Sequences of differential operators: exponentials, hypercyclicity and equicontinuity

by

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Abstract

In this paper, an eigenvalue criterion for hypercyclicity due to the first author is improved. As a consequence, some new sufficient conditions for a sequence of infinite order linear differential operators to be hypercyclic on the space of holomorphic functions on certain domains of $\mathbb{C}^N$ are shown. Moreover, several necessary conditions are furnished. The equicontinuity of a family of operators as before is also studied, and it is even characterized if the domain is $\mathbb{C}^N$. The results obtained extend or improve earlier work of several authors.

1 Introduction, notation and preliminary results.

Throughout this paper we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}$ the real line, by $\mathbb{C}$ the field of complex numbers, and by $\mathbb{N}_0$ the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let X, Y be two linear topological spaces, $T_i : X \to Y$ ($i \in I :=$ an arbitrary index set) a family of continuous linear mappings, and $x \in X$. Then x is said to be *hypercyclic* or *universal* for (T_i) whenever its orbit $\{T_i x : i \in I\}$ under (T_i) is dense in Y. The family (T_i) is called *hypercyclic* whenever it has a hypercyclic vector. Note that if (T_i) is hypercyclic then it is not equicontinuous, but the converse is false in general. In the case $I = \mathbb{N}$, it is clear that, in order that a *sequence* (T_n) can be hypercyclic, Y must be separable. If $T : X \to X$ is an operator ($=$ continuous linear selfmapping) on X, then a vector $x \in X$ is said to be *hypercyclic* for T if and only if it is hypercyclic for the sequence (T^n) of iterates of T, i.e., $T^n = T \circ T \circ \cdots \circ T$ (n–fold). The operator T is *hypercyclic* when there is a hypercyclic vector for T. The symbols $HC(T)$ and $HC((T_i))$ will denote, respectively, the set of hypercyclic vectors of an operator T and of a family $T_i : X \to Y$ ($i \in I$) of continuous

linear mappings. In the last two decades an extensive literature about the topic of hypercyclicity has been developed; a good survey for the whole history is [Gr1].

Let G be a nonempty open subset of $\mathbb{C}^N$ ($N \in \mathbb{N}$). We say that G is a domain when, in addition, it is connected. A domain $G \subset \mathbb{C}^N$ is said to be a *Runge domain* (see [Hor] or [Kra]) if and only if each holomorphic function on G can be uniformly approximated by polynomials on compact subsets of G. Note that, if $N = 1$, then G is a Runge domain if and only if it is simply connected. By $H(G)$ we denote, as usual, the Fréchet space of holomorphic functions on G, endowed with the compact-open topology. Recall that the family $\{V(K, \varepsilon) : \varepsilon > 0, K$ is a compact subset of $G\}$ is a neighbourhood basis for the origin in $H(G)$. Here $V(K, \varepsilon) := \{f \in H(G) : ||f||_K < \varepsilon\}$. For $A \subset \mathbb{C}^N$ we have denoted $||g||_A := \sup\{|g(z)| : z \in A\}$ whenever g is a complex function defined on the set A.

G. Godefroy and J.H. Shapiro [GoS, Section 5] proved in 1991 the following generalization of the classical approximation theorems by translates and derivatives of a single entire function due respectively to Birkhoff [Bir] and MacLane [Mac]: If T is an operator on the space $H(\mathbb{C}^N)$ of entire functions on $\mathbb{C}^N$ that commutes with each of the translation operators τ_a ($a \in \mathbb{C}^N$) given by $\tau_a f(z) = f(z + a)$ ($f \in H(\mathbb{C}^N)$, $z \in \mathbb{C}^N$), and is not a scalar multiple of the identity, then $HC(T)$ is a dense G_δ-subset of $H(\mathbb{C}^N)$; in addition, $HC(T)$ contains all nonzero functions of a dense, T–invariant, linear submanifold of $X := H(\mathbb{C}^N)$. P. Bourdon [Bou] and D. Herrero [Her] proved independently that every hypercyclic operator T on any Banach space X (in fact, on any real or complex locally convex space X; see [Ans] and [Bes]) has the same property. The first author of the present paper [Be4] has recently shown that if X and Y are two separable metrizable linear topological spaces and if $T_n : X \to Y$ ($n \in \mathbb{N}$) is a sequence of continuous linear mappings for which there is an increasing sequence (n_j) of positive integers with the property that $HC((T_{m_j}))$ is dense for every subsequence (m_j) of (n_j), then $HC((T_n)) \cup \{0\}$ contains a dense linear submanifold of X.

Given $N \in \mathbb{N}$, denote by D_j ($1 \le j \le N$) complex partial differentiation with respect to the j-th coordinate. A multi–index is an N–tuple $p = (p_1, ..., p_N)$ of nonnegative integers. Denote $|p| = p_1 + \cdots + p_N$, $p! = p_1! \cdots p_N!$, $D^p = D_1^{p_1} \circ \cdots \circ D_N^{p_N}$ (with $D_j^0 = I =$ the identity operator for every $j \in \{1, ..., N\}$), and $|z| = (|z_1|^2 + \cdots + |z_N|^2)^{1/2}$, $z^p = z_1^{p_1} \cdots z_N^{p_N}$, $zw = z_1 w_1 + \cdots + z_N w_N$ if $z = (z_1, ..., z_N)$, $w = (w_1, ..., w_N)$. An entire function $\Phi(z) = \sum_{|p| \ge 0} a_p z^p$ is said to be of exponential type whenever there exist positive constants A and B such that $|\Phi(z)| \le A e^{B|z|}$

$(z \in \mathbb{C}^N)$. For later references, we denote by $\mathcal{E}$ the class of all entire functions of exponential type. An entire function Φ is said to be of subexponential type if and only if, given $\varepsilon > 0$, there is a positive constant $A = A(\varepsilon)$ such that $|\Phi(z)| \leq Ae^{\varepsilon|z|}$ $(z \in \mathbb{C}^N)$. Every entire function of subexponential type is obviously in $\mathcal{E}$. It is easy to realize (see, for instance, [Val], [Dic] or [Be3]) that if $G \subset \mathbb{C}^N$ is a nonempty open subset and Φ is an entire function as above with subexponential type, then the series $\Phi(D) = \sum_{|p| \geq 0} a_p D^p$ defines an operator on $H(G)$. If $G = \mathbb{C}^N$, the same result holds just by assuming that Φ is of exponential type. So $\Phi(D)$ defines, under the latter conditions, an infinite order linear differential operator with constant coefficients. It is shown in [GoS] that, given an operator L on $H(\mathbb{C}^N)$, then L commutes with every translation operator τ_a $(a \in \mathbb{C}^N)$ if and only if L commutes with each D_k $(1 \leq k \leq N)$ if and only if $L = \Phi(D)$ for some entire function Φ in $\mathcal{E}$.

As a consequence of an eigenvalue criterion for hypercyclicity [Be3, Theorem 7], the first author obtained some extensions of Godefroy–Shapiro's result [Be3, Theorems 8–9], this time about the hypercyclicity of a *sequence* of operators $(\Phi_n(D))$ defined on the space of holomorphic functions on a Runge domain G of $\mathbb{C}^N$. Furthermore, conditions about the equicontinuity of a sequence $(c_n D^n)$, where $(c_n) \subset \mathbb{C}$ (note that this is the special case $\Phi_n(z) = c_n z^n$), are shown in [Be1] and [Be2] (see also [Cal], when each c_n is replaced to a holomorphic fuction $c_n(z)$).

