $\{n(k) : k \in \mathbb{N}\} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} |a_{n(k)}|^{1/n(k)} = 1. \tag{1}$$

We can extract a sequence $\{m(1) < m(2) < \cdots\} \subset \{n(k) : k \in \mathbb{N}\}$ with

$$m(k+1) > 2m(k) \quad (k \in \mathbb{N}). \tag{2}$$

Now we divide the sequence $\{m(k) : k \in \mathbb{N}\}$ into infinitely many strictly increasing sequences $A_j = \{p(j,k) : k \in \mathbb{N}\}$ $(j \in \mathbb{N})$ so that they are pairwise disjoint. Due to property (A), each series

$$F_j(z) = \sum_{k=1}^{\infty} a_{p(j,k)} z^{p(j,k)}$$

defines a function belonging to X. But from (1) we have clearly that

$$\lim_{k\to\infty} |a_{p(j,k)}|^{1/p(j,k)} = 1 \quad (j \in \mathbb{N}) \tag{3}$$

whereas by (2) we obtain that every F_j possesses Hadamard gaps. Consider the linear span

$$L(F) := \mathrm{span}\,\{F_j : j \in \mathbb{N}\}.$$

Then, obviously, $L(F)$ is a linear manifold contained in X. Moreover, $L(F)$ is infinite-dimensional because the functions F_j $(j \in \mathbb{N})$ are linearly independent due to the fact that $\text{supp}\,(F_j) \cap \text{supp}\,(F_l) \subset A_j \cap A_l = \emptyset$ whenever $j \neq l$. Furthermore, it is evident that if

$$h := \sum_{j=1}^{N} c_j F_j \in L(F) \;\; (c_j \in \mathbb{C},\, j = 1, ..., N) \tag{4}$$

then $\text{supp}\,(h) \subset \bigcup_{j=1}^{N} \text{supp}\,(F_j) \subset \text{supp}\,(F)$, hence $h \in H_{\text{supp}\,(F)}(\mathbb{D})$.

Finally, assume that $h \in L(F) \setminus \{0\}$. Without loss of generality, we can suppose that h is as in (4) with $c_N \neq 0$. By (3), the radius of convergence of the power series defining $c_N F_N$ is 1. But the same is true for h because the corresponding radii for $c_j F_j$ $(j = 1, ..., N-1)$ are ≤ 1 and the supports of the $c_j F_j$ $(j = 1, ..., N)$ are pairwise disjoint. On the other hand, if $\text{supp}\,(h) = \{p(1) < p(2) < \cdots\}$ $(\subset \{m(k) : k \in \mathbb{N}\})$ then from (2) we have that $p(k+1) > 2p(k)$ for all $k \in \mathbb{N}$. Thus the Hadamard lacunary theorem asserts that $h \in H_e(\mathbb{D})$. □

It should be noted that Lemma 4.1 yields the following result for the special case $X = H(\mathbb{D})$: Given an infinite subset $Q \subset \mathbb{N}_0$, then there exists an infinite-dimensional linear manifold $M(Q) \subset H_Q(\mathbb{D})$ such that $M(Q) \subset H_e(\mathbb{D})$. Indeed, one can choose a sequence $\{n(j) : j \in \mathbb{N}\} \subset Q$ with $n(j+1) > 2n(j)$ for all $j \in \mathbb{N}$. Therefore the function $F(z) := \sum_{j=1}^{\infty} z^{n(j)}$ is holomorphic in $\mathbb{D}$, has radius of convergence 1 and possesses Hadamard gaps, so the Hadamard lacunary theorem tells us that $F \in H_e(\mathbb{D})$. Hence we can take $M(Q) = L(F)$.

Theorem 4.2. *Assume that X is a metrizable topological vector space with $X \subset H(\mathbb{D})$. Suppose that at least one of the following conditions holds:*

(a) *X is Baire and satisfies the properties* (A), (B), (C) *and* (D).

(b) *X satisfies the property* (B) *and there is a subset of X for which* (A) *and* (E) *hold.*

Then there is a dense linear manifold M in X such that $M \setminus \{0\} \subset H_e(\mathbb{D})$.