Our aim in this paper is to provide with a more general eigenvalue criterion and, as a consequence, new sufficient conditions for the hypercyclicity of a sequence of infinite order linear differential operators. In addition, necessary conditions are established, and some special cases are analyzed. Necessary conditions and sufficient conditions for its equicontinuity are also furnished, and in particular we characterize completely the equicontinuity in $H(\mathbb{C}^N)$.

2   Eigenvalues, exponentials, hypercyclicity and equicontinuity.

Likewise in [GoS, Section 5] and [Be3, Theorems 8–9], the key of the proof of hypercyclicity is to provide a good supply of eigenvectors of the corresponding operators. Recall that, in a linear topological space, a subset is said to be total whenever its linear span is dense. If T is an operator and e is an eigenvector, then

we denote by $\lambda(T, e)$ its corresponding eigenvalue. Next, we state as a lemma the following rather general hypercyclicity criterion, which can be found in [Gr1].

Lemma 2.1 *Assume that X is a Baire topological vector space, Y is a separable metrizable topological vector space and $T_n : X \to Y$ $(n \in \mathbb{N})$ are continuous linear mappings. Suppose that there are dense subsets X_0 of X and Y_0 of Y and mappings $S_n : Y_0 \to X$ such that*

(a) for every $x \in X_0$, there exists an increasing sequence $(n_k) = \{n_1 < n_2 < ...\}$ of positive integers with $T_{n_k}x \to 0$ $(k \to \infty)$,

(b) for every $y \in Y_0$, (S_ny) converges, and

(c) for every $y \in Y_0$, $T_n(S_ny) \to y$ $(n \to \infty)$.

Then $HC((T_n))$ is residual.

As noted in [Gr1, Remark 2], if all the limits in (b) are zero then we may weaken (a) to be *for every $x \in X_0$, there exists an increasing sequence $(n_k) = \{n_1 < n_2 < ...\}$ of positive integers such that $(T_{n_k}x)$ converges.* Furthermore, the quantifier "$\exists(n_k)$" can be shifted from (a) to (b) or (c).

Under the same hypothesis for X and Y, it can be proved (see, for instance, [Be2]) that the following condition is also sufficient in order that $HC((T_n))$ be residual: there exist dense subsets X_0 of X and Y_0 of Y satisfying that for every $x \in X_0$ and every $y \in Y_0$ there exists an increasing sequence (n_k) of positive integers and a sequence $(x_k) \subset X$ such that $x_k \to 0$, $T_{n_k}x \to 0$ and $T_{n_k}x_{n_k} \to y$ as $k \to \infty$.

By using the latter result, the next eigenvalue criterion can be proved (see [Be3, Theorem 7]): Let X be a separable F–space and (T_n) a sequence of operators on X. Assume that there are two total subsets $\mathcal{A}$, $\mathcal{B}$ of X satisfying that for every pair of finite subsets $\mathcal{F}_1 \subset \mathcal{A}$ and $\mathcal{F}_2 \subset \mathcal{B}$ there is an increasing sequence (n_k) in $\mathbb{N}$ such that every element in $\mathcal{F}_1 \cup \mathcal{F}_2$ is an eigenvector for each T_{n_k} in such a way that $\lambda(T_{n_k}, a) \to 0$ $(k \to \infty)$ for all $a \in \mathcal{F}_1$ and $\lambda(T_{n_k}, b) \to \infty$ $(k \to \infty)$ for all $b \in \mathcal{F}_2$. Then $HC((T_n))$ is residual.

If we employ Lemma 2.1 (and the note after it) instead of the just mentioned result then the following eigenvalue criterion can be obtained. The proof is left to the interested reader.

Theorem 2.2 *Let X be a separable F–space and (T_n) be a sequence of operators on X. Assume that there are two total subsets $\mathcal{A}$, $\mathcal{B}$ of X satisfying at least one of the following conditions:*

(A) For every finite subset $\mathcal{F} \subset \mathcal{A}$ there is an increasing sequence (n_k) in $\mathbb{N}$ such that every element in $\mathcal{F}$ is an eigenvector for each T_{n_k} in such a way that $\lambda(T_{n_k}, a) \to 0$ $(k \to \infty)$ for all $a \in \mathcal{F}$. In addition, every element in $\mathcal{B}$ is an eigenvector for each T_n in such a way that for every $b \in \mathcal{B}$ the sequence $(\lambda(T_n, b))$ converges to a nonzero scalar.

(B) For every finite subset $\mathcal{F} \subset \mathcal{A}$ there is an increasing sequence (n_k) in $\mathbb{N}$ in such a way that for every $a \in \mathcal{F}$ the sequence $(\lambda(T_{n_k}, a))$ converges. In addition, every element in $\mathcal{B}$ is an eigenvector for each T_n in such a way that, for every $b \in \mathcal{B}$, $(\lambda(T_n, b)) \to \infty$ $(n \to \infty)$.

(C) Every element in $\mathcal{A}$ is an eigenvector for each T_n in such a way that $\lambda(T_n, a) \to 0$ $(n \to \infty)$ for every $a \in \mathcal{A}$. In addition, for every finite subset $\mathcal{F} \subset \mathcal{B}$ there is an increasing sequence (n_k) in $\mathbb{N}$ such that every element in $\mathcal{F}$ is an eigenvector for each T_{n_k} in such a way that for every $b \in \mathcal{F}$ the sequence $(\lambda(T_{n_k}, b))$ converges to a nonzero scalar.

(D) Every element in $\mathcal{A}$ is an eigenvector for each T_n in such a way that for every $a \in \mathcal{A}$ the sequence $(\lambda(T_n, a))$ converges. In addition, for every finite subset $\mathcal{F} \subset \mathcal{B}$ there is an increasing sequence (n_k) in $\mathbb{N}$ such that every element in $\mathcal{F}$ is an eigenvector for each T_{n_k} in such a way that $(\lambda(T_{n_k}, b)) \to \infty$ $(k \to \infty)$ for every $b \in \mathcal{F}$.

Then $HC((T_n))$ is residual.

In other order of ideas, recall that $\mathcal{E}$ denotes the class of entire functions on $\mathbb{C}^N$ of exponential type. We say that a subset $S \subset \mathbb{C}^N$ is an $\mathcal{E}$–*unicity set* whenever the following property holds: if $f \in \mathcal{E}$ and $f(z) = 0$ for all $z \in S$ then $f \equiv 0$. Note that, by the identity principle for holomorphic functions, if f is an *arbitrary* entire function vanishing at S and S is a nonempty open set (or even just a set with at least an accumulation point if $N = 1$) then $f \equiv 0$. This is not necessary for the class $\mathcal{E}$; for instance, if $N = 1$ and $\chi := \limsup_{r \to \infty} \frac{\log n(r)}{\log r} > 1$, where $n(r)$ is the number of points of $S \cap \{|z| \le r\}$, then S is an $\mathcal{E}$–unicity set (e.g., $S = \{n^{1/2} : n \in \mathbb{N}\}$, which gives $\chi = 2$). Indeed, if $f \not\equiv 0$, the latter condition

would imply that the convergence exponent of the sequence of zeros of f is strictly greater that the growth order of f, which is clearly impossible. The next lemma will be useful later. Its proof is classical, but we include it for the sake of completeness. If $c \in \mathbb{C}^N$ then we denote $e_c(z) = \exp(cz)$.

Lemma 2.3 *If S is an $\mathcal{E}$–unicity set then $M(S) := \{e_c : c \in S\}$ is total in $H(\mathbb{C}^N)$.*

   Proof. Fix a functional $L \in H(\mathbb{C}^N)^*$ ($=$ the topological dual space of $H(\mathbb{C}^N)$) such that $L(e_c) = 0$ for all $c \in S$. Consider the Laplace transform $\tilde{L}$ of L (see [Hor, p. 100]) given by $\tilde{L}(z) = L(e_z)$ ($z \in \mathbb{C}^N$). Then it is easy to show that $\tilde{L}$ is an entire function on $\mathbb{C}^N$ of exponential type which vanishes at S. Since S is an $\mathcal{E}$–unicity set, we get $\tilde{L} \equiv 0$. Then $(D^p \tilde{L})(0) = 0$ for all $p \in \mathbb{N}_0^N$. But it is easy to show by induction that $(D^p \tilde{L})(0) = L(\alpha_p)$, where $\alpha_p(t) = t^p$ ($t \in \mathbb{C}^N$). By linearity, L vanishes at every polynomial, so $L \equiv 0$ because the set of polynomials is dense in $H(\mathbb{C}^N)$. Summarizingly, if $L(f) = 0$ for all $f \in M(S)$ then $L(f) = 0$ for all $f \in H(\mathbb{C}^N)$. By the Hahn–Banach theorem, the linear span of $M(S)$ is dense in $H(\mathbb{C}^N)$ or, equivalently, $M(S)$ is total. ∎

   Next, we state here eight conditions that may or may not be satisfied by a sequence $(\Phi_n) \subset H(\mathbb{C}^N)$. Recall that if $\Phi(z) = \sum_{|p| \geq 0} a_p z^p \in H(\mathbb{C}^N)$ and Φ is not identically zero, its multiplicity for the zero at the origin is $m(\Phi) = \min\{|p| : a_p \neq 0\}$. Note that $\Phi(D)e_c = \Phi(c)e_c$ for all $c \in \mathbb{C}^N$, so e_c is an eigenvector of $\Phi(D)$ with eigenvalue $\Phi(c)$.

(P) There are two $\mathcal{E}$–unicity sets A, B in $\mathbb{C}^N$ such that for every pair of finite subsets $F_1 \subset A$ and $F_2 \subset B$ there exists an increasing sequence $(n_k) \subset \mathbb{N}$ with $\Phi_{n_k}(a) \to 0$ ($k \to \infty$) for all $a \in F_1$ and $\Phi_{n_k}(b) \to \infty$ ($k \to \infty$) for all $b \in F_2$.

(Q) There is an $\mathcal{E}$–unicity set B in $\mathbb{C}^N$ such that for every finite subset $F \subset B$ there exists an increasing sequence $(n_k) \subset \mathbb{N}$ with $m(\Phi_{n_k}) \to \infty$ ($k \to \infty$) and $\Phi_{n_k}(b) \to \infty$ ($k \to \infty$) for all $b \in F$.

(R) There are two $\mathcal{E}$–unicity sets A, B in $\mathbb{C}^N$ such that for every finite subset $F \subset A$ there exists an increasing sequence $(n_k) \subset \mathbb{N}$ with $\Phi_{n_k}(a) \to 0$ ($k \to \infty$) for all $a \in F$, and for each $b \in B$ the sequence $(\Phi_n(b))$ converges to a nonzero complex number.

(S) There is an $\mathcal{E}$–unicity set B in $\mathbb{C}^N$ such that for each $b \in B$ the sequence $(\Phi_n(b))$ converges to a nonzero complex number, and there exists an increasing sequence $(n_k) \subset \mathbb{N}$ with $m(\Phi_{n_k}) \to \infty$ $(k \to \infty)$.

(T) There are two $\mathcal{E}$–unicity sets A, B in $\mathbb{C}^N$ such that for every finite subset $F \subset A$ there exists an increasing sequence $(n_k) \subset \mathbb{N}$ satisfying that for every $a \in F$ the sequence $(\Phi_{n_k}(a))$ converges. In addition, $\Phi_n(b) \to \infty$ $(n \to \infty)$ for every $b \in B$.

(U) There are two $\mathcal{E}$–unicity sets A, B in $\mathbb{C}^N$ such that $\Phi_n(a) \to 0$ $(n \to \infty)$ for all $a \in A$, and for each finite subset $F \subset B$ there exists an increasing sequence $(n_k) \subset \mathbb{N}$ satisfying that for every $b \in F$ the sequence $(\Phi_{n_k}(b))$ converges to a nonzero complex number.

(V) There is an $\mathcal{E}$–unicity set B in $\mathbb{C}^N$ such that for each finite subset $F \subset B$ there exists an increasing sequence $(n_k) \subset \mathbb{N}$ satisfying that for every $b \in F$ the sequence $(\Phi_{n_k}(b))$ converges to a nonzero complex number. In addition, $m(\Phi_n) \to \infty$ $(n \to \infty)$.

(W) There are two $\mathcal{E}$–unicity sets A, B in $\mathbb{C}^N$ such that for every $a \in A$ the sequence $(\Phi_n(a))$ converges, and for every finite subset $F \subset B$ there is an increasing sequence $(n_k) \subset \mathbb{N}$ with $\Phi_{n_k}(b) \to \infty$ $(k \to \infty)$ for all $b \in F$.

We are now ready to state our next result. In the remaining of this paper, Φ and Φ_i $(i \in I :=$ an arbitrary index set$)$ will denote entire functions of subexponential type if $G \ne C^N$ and of exponential type if $G = \mathbb{C}^N$, G being a given domain in $\mathbb{C}^N$. Thus, the operators $\Phi(D)$, $\Phi_i(D)$ $(i \in I)$ are well defined on $H(G)$.

Theorem 2.4 *Suppose that G is a Runge domain of $\mathbb{C}^N$ and that (Φ_n) satisfies at least one of the conditions (P)–(W). Then $HC((\Phi_n(D)))$ is residual in $H(G)$.*

Proof. Recall that, by Lemma 2.3, the set $M(S)$ is total in $H(\mathbb{C}^N)$ (hence in $H(G)$, because G is Runge) whenever S is an $\mathcal{E}$–unicity set. Recall also that the set $\{z^p : p \in \mathbb{N}_o^N\}$ is total in $H(G)$, because that set spans $\{$polynomials$\}$. Take $X = H(G)$ and $T_n = \Phi_n(D)$ $(n \in \mathbb{N})$. Then: Apply the result mentioned just before Theorem 2.2 on $\mathcal{A} = M(A)$, $\mathcal{B} = M(B)$ if (Φ_n) satisfies (P), and on $\mathcal{A} = \{z^p : p \in \mathbb{N}_0^N\}$, $\mathcal{B} = M(B)$ if (Φ_n) satisfies (Q). Apply condition (A) of Theorem 2.2 on $\mathcal{A} = M(A)$, $\mathcal{B} = M(B)$ if (Φ_n) satisfies (R), and on $\mathcal{A} = \{z^p : p \in \mathbb{N}_0^N\}$, $\mathcal{B} = M(B)$ if (Φ_n) satisfies (S). Apply condition (B) of Theorem 2.2

on $\mathcal{A} = M(A)$, $\mathcal{B} = M(B)$ if (Φ_n) satisfies (T). Apply condition (C) of Theorem 2.2 on $\mathcal{A} = M(A)$, $\mathcal{B} = M(B)$ if (Φ_n) satisfies (U), and on $\mathcal{A} = \{z^p : p \in \mathbb{N}_0^N\}$, $\mathcal{B} = M(B)$ if (Φ_n) satisfies (V). Finally, apply condition (D) of Theorem 2.2 on $\mathcal{A} = M(A)$, $\mathcal{B} = M(B)$ if (Φ_n) satisfies (W).