Proof. Let us denote by d a distance on X which is translation-invariant and compatible with the topology of X. If we start from (a) then we can apply Theorem 3.1 on $G = \mathbb{D}$ to obtain that $X \cap H_e(\mathbb{D})$ is residual in X. In particular, such subset is nonempty and we can pick a function $F \in X \cap H_e(\mathbb{D})$, hence $F \in X \setminus H(\overline{\mathbb{D}})$. If (b) is assumed then, by the property (E), we obtain the existence of a function $F \in Y \setminus H(\overline{\mathbb{D}})$ for some subset $Y \subset X$ satisfying, in addition, stability under projections. Thus, we may start in both cases with a function $F \in X \setminus H(\overline{\mathbb{D}})$ whose all projections $P_Q(F)$ $(Q \subset \mathbb{N}_0)$ are in X. Moreover, due to (B), there is a sequence $\{g_n : n \in \mathbb{N}\} \subset H(\overline{\mathbb{D}}) \cap X$ that is dense in X.

Consider the linear manifold $L(F) = \text{span}\,\{F_n : n \in \mathbb{N}\}$ provided in the proof of Lemma 4.1. Recall that by construction we had in fact that

$$F_n = P_{A_n}(F) \quad (n \in \mathbb{N})$$

for certain sets $A_n \subset \mathbb{N}$. Then $F_n \in X$ for all $n \in \mathbb{N}$.

Let us fix an $n \in \mathbb{N}$. The continuity of the multiplication by scalars in the topological vector space X gives the existence of a constant $\varepsilon_n > 0$ for which $d(\varepsilon_n F_n, 0) < 1/n$. Now we define

$$f_n := g_n + \varepsilon_n F_n \quad \text{and} \quad M := \text{span}\,\{f_n : n \in \mathbb{N}\}.$$

We have that $f_n \in X$ for all n because g_n, $F_n \in X$, whence M is a linear manifold contained in X. Furthermore, the translation-invariance of d implies $d(f_n, g_n) = d(\varepsilon_n F_n, 0) < 1/n$, so $d(f_n, g_n) \to 0$ as $n \to \infty$. This and the density of $\{g_n : n \in \mathbb{N}\}$ imply the density of $\{f_n : n \in \mathbb{N}\}$, which in turn implies, trivially, that M is dense in X.

Finally, take a function $f \in M \setminus \{0\}$. Then there exist $N \in \mathbb{N}$ and complex constants $c_1, ..., c_N$ with $c_N \neq 0$ such that $f = c_1 g_1 + \cdots + c_N g_N + h$, where

$$h := \sum_{j=1}^{N} c_j \varepsilon_j F_j \in L(F) \setminus \{0\}.$$

By Lemma 4.1, $h \in H_e(\mathbb{D})$. But the function $g := c_1 g_1 + \cdots + c_N g_N$ is holomorphically continuable on $D(0, R)$ for some $R > 1$ (in fact, for $R = \min_{1 \le n \le N} R_n$, where R_n is the radius of convergence of the Taylor series of g_n). Consequently, the sum $f = g + h$ can be holomorphically continued beyond *no* point of $\partial\mathbb{D}$, that is, $f \in H_e(\mathbb{D})$, as required. □

Remarks 3.2 contain examples of spaces X on which Theorem 4.2 can be applied, namely, $H(\mathbb{D})$, $H^2(\beta)$, $A^N(\mathbb{D})$ $(N \in \mathbb{N}_0 \cup \{\infty\})$, X_α $(0 < \alpha < 1)$,

H^p, B^p $(0 < p < \infty)$. Suffice it to say that $A^\infty(\mathbb{D})$ is a subset of each space $A^N(\mathbb{D})$, H^p, B^p and that $A^\infty(\mathbb{D})$ *does* satisfy (A) and (E).

Next, we focus our attention on the search of large closed linear manifolds of noncontinuable holomorphic functions. As the following theorem shows, all the spaces $H(\mathbb{D})$, $H^2(\beta)$, $A^N(\mathbb{D})$, X_α, H^p, B^p, H^∞, A^{-q}, $A^{-\infty}$ enjoy the existence of such linear manifolds.

Theorem 4.3. *Assume that X is a topological vector space with $X \subset H(\mathbb{D})$. Suppose that at least one of the following conditions holds:*

(a) *X is Baire and satisfies the properties* (A), (C) *and* (D)*.*

(b) *X satisfies the property* (C) *and there is a subset of X for which* (A) *and* (E) *hold.*

Then there is a infinite-dimensional closed linear manifold $M \subset X$ such that $M \setminus \{0\} \subset H_e(\mathbb{D})$.