Let us furnish several examples that illustrate Theorem 2.4. The reader will realize that none of the examples below can be derived from Theorems 8, 9 of [Be3]. But before this we should fix some subsets. Consider $S = \{n^{1/2} : n \in \mathbb{N}\}$ and let (r_j) be any sequence of positive real numbers such that the plane disks $\{|z - j^{1/2}| < r_j\}$ $(j \in \mathbb{N})$ be pairwise disjoint, for instance, $r_j = 1/6j$. Define the compacts sets $K_n := (L_n \cup S) \cap I_n$ $(n \in \mathbb{N})$, where

$$I_n := [-n, n] \times [-n, n]$$

and

$$L_n := \mathbb{C} \setminus [((0, +\infty) \times (-1/n, 0)) \cup \bigcup_{j=1}^{\infty} \{|z - j^{1/2}| < r_j/n\}].$$

It is easy to see that each K_n has connected complement. Define the functions $f_n, g_n : K_n \to \mathbb{C}$ $(n \in \mathbb{N})$ as

$$f_n(z) = \begin{cases} 1 & (z \in L_n \cap I_n) \\ n & (z \in S \cap I_n) \end{cases}$$

and

$$g_n(z) = \begin{cases} 1 & (z \in L_n \cap I_n) \\ 0 & (z \in S \cap I_n). \end{cases}$$

It is clear that every f_n and every g_n is holomorphic on some open subset containing K_n and depending on n. Then Runge's theorem guarantees the existence of polynomials P_n, Q_n satisfying

$$||P_n - f_n||_{K_n} < 1/n \quad \text{and} \quad ||Q_n - g_n||_{K_n} < 1/n \quad (n \in \mathbb{N}).$$

Since $L_n \cap I_n$ $(S \cap I_n)$ grows up to $\mathbb{C} \setminus S$ (up to S, respectively) as n tends to infinity, the latter two inequalities lead us to the following facts of point convergence: $P_n \to 1$ on $\mathbb{C} \setminus S$, $P_n \to \infty$ on S, $Q_n \to 1$ on $\mathbb{C} \setminus S$ and $Q_n \to 0$ on S as $n \to \infty$. EXAMPLE 1. There is a residual set of entire functions f on $\mathbb{C}$ such that each entire function can be locally uniformly approximated by entire functions of the form

$$\sum_{j=0}^{n} A_{jn} f^{(j)} \quad (n \in \mathbb{N}),$$

where $A_{nn} = 1$ and

$$A_{jn} = (-1)^{n-j} \sum_{1 \leq i_1 < i_2 < \cdots < i_{n-j} \leq n} (i_1 \cdots i_{n-j})^{1/2} \quad (0 \leq j \leq n-1).$$

Indeed, it suffices to apply the latter theorem with condition (P) or (T) on $A = S$, $B = \mathbb{C} \setminus S$, $\Phi_n(z) = \prod_{j=1}^{n} (z - j^{1/2})$ ($n \in \mathbb{N}$) (use Cardano–Vieta's relations).

EXAMPLE 2. The set $HC((P_n(D)))$ is residual in $H(\mathbb{C})$ because Theorem 2.4 can be applied with condition (T) or (W) on $A = \mathbb{C} \setminus S$, $B = S$.

EXAMPLE 3. The set $HC((Q_n(D)))$ is residual in $H(\mathbb{C})$ because Theorem 2.4 can be applied with condition (R) or (U) on $A = S$, $B = \mathbb{C} \setminus S$.

Analogous properties to (P)–(W) regarding the densely hereditary hypercyclicity of $(\Phi_n(D))$ can be formulated as in [Be4, Section 3]. This would yield sufficient conditions for the existence of dense $(\Phi_n(D))$–hypercyclic linear submanifolds in $H(G)$.

In his paper, Birkhoff [Bir] essentially proved that given an unbounded sequence $(a_n) \subset \mathbb{C}$ there exists an entire function in $\mathbb{C}$ such that the set of translates $\{f(z + a_n) : n \in \mathbb{N}\}$ is dense in $H(\mathbb{C})$, i.e., the sequence (τ_{a_n}) is hypercyclic (as a matter of fact, the sequence (a_n) depended on the particular entire function to be approximated; in [Luh] this dependence is dropped). His constructive proof can be adapted to $\mathbb{C}^N$: see, for instance, [Abe] and [AbZ]; see also [ArG] for corresponding results for harmonic functions on $\mathbb{R}^N$. As a quick application of the latter theorem, we will obtain this Birkhoff theorem in several variables.

Theorem 2.5 *Assume that* $S \subset \mathbb{C}^N$. *Then the following conditions are equivalent:*

(a) S *is unbounded.*

(b) The family of operators $(\tau_a)_{a \in S}$ *is hypercyclic on* $H(\mathbb{C}^N)$.

(c) $HC((\tau_a)_{a \in S})$ *is residual in* $H(\mathbb{C}^N)$.

(d) $(\tau_a)_{a \in S}$ *is not equicontinuous on* $H(\mathbb{C}^N)$.

Proof. The implications (c) $\Rightarrow$ (b) $\Rightarrow$ (d) are trivial. If S is bounded, take $M \in (0, +\infty)$ with $|a| \leq M$ for all $a \in S$. Given a basic neighbourhood $V(K, \varepsilon)$ for the origin in $H(\mathbb{C}^N)$, it is clear that

$$\bigcup_{a \in S} \tau_a(V(L, \delta)) \subset V(K, \varepsilon),$$

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where $\delta = \varepsilon$ and $L = \{z + w : z \in K, |w| \leq M\}$. Hence $(\tau_a)_{a \in S}$ is equicontinuous, so (d) implies (a).

As for (a) $\Rightarrow$ (c), we will try to apply Theorem 2.4 under the condition (P) (it is also possible to use Theorem 8 of [Be3]). By hypothesis, there is a sequence $(a_n) \subset S$ with $a_n \to \infty$ $(n \to \infty)$. Assume that $a_n = (a_{n1}, ..., a_{nN})$ and $b_{nj} = \mathrm{Re}\, a_{nj}$, $c_{nj} = \mathrm{Im}\, a_{nj}$ $(j = 1, ..., N; n \in \mathbb{N})$. By taking a subsequence if necessary, and possibly by using a permutation of the variables $z_1, ..., z_N$ together with a rotation on the variable z_1 (the latter two operations generate fixed automorphisms of $H(\mathbb{C}^N)$ which preserve hypercyclicity), we can suppose without loss of generality that $b_{n1} \to \infty$ $(n \to \infty)$ and that there is a $2N$–tuple $(\varepsilon_1, \delta_1, ..., \varepsilon_N, \delta_N) \in \{0, 1\}^{2N}$ such that $(a_n) \subset \Pi(\varepsilon_1, \delta_1, ..., \varepsilon_N, \delta_N) := \{z = (b_1 + ic_1, ..., b_N + ic_N) \in \mathbb{C}^N :$ $(-1)^{\varepsilon_1} b_1 \geq 0, (-1)^{\delta_1} c_1 \geq 0, ..., (-1)^{\varepsilon_N} b_N \geq 0, (-1)^{\delta_N} c_N \geq 0\}$. Take

$$A = \mathrm{int}\, \Pi(1 - \varepsilon_1, \delta_1, 1 - \varepsilon_2, \delta_2, ..., 1 - \varepsilon_N, \delta_N)$$

and

$$B = \mathrm{int}\, \Pi(\varepsilon_1, 1 - \delta_1, \varepsilon_2, 1 - \delta_2, ..., \varepsilon_N, 1 - \delta_N),$$

where "int" denotes the interior of the corresponding set. Trivially, A and B are nonempty open subsets of $\mathbb{C}^N$. Now, note that $\tau_{a_n} = \Phi_n(D)$, where $\Phi_n(z) = e^{a_n z}$ $(n \in \mathbb{N})$. For any $z = (z_1 = x_1 + iy_1, ..., z_N = x_N + iy_N) \in \mathbb{C}^N$ we have that

$$|\Phi_n(z)| = \exp\Big(\sum_{j=1}^{N}(b_{nj}x_j - c_{nj}y_j)\Big)$$

$$= \exp\big(b_{n1}x_1\big) \cdot \exp\Big(\sum_{j=2}^{N} b_{nj}x_j - \sum_{j=1}^{N} c_{nj}y_j\Big).$$

Observe that $b_{n1}x_1 \to +\infty$ $(n \to \infty)$ and $\sum_{j=2}^{N} b_{nj}x_j - \sum_{j=1}^{N} c_{nj}y_j \geq 0$ for all $z \in B$, and that $b_{n1}x_1 \to -\infty$ $(n \to \infty)$ and $\sum_{j=2}^{N} b_{nj}x_j - \sum_{j=1}^{N} c_{nj}y_j \leq 0$ for all $z \in A$. Thus, $\Phi_n \to 0$ $(n \to \infty)$ pointwise on A and $\Phi_n \to \infty$ $(n \to \infty)$ pointwise on B. This finishes the proof because if $HC((\tau_{a_n}))$ is residual then, trivially, $HC((\tau_a)_{a \in S})$ is residual.

Note that for the case $S = \{na : n \in \mathbb{N}\}$ $(a \in \mathbb{C}^N \setminus \{0\}$ fixed), the property (c) is derived from [GoS, Section 5]. This property is also obtained for an unbounded sequence $S = \{a_n : n \in \mathbb{N}\}$ in the case $N = 1$ by a different universality proof in [GeS].

A slight improvement of Godefroy–Shapiro's theorem is possible as application of Theorem 2.4 by introducing a multiplicative complex sequence. Note that the

condition on (c_n) in the next result implies that $((n!|c_n|)^{1/n})$ is unbounded (compare to Theorem 2.12), and that Godefroy–Shapiro's result is the special case $G = \mathbb{C}^N$, $c_n \equiv 1$ under condition (b).

Theorem 2.6 *Let (c_n) be a complex sequence. Assume that G is a Runge domain in $\mathbb{C}^N$ and that Φ is nonconstant. Suppose that at least one of the following properties is satisfied:*

(a) $(|c_n|^{1/n})$ does not converge to zero and $\Phi(0) = 0$.

(b) $0 < \liminf_{n\to\infty} |c_n|^{1/n} \leq \limsup_{n\to\infty} |c_n|^{1/n} < +\infty$.

Then the set $HC((c_n\Phi^n(D)))$ is residual in $H(G)$.

Proof. Consider the sequence of entire functions $\Phi_n(z) = c_n\Phi(z)^n$ ($n \in \mathbb{N}$; $z \in \mathbb{C}$). Under the hypotheses of (a), there are $M \in (0, +\infty)$ and an increasing sequence (n_k) of positive integers with $|c_{n_k}| \geq M^{n_k}$ ($k \in \mathbb{N}$). Since Φ is nonconstant, there is a nonempty open subset $B \subset \mathbb{C}^N$ on which $|\Phi(z)| > 2/M$, hence

$$|\Phi_{n_k}(z)| > 2^{n_k} \to \infty \quad (k \to \infty)$$

for all $z \in B$. Moreover $m(\Phi_{n_k}) = n_k \cdot m(\Phi) \to \infty$ ($k \to \infty$) because $m(\Phi) > 0$. Consequently, Theorem 2.4 can be applied because condition (Q) is satisfied, and we are done. Under the hypothesis of (b), we have that

$$M^{n_k} \leq |c_{n_k}| \leq M_1^{n_k} \quad (k \in \mathbb{N})$$

for some positive finite constants M, M_1 and some increasing sequence $(n_k) \subset \mathbb{N}$. Choose the set B as in the first case. Choose also a nonempty open subset $A \subset \mathbb{C}^N$ on which $|\Phi(z)| < 1/2M_1$. Therefore

$$|\Phi_{n_k}(z)| \leq 1/2^{n_k} \to 0 \quad (k \to \infty)$$

for all $z \in A$. Condition (P) in Theorem 2.4 is satisfied this time, and the proof is finished.

Additional sufficient conditions for hypercyclicity will be furnished later (see Theorems 2.12, 2.13, 2.15). Next, we state a necessary condition for the hypercyclicity of $(\Phi_n(D))$. We need some notation first. If $a = (a_1, ..., a_N) \in \mathbb{C}^N$ and $r > 0$, we denote by $D(a, r)$ the closed polydisk with center a and radius r, i.e., $D(a, r) = \{z \in \mathbb{C}^N : |z_j - a_j| \leq r, 1 \leq j \leq N\}$. We consider the distance

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$d(z, a) = \max\{|z_1 - a_1|, ..., |z_N - a_N|\}$ $(z, a \in \mathbb{C}^N)$. The inscribed radius of G is $\rho(G) = \sup_{b \in G} \inf_{a \notin G} d(a, b) = \sup\{r > 0 :$ there is a polydisk D of radius r with $D \subset G\}$. For future references we point out here that the circumscribed radius of G is defined as $R(G) = \inf_{a \in \mathbb{C}^N} \sup_{b \in G} d(a, b) = \inf\{r > 0 :$ there is a polydisc D of radius r with $G \subset D\}$. For each sequence $(\Phi_n(z) = \sum_{|p| \geq 0} c_{pn} z^p) \subset H(\mathbb{C}^N)$, where each Φ_n is of exponential type, we associate the number in $[0, +\infty]$ given by

$$\alpha = \alpha((\Phi_n)) := \limsup_{n \to \infty} \big(\sup_{|p| > 0} (p! \cdot |c_{pn}|)^{1/|p|} \big).$$

Note that for each $n \in \mathbb{N}$ the number $\sup_{|p| > 0} (p! \cdot |c_{pn}|)^{1/|p|}$ is finite.