Proof. Due to (a) or (b), we get as in the first part of the proof of Theorem 4.2 the existence of a function $F \in X \setminus H(\overline{\mathbb{D}})$. From now on we will follow the same notation as that in the proof of Lemma 4.1. It is clear that the sequence $\{n(k) : k \in \mathbb{N}\}$ selected there may be choosen to satisfy $a_{n(k)} \neq 0$ for all $k \in \mathbb{N}$. Also, we denote $Q := \bigcup_{j\in\mathbb{N}} A_j$.

Let us consider again the linear manifold $L(F) = \operatorname{span}\{F_n : n \in \mathbb{N}\}$ constructed in that lemma. Recall that it is infinite-dimensional. Then its closure

$$M := \overline{L(F)}$$

in X is an infinite-dimensional closed linear manifold. All that should be proved is $M \setminus \{0\} \subset H_e(\mathbb{D})$.

To this end, we observe that the conclusion will follow as soon as we demonstrate the following three properties:

(i) The set Λ contains $L(F)$, where $\Lambda := \{f(z) = \sum_{n\in Q} c_n z^n \in X :$ there exists $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$ such that $c_{p(j,k)} = \lambda_j a_{p(j,k)}$ for all $j, k \in \mathbb{N}\}$.

(ii) Λ is closed in X.

(iii) $\Lambda \setminus \{0\} \subset H_e(\mathbb{D})$.

Indeed, (i) together with (ii) would imply that $M \subset \Lambda$, whence $M \setminus \{0\} \subset \Lambda \setminus \{0\} \subset H_e(\mathbb{D})$ by (iii), and we would be done.

Property (i) is trivial: It suffices to choose $\lambda_j = 0$ $(j > N)$ for each prescribed function $f = \sum_{j=1}^N \lambda_j F_j \in L(F)$. As for (ii), assume that

$$\{f_\alpha(z) := \sum_{n \in Q} c_n^{(\alpha)} z^n\}_{\alpha \in I} \subset \Lambda$$

is a net with $f_\alpha \to f$ in X. It must be shown that $f \in \Lambda$. Suppose that f has a Taylor expansion $f(z) = \sum_{n=0}^\infty c_n z^n$ $(z \in \mathbb{D})$. Due to (C), we have that $f_\alpha^{(n)}(0) \to f^{(n)}(0)$ for each $n \in \mathbb{N}_0$, so $c_n^{(\alpha)} \to c_n$. Then $c_n = 0$ for all $n \notin Q$ and $f(z) = \sum_{n \in Q} c_n z^n$. Moreover, for every $\alpha \in I$ there exists a sequence $\{\lambda_j^{(\alpha)}\}_{j=1}^\infty \subset \mathbb{C}$ such that $c_{p(j,k)}^{(\alpha)} = \lambda_j^{(\alpha)} a_{p(j,k)}$ for all $j, k \in \mathbb{N}$. Again by (C), we get $c_{p(j,k)}^{(\alpha)} \to c_{p(j,k)}$, hence $\lambda_j^{(\alpha)} \to c_{p(j,k)}/a_{p(j,k)}$ for all j, k. But by the uniqueness of the limit, there must be constants $\lambda_j \in \mathbb{C}$ $(j \in \mathbb{N})$ satisfying $\lambda_j = c_{p(j,k)}/a_{p(j,k)}$, or equivalently, $c_{p(j,k)} = \lambda_j a_{p(j,k)}$ for all $j, k \in \mathbb{N}$. Then $f \in \Lambda$.

Finally, assume that $f \in \Lambda \setminus \{0\}$ and that f has a Taylor expansion about the origin as in the definition of Λ, see (i). Then there exists $J \in \mathbb{N}$ with $\lambda_J \neq 0$. Of course, $\limsup_{n\to\infty} |c_n|^{1/n} \le 1$. But by (1),

$$\lim_{k\to\infty} |c_{p(J,k)}|^{1/p(J,k)} = \lim_{k\to\infty} |\lambda_J|^{1/p(J,k)} \cdot \lim_{k\to\infty} |a_{p(J,k)}|^{1/p(J,k)} = 1.$$

Therefore $\limsup_{n\to\infty} |c_n|^{1/n} = 1$, that is, the radius of convergence of the Taylor expansion of f is 1. On the other hand, the set Q consisted of the integers of the sequence $\{m(1) < m(2) < \cdots\}$, which had Hadamard gaps by virtue of (2). Hence (again) Hadamard's lacunary theorem comes in our help yielding $f \in H_e(\mathbb{D})$. This shows (iii) and finishes the proof. □

Remarks 4.4. 1. If the last proof is viewed closely then one realizes that in condition (b) the property (C) can be replaced by a weaker one, namely: All evaluation functionals $f \in X \mapsto f^{(k)}(0) \in \mathbb{C}$ $(k \in \mathbb{N}_0)$ are continuous.