Theorem 2.7 *Suppose that $G \subset \mathbb{C}^N$ is a domain. Assume that $\Phi_n(z) = \sum_{|p| \geq 0} c_{pn} z^p$ $(n \in \mathbb{N})$ are entire functions such that the sequence $(\Phi_n(0))$ is bounded. If the sequence $(\Phi_n(D))$ is hypercyclic on $H(G)$ then $\rho(G) \leq \alpha$.*

Proof. For each $n \in \mathbb{N}$ denote $K_n = \sup_{|p| > 0} (p! \cdot |c_{pn}|)^{1/|p|}$, in such a way that $\alpha = \limsup_{n \to \infty} K_n$. Let $\beta \in (0, +\infty)$ with $|c_{0n}| = |\Phi_n(0)| \leq \beta$ $(n \in \mathbb{N})$. Assume, by the way of contradiction, that $\limsup_{n \to \infty} K_n < \rho(G)$ and that $f \in H(G)$ is hypercyclic for $(\Phi_n(D))$. Fix positive real numbers r, R with $\limsup_{n \to \infty} K_n < r < R < \rho(G)$. Then there exists a polydisk $D(a, R) \subset G$. By Cauchy's inequalities (see [Hor, Theorem 2.2.7]) we get

$$|(D^p f)(a)| \leq p! \cdot \frac{||f||_{D(a,R)}}{R^{|p|}} \quad (|p| \geq 0).$$

In addition, there is $m \in \mathbb{N}$ with $K_n \leq r$ for all $n \geq m$. Therefore

$$|c_{pn}| \leq \frac{r^{|p|}}{p!} \quad (|p| > 0; \ n \geq m),$$

hence

$$|(\Phi_n(D)f)(a)| = |\sum_{|p| \geq 0} c_{pn} D^p f(a)|$$

$$= |c_{0n} f(a) + \sum_{|p| > 0} c_{pn} D^p f(a)| \leq \beta |f(a)| + \sum_{|p| > 0} \frac{r^{|p|}}{p!} \cdot p! \cdot \frac{||f||_{D(a,R)}}{R^{|p|}}$$

$$\leq ||f||_{D(a,R)} \cdot (\beta + \sum_{|p| \geq 0} (r/R)^{|p|}) = ||f||_{D(a,R)} \cdot (\beta + (\frac{R}{R-r})^N),$$

for every $n \geq m$. Consequently, the sequence $\{(\Phi_n(D)f)(a) : n \in \mathbb{N}\}$ is bounded, which is absurd. The proof is finished.

Observe that the latter theorem extends Theorem 2 in [Be2], which asserted that if G is domain in $\mathbb{C}$ and $(c_n D^n)$ is hypercyclic in $H(G)$ then $\limsup_{n\to\infty} (n!|c_n|)^{1/n} \geq \rho(G)$; note that this is just the case $N = 1$, $\Phi_n(z) = c_n z^n$. Observe also that Theorem 2.7 implies in particular that if $(\Phi_n(D))$ is hypercyclic in $H(\mathbb{C}^N)$ then either $\{\Phi_n(0) : n \in \mathbb{N}\}$ is unbounded or $\{\sup_{|p|>0} (p!|c_{pn}|)^{1/|p|} : n \in \mathbb{N}\}$ is unbounded.

A corresponding sufficient condition for equicontinuity can be formulated, but the sequence $(\Phi_n(D))$ may be replaced to a general family $\{\Phi_i(D) : i \in I\}$ of differential operators. This will be achieved in Theorem 2.9. Before this, we need a definition and an auxiliary statement. A *polydomain* in $\mathbb{C}^N$ is a product $G = G_1 \times \cdots \times G_N$ of domains in $\mathbb{C}$. The following lemma is a generalization of Theorem 13.5 in [Rud]. Its proof can be made by induction and it is left to the reader.

Lemma 2.8 *If G is a polydomain in $\mathbb{C}^N$ and $K \subset G$ is a compact subset, then there are cycles $\gamma_1, ..., \gamma_N$ with $\gamma_1 \times \cdots \times \gamma_N \subset G \setminus K$ such that for every $f \in H(G)$, every $p = (p_1, ..., p_N) \in \mathbb{N}_0^N$ and every $z = (z_1, ..., z_N) \in K$, the following Cauchy formula holds:*

$$D^p f(z) = \frac{p!}{(2\pi i)^N} \oint_{\gamma_1} \oint_{\gamma_2} \cdots \oint_{\gamma_N} \frac{f(t_1, ..., t_N)}{\prod_{j=1}^N (t_j - z_j)^{1+p_j}} \, dt_1 \ldots dt_N.$$

Recall the following well-known characterizations (see, for instance, [Boa]): an entire function $\Phi(z) = \sum_{|p|\geq 0} C_p z^p$ is of exponential type if and only if $\sup_{|p|>0} (p!|C_p|)^{1/|p|}$ is finite, and it is of subexponential type if and only if $\lim_{|p|\to\infty} (p!|C_p|)^{1/|p|} = 0$.

Theorem 2.9 *Suppose that G is a domain in $\mathbb{C}^N$ and that $\{\Phi_i(z) = \sum_{|p|\geq 0} c_{pi} z^p : i \in I\}$ is a family of entire functions. Then the following statements hold:*

(a) *If G is a polydomain and there is a majorant entire function $\Phi(z) = \sum_{|p|\geq 0} C_p z^p$ for the family (Φ_i) (i.e., $C_p \geq 0$ and $|c_{pi}| \leq C_p$ for all $p \in \mathbb{N}_0^N$ and all $i \in I$) with subexponential type then the family of operators $(\Phi_i(D))$ is equicontinuous on $H(G)$.*

(b) *If $(\Phi_i(D))$ is equicontinuous on $H(G)$ then the sequence (Φ_i) admits a majorant entire function with exponential type.*

Proof. By the remark just above this theorem, we have that (Φ_i) admits a majorant entire function of subexponential (exponential) type if and only if $\{c_{pi} :$

$i \in I\}$ is bounded for each p and $\lim_{|p| \to \infty} (p! \sup_i |c_{pi}|)^{1/|p|} = 0$ (if and only if $\{c_{0i} : i \in I\}$ is bounded and $\sup_{|p|>0, i \in I} (p!|c_{pi}|)^{1/|p|}$ is finite, respectively). In this proof we denote $T_i = \Phi_i(D)$ $(i \in I)$.

Let us prove (a). By hypothesis, $G = G_1 \times \cdots \times G_N$, where each G_j is a domain in $\mathbb{C}$. Fix $\varepsilon > 0$ and a compact subset $K \subset G$. For the corresponding basic neighbourhood $V(\varepsilon, K)$ of the origin in $H(G)$ we must find $\delta > 0$ and a compact subset $L \subset G$ satisfying

$$\bigcup_{i \in I} T_i(V(\delta, L)) \subset V(\varepsilon, K). \tag{1}$$

Lemma 2.8 allows us to choose a "polycycle" $\gamma = \gamma_1 \times \cdots \times \gamma_N \subset G \backslash K$ such that the Cauchy formula of its statement holds for $p \in \mathbb{N}_0^N$, $z \in K$ and $f \in H(G)$. Since G is a polydomain we may suppose without loss of generality that $K = K_1 \times \cdots \times K_N$, where each K_j is a compact subset of G_j. Let us set

$$\mu := \inf\{|t_j - z_j| : t_j \in \gamma_j; z_j \in K_j; j = 1, ..., N\} > 0.$$

By hypothesis, $\lim_{|p| \to \infty} (p! \sup_{i \in I} |c_{pi}|)^{1/|p|} = 0$, so there is $m \in \mathbb{N}$ with $(p!|c_{pi}|)^{1/|p|} \leq \mu/2$ for all $i \in I$ and all p with $|p| > m$. But each family $\{c_{pi} : i \in I\}$ $(|p| \leq m)$ is bounded, so there is a positive finite constant M such that

$$\frac{p!|c_{pi}|2^{|p|}}{\mu^{|p|}} \leq M \quad (i \in I, |p| \geq 0).$$

Choose $L = \gamma$ and

$$\delta = \frac{(\pi\mu)^N \cdot \varepsilon}{M \cdot \prod_{j=1}^N \text{length}(\gamma_j)}.$$