2. If X is a *Baire* topological vector space with $X \subset H(\mathbb{D})$ satisfying condition (b) of the last theorem then we also have that $H_e(\mathbb{D}) \cap X$ is residual in X. Indeed, using (A) and (E) we can construct a function $f \in X$ with lacunary Taylor expansion and radius of convergence 1, so $f \in H_e(\mathbb{D}) \cap X$. Then (D) is satisfied and Theorem 3.1 applies.

5 Noncontinuability on more general domains

The conclusion of Theorem 4.2 holds for *any* domain $G \subset \mathbb{C}$ when X is the full space $H(G)$. The proof will be rather different from that of the mentioned theorem.

Theorem 5.1. *Let $G \subset \mathbb{C}$ be a domain. Then there is a dense linear manifold M in $H(G)$ such that $M \setminus \{0\} \subset H_e(G)$.*

Proof. The case $G = \mathbb{C}$ is trivial, so we may assume $G \neq \mathbb{C}$. Denote by G_* the one-point compactification of G. Let us fix an increasing sequence $\{K_n : n \in \mathbb{N}\}$ of compact subsets of G such that each compact subset of G is contained in some K_n and each connected component of the complement of every K_n contains some connected component of the complement of G (see [3, Chapter 7]). Choose a countable dense subset $\{g_n : n \in \mathbb{N}\}$ of the (separable) space $H(G)$.

Choose also a sequence $\{a_n : n \in \mathbb{N}\}$ of distinct points of G such that it has no accumulation point in G and each prime end (see [4, Chapter 9]) of ∂G is an accumulation point of the sequence. More precisely, the sequence $\{a_n\}$ should have the following property: For every $a \in G$ and every $r > \operatorname{dist}(a, \partial G)$, the intersection of $\{a_n\}$ with the connected component of $D(a, r) \cap G$ containing a is infinite. An example of the required sequence may be defined as follows. Let $A = \{\alpha_k\}$ be a dense countable subset of G. For each $k \in \mathbb{N}$ choose $b_k \in \partial G$ such that $|b_k - \alpha_k| = \operatorname{dist}(\alpha_k, \partial G)$. For every $k \in \mathbb{N}$ let $\{a_{kl} : l \in \mathbb{N}\}$ be a sequence of points of the line interval joining α_k with the corresponding point b_k such that $|a_{kl} - b_k| < 1/(k + l)$ $(k, l \in \mathbb{N})$. Each one-fold sequence $\{a_n\}$ (without repetitions) consisting of all distinct points of the set $\{a_{kl} : k, l \in \mathbb{N}\}$ has the required property.

Now consider for each $N \in \mathbb{N}$ the set $A_N := K_N \cup \{a_n : n \in \mathbb{N}\}$. We have:

- The set A_N is closed in G because the set $\{a_n : n \in \mathbb{N}\}$ does not cluster in G.
- The set $G_* \setminus A_N$ is connected due to the shape of K_N (recall that in G_* the whole boundary ∂G collapses to a unique point, say ω) and to the denumerability of $\{a_n : n \in \mathbb{N}\}$.

- The set $G_*\backslash A_N$ is locally connected at ω, again by the denumerability of $\{a_n : n \in \mathbb{N}\}$ and by the fact that one can suppose that neighbourhoods of ω do not intersect K_N.

On the other hand, the function $h_n : A_N \to \mathbb{C}$ defined as

$$h_N(z) = \begin{cases} g_N(z) & \text{if } z \in K_N \\ n^N & \text{if } z = a_n \text{ and } a_n \notin K_N \end{cases}$$

is continuous on A_N and holomorphic on A_N^0 $(= K_N^0)$. Hence the Arakelian approximation theorem (see [8, pages 136–144]) guarantees the existence of a function $f_N \in H(G)$ such that

$$|f_N(z) - h_N(z)| < \frac{1}{N} \text{ for all } z \in A_N. \tag{5}$$

We define

$$M := \text{span}\,\{f_N : N \in \mathbb{N}\}.$$

Then M is a linear manifold contained in $H(G)$. It is dense because $\{f_N : N \in \mathbb{N}\}$ is dense, which in turn is true from (5) (recall that $h_N = g_N$ on K_N), from the denseness of $\{g_N : N \in \mathbb{N}\}$ and from the property that for a prescribed compact set $K \subset G$ we have $K \subset K_N$ whenever N is large enough.