If $i \in I$, $f \in V(\delta, L)$ and $z \in K$ then we have

$$|(T_i f)(z)| = |\sum_{|p| \geq 0} c_{pi} D^p f(z)|$$

$$= |\sum_{|p| \geq 0} \frac{p! c_{pi}}{(2\pi i)^N} \oint_{\gamma_1} \cdots \oint_{\gamma_N} \frac{f(t_1, ..., t_N)}{\prod_{j=1}^N (t_j - z_j)^{1+p_j}} dt_1 \ldots dt_N|$$

$$\leq \sum_{|p| \geq 0} \frac{p!|c_{pi}|}{(2\pi)^N} \cdot \frac{\delta}{\mu^{N+|p|}} \cdot \prod_{j=1}^N \text{lenght}(\gamma_j)$$

$$\leq \frac{\varepsilon}{2^N} \cdot \sum_{|p| \geq 0} (1/2)^{|p|} = \frac{\varepsilon}{2^N} \cdot (\sum_{k=0}^\infty 1/2^k)^N = \varepsilon.$$

This proves (1), as required.

Now, we prove (b). Since (T_i) is equicontinuous we can find $\delta > 0$ and a compact subset $L \subset G$ such that

$$\bigcup_{i \in I} T_i(V(L, \delta)) \subset V(\{a\}, 1), \tag{2}$$

where a is any fixed point of G. Consider the family of monomials f_p $(p \in \mathbb{N}_0^N)$ given by

$$f_p(z) = \delta \cdot (\frac{z - a}{R})^p,$$

where $R = \sup\{|t - a| : t \in L\}$. Since L can be chosen distinct from $\{a\}$, we have that $0 < R < +\infty$. Then $f_p \in V(\delta, L)$ for all p, whence $T_i f_p \in V(1, \{a\})$ by (2), that is,

$$|(\Phi_i(D) f_p)(a)| \leq 1 \quad (i \in I, \, p \in \mathbb{N}_0^N).$$

But $(\Phi_i(D) f_p)(a) = \sum_{|q| \geq 0} c_{qi} D^q f_p(a) = c_{pi} \delta \cdot p! \cdot \frac{1}{R^{|p|}}$, because $D^q f_p(z) \equiv 0$ if $|q| \geq |p|$ with $q \neq p$ and $D^q f_p(a) = 0$ if $|q| < |p|$. Hence $|c_{pi} \delta \cdot p! \cdot R^{-|p|}| \leq 1$, whence

$$|c_{0i}| \leq 1/\delta \quad (i \in I)$$

and

$$\sup_{|p| > 0, \, i \in I} (p! |c_{pi}|)^{1/|p|} \leq R \cdot \sup_{|p| > 0} (1/\delta)^{1/|p|} < +\infty.$$

The latter two inequalities show that (Φ_i) admits a majorant entire function of exponential type, as required. The proof is finished.

In the special case of a sequence (Φ_n) we get a generalization of the part "if" of [Be2, Theorem 3], because we consider the number $\alpha = \alpha((\Phi_n))$ defined before Theorem 2.7. Theorem 3 of [Be2] asserted that if G is a domain in $\mathbb{C}$ and if $\limsup_{n \to \infty}(n! |c_n|)^{1/n} = 0$ then the family $(c_n D^n)$ is equicontinuous on $H(G)$, and the converse is true under the assumption that $G \neq \mathbb{C}$. Observe that its part "if" is again the particular case $\Phi_n(z) = c_n z^n$ of the next result.

Corollary 2.10 *Assume that $G \subset \mathbb{C}^N$ is a polydomain, that the sequence $(\Phi_n(0))$ is bounded and that $\alpha = 0$. Then $(\Phi_n(D))$ is equicontinuous on $H(G)$.*

Proof. Let us suppose that $\Phi_n(z) = \sum_{|p| \geq 0} c_{pn} z^p$ $(n \in \mathbb{N})$. Then $\{c_{0n} : n \in \mathbb{N}\}$ is bounded and $\lim_{n \to \infty} \sup_{|p| > 0}(p! |c_{pn}|)^{1/|p|} = 0$, from which it is easily derived that $\{c_{pn} : n \in \mathbb{N}\}$ is bounded for each multi–index p and that $\lim_{|p| \to \infty}(p! \sup_n |c_{pn}|)^{1/|p|} = 0$ (for this, use the fact that for each fixed n one has $(p! |c_{pn}|)^{1/|p|} \to 0$ as $|p| \to \infty$). But this is to say that (Φ_n) has a majorant

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entire function with subexponential type, so part (a) of Theorem 2.9 yields the desired result.

For $G = \mathbb{C}^N$ we are able to characterize the equicontinuous families of differential operators.

Theorem 2.11 *The family of operators $\{\Phi_i(D) : i \in I\}$ is equicontinuous on $H(\mathbb{C}^N)$ if and only if (Φ_i) admits a majorant entire function of exponential type.*

Proof. The part "only if" is due to Theorem 2.9(b). As for the converse, we can follow step by step the proof of part (a) of Theorem 2.9 with the sole exception that we may choose the polycycle γ far enough from the compact set K (so μ can be choosen as large as desired) in such a way that

$$\sup_{|p|>0,\, i \in I} (p! \, |c_{pi}|)^{1/|p|} \leq \mu/2.$$

The constant M may be choosen as $M = \max\{1, \sup_{i \in I} |c_{0i}|\}$. The proof is finished.

In [Be2, Theorem 1] it has been established that if $G \subset \mathbb{C}$ is a simply connected domain and (c_n) is a complex sequence with $R(G) \leq \limsup_{n \to \infty} (n! \, |c_n|)^{1/n}$ then $HC((c_n D^n))$ is residual in $H(G)$. A slight generalization can be obtained in the N–dimensional case. The proof is very similar to the 1–dimensional one, so we omit it.

Theorem 2.12 *Assume that $G \subset \mathbb{C}^N$ is a Runge domain and that $(p(n))$ is a sequence of multi–indexes with $|p(n)| \to \infty$ $(n \to \infty)$. If (c_n) is a complex sequence with*

$$R(G) \leq \limsup_{n \to \infty} (p(n)! \, |c_n|)^{1/|p(n)|}$$

then the set $HC((c_n D^{p(n)}))$ is residual in $H(G)$.

As a consequence of Theorems 2.11, 2.12 we can get a characterization of equicontinuity and hypercyclicity of the same sequence in $H(\mathbb{C}^N)$. This is achieved in the next result, which in turn is an N–dimensional extension of [Be2, Theorem 4] (see also [Be1]).

Theorem 2.13 *Assume that (c_n) is a complex sequence and that $(p(n))$ is a sequence of nonzero multi–indexes such that $|p(n)| \to \infty$ $(n \to \infty)$. Then the following properties are equivalent:*

(a) *The sequence* $((p(n)!\,|c_n|)^{1/|p(n)|})$ *is bounded.*

(b) *There is no hypercyclic entire function for* $(c_n D^{p(n)})$.

(c) *The set* $HC((c_n D^{p(n)}))$ *is not residual in* $H(\mathbb{C}^N)$.

(d) *The sequence* $(c_n D^{p(n)})$ *is equicontinuous on* $H(\mathbb{C}^N)$.