Now, fix a function $f \in M \setminus \{0\}$, so $f = \sum_{j=1}^N c_j f_j$ for some N and some complex constants c_j $(j = 1, ..., N)$ with $c_N \neq 0$. By (5) we get

$$|f_j(a_n) - n^j| < 1 \text{ for all } n \geq n_0 \ (j = 1, ..., N),$$

for some $n_0 \in \mathbb{N}$ since each K_j may contain only finitely many points a_n. Therefore

$$|f(a_n) - \sum_{j=1}^N c_j n^j| < \alpha \ \ (n \geq n_0),$$

where $\alpha := \sum_{j=1}^N |c_j| < +\infty$. Then $f(a_n) \to \infty$ $(n \to \infty)$. Given an arbitrary point $a \in G$ the radius of convergence $\rho(f, a)$ is equal to $\text{dist}(a, \partial G)$. Indeed, if this were not the case, we could choose r with $\text{dist}(a, \partial G) < r < \rho(f, a)$ and, by the construction of $\{a_n : n \in \mathbb{N}\}$, there would exist a sequence $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ for which $a_{n_k} \in G \cap D(a, r)$ $(k \in \mathbb{N})$. On the other hand, the sum $S(z)$ of the Taylor series of f with center a is bounded on $D(a, r)$. But $S = f$ on $G \cap D(a, r)$, so $S(a_{n_k}) = f(a_{n_k}) \to \infty$ $(k \to \infty)$, which is absurd. Consequently, f has no analytic continuation across any boundary point of G. This finishes the proof. □

An elementary modification of the last proof reveals that a slight improvement of Theorem 5.1 can be obtained: For a prescribed function $\varphi : G \to (0,+\infty)$ there exists a dense linear manifold M_φ in $H(G)$ such that every $f \in M_\varphi \setminus \{0\}$ satisfies

$$\limsup_{z\to t} \frac{|f(z)|}{\varphi(z)} = +\infty \text{ for all prime end } t \text{ of } \partial G.$$

We conclude this paper with a number of comments and questions.

Remarks 5.2. 1. The Kierst-Szpilrajn theorem –that is, the conclusion of Theorem 1.1 or 3.1– remains true for a wide class of 'natural' Fréchet spaces (see e.g. [11, Proposition 1.7.6]). A specially interesting case is that of non-continuable holomorphic functions which are very regular on the boundary, for which a positive answer is known even in several dimensions (see [17]). Namely, let G be a bounded open subset of $\mathbb{C}^p$ such that $G = \overline{G}^0$, and the compact set $\overline{G}$ is polynomially convex. Then G is a domain of holomorphy of a function $f \in A^0(G)$. If, moreover, $\overline{G}$ has *Markov property* then G is a domain of holomorphy of a function $f \in A^\infty(G)$. Let us recall that for $p = 1$, $A \subset \mathbb{C}$ has Markov property, if there exists a positive constant c such that $\operatorname{diam}(S) \geq c$ for each connected component S of A.
2. From the last remark we obtain in particular that if G is a Jordan domain then $A^N(G) \cap H_e(G) \neq \emptyset$ for all $N \in \mathbb{N}_0 \cup \{\infty\}$ (a nice, rather constructive proof for the case $N = 0$ can be found in [14, Theorem 2]). According Remark 3.2.3 the space $X = A^N(G)$ satisfies (C), and by Remark 3.2.4 it also enjoys (D). Hence Theorem 3.1 applies, so obtaining that the set $A^N(G) \cap H_e(G)$ is residual in $A^N(G)$. Note that as observed in [14, Section 3], if no assumption is imposed on G then even in the case of a bounded simply connected domain G the set $A^0(G) \cap H_e(G)$ (so each $A^N(G) \cap H_e(G)$) may well be empty; consider for instance $G = \mathbb{D} \setminus [0,1]$.
3. In view of Theorem 3.1, it would be interesting to know whether there exists a non-metrizable Baire topological vector space $X \subset H(G)$ satisfying the condition (C).
4. Finally, we want to pose here the following question: Are there analogues of Theorem 5.1 for *subspaces* $X \subset H(G)$, e.g. for $X = A^\infty(G)$ where G is bounded?

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