Proof. It is evident that (b) implies (c) and that (d) implies (b). Since $R(\mathbb{C}^N) = +\infty$, we obtain from Theorem 2.12 that (c) implies (a). Assume that (a) holds. Then we can apply Theorem 2.11 with $I = \mathbb{N}$ and $\Phi_n(z) = c_n z^{p(n)}$. Indeed, there is a constant M with $p(n)!|c_n| \le M^{|p(n)|}$ for all $n \in \mathbb{N}$, hence the function

$$\Phi(z) = \sum_{n=1}^{\infty} \frac{M^{|p(n)|}}{p(n)!} z^{p(n)} \quad (z \in \mathbb{C}^N)$$

is a majorant entire function for (Φ_n) with exponential type. Then (d) is true and the proof is finished.

We point out that in [Gr2, Corollary to Theorem 4] the part about hypercyclicity of [Be2, Theorem 4] is extended for the case $N = 1$ to sequences of weighted pseudo-shifts in the space $H(\mathbb{C})$.

The part "only if" of [Be2, Theorem 3] is able to be extended in the same way to the N–dimensional case, as the following theorem shows.

Theorem 2.14 *Let $G = G_1 \times \cdots \times G_N \subset \mathbb{C}^N$ be a polydomain with $G_j \ne \mathbb{C}$ ($j = 1, ..., N$). Assume that (c_n) is a complex sequence and that $(p(n))$ is a sequence of nonzero multi–indexes such that the sequence of operators $(c_n D^{p(n)})$ is equicontinuous on $H(G)$. Then*

$$\lim_{n \to \infty} (p(n)!\,|c_n|)^{1/|p(n)|} = 0.$$

Proof. Consider the number $\alpha := \limsup_{n \to \infty} (p(n)!|c_n|)^{1/|p(n)|}$. By the way of contradiction, assume that $\alpha > 0$. Fix a point $a = (a_1, ..., a_N) \in G$. Then $a_j \in G_j$ and there exist points $b_j \in \mathbb{C} \setminus G_j$ ($j = 1, ..., N$) such that $|a_j - b_j| = \inf\{|a_j - t| : t \in \mathbb{C} \setminus G_j\}$. Denote $R = \min\{|a_j - b_j| : j = 1, ..., n\} > 0$. Fix $r \in (0, R)$ with $R - r < \alpha$. Put $K = D(a, r)$. Then K is a compact subset of G. Let L be any compact subset of G and δ a positive number. Let $m > 0$ be so small that

$$\frac{m}{(\inf\{|z_j - b_j| : j \in \{1, ..., N\}, z = (z_1, ..., z_n) \in L \cup K\})^N} < \delta.$$

Consider the function

$$f(z) = \frac{m}{\prod_{j=1}^{N}(z_j - b_j)}.$$

Then $f \in H(G)$ and, in addition, f belongs to $V(\delta, L)$. Furthermore, for $z = (z_1, ..., z_N) \in G$,

$$|(T_n f)(z)| = \frac{p(n)!\, m|c_n|}{\prod_{j=1}^{N} |z_j - b_j|^{1+p_j(n)}},$$

where $p(n) = (p_1(n), ..., p_N(n))$ and $T_n = c_n D^{p(n)}$. Since $\inf\{|t - b_j| : |t - a_j| < r\} \geq R - r$ for every $j \in \{1, ..., N\}$, we get

$$\sup\{|(T_n f)(z)| : z \in K\} \leq \frac{p(n)!|c_n|m}{(R-r)^{N+|p(n)|}} = \frac{m}{(R-r)^N} \cdot \frac{p(n)!|c_n|}{(R-r)^{|p(n)|}}.$$

But $\dfrac{p(n_k)!|c_{n_k}|}{(R-r)^{|p(n_k)|}} \to \infty$ $(k \to \infty)$ for some increasing sequence $(n_k) \subset \mathbb{N}$, because $\alpha > R - r$. Hence $\sup\{|T_n f(z)| : z \in K\} = \infty$. Therefore

$$\bigcup_{n \in \mathbb{N}} T_n(V(\delta, L)) \not\subset V(1, K),$$

which implies that (T_n) is not equicontinuous. The proof is finished.

Our final result comes back to hypercyclicity and looks slightly different from the others. It puts the emphasis on the first nonzero Taylor coefficient of each Φ_n. This time the setting is the complex plane $\mathbb{C}$. Observe that MacLane's theorem is again recovered if we choose $\Phi_n(z) = z^n$ for each $n \in \mathbb{N}$.

Theorem 2.15 *Assume that $\left(\Phi_n(z) = \sum_{j=0}^{\infty} c_{jn} z^j\right)$ is a sequence of nonzero entire functions and denote $p(n) := m(\Phi_n)$ $(n \in \mathbb{N})$. Assume that the following three conditions are fulfilled:*

(a) $p(n) \to \infty$ as $n \to \infty$.

(b) $p(n)|c_{p(n),n}|^{k/p(n)} \to \infty$ as $n \to \infty$ for every $k \in \mathbb{N}$.

(c) Each sequence $\{c_{j+p(n),n} : n \in \mathbb{N}\}$ $(j \in \mathbb{N})$ is bounded.

Then the set $HC((\Phi_n(D)))$ is residual in $H(G)$ for any simply connected domain $G \subset \mathbb{C}$.

Proof. We are trying to apply Lemma 2.1 with $X = H(G) = Y$, $X_0 = \{\text{polynomials}\} = Y_0$ and $T_n = \Phi_n(D)$ $(n \in \mathbb{N})$. If P is a polynomial then by

(a) there exists $n_0 \in \mathbb{N}$ with $p(n) > \mathrm{degree}\,(P)$ for all $n \geq n_0$, hence $D^j P = 0$ for all $j \geq p(n)$ $(n \geq n_0)$. Therefore $T_n P = 0$ eventually and condition (a) of Lemma 2.1 is satisfied. Now fix m and n in $\mathbb{N}$ and try to solve the equation $T_n f = z^m$. Observe that $T_n = \Psi_n(D) \circ D^{p(n)}$, where $\Psi_n(z) = \sum_{j=0}^{\infty} a_{jn} z^j$ and $a_{jn} = c_{j+p(n),n}$, so $a_{0n} \neq 0$ for all $n \in \mathbb{N}$. Consider the equation

$$\Psi_n(D)g = z^m, \tag{1}$$

where g is a polynomial of degree not greater than m, say, $g(z) = \sum_{k=0}^{m} b_{kn} z^k$. It is easy to see that such a polynomial solution exists. Indeed, (1) is equivalent to

$$\sum_{j=0}^{m} a_{jn} \Big(\sum_{k=0}^{m} b_{kn} z^k \Big)^{(j)} = z^m,$$

which in turn is the same as the system

$$\begin{cases} \sum_{j=k}^{m} a_{j-k,n} b_{jn} \cdot \frac{j!}{k!} = 0 & (k = 0, 1, ..., m-1) \\ a_{0n} b_{mn} = 1. \end{cases}$$

This is a recurrent square system with determinant $a_{0n}^{m+1} \neq 0$, so it has a unique solution $(b_{0n}, ..., b_{mn})$ and Cramer's rule yields

$$b_{kn} = \frac{1}{a_{0n}^{m+1}} \cdot \sum_{j=1}^{m} P_{jkm}(a_{1n}, ..., a_{mn})\, a_{0n}^{j} \tag{2}$$

for $k \in \{0, 1, ..., m\}$, where P_{jkm} $(j = 1, ..., m)$ are polynomials of m complex variables not depending on n. From (c), there is a finite positive constant M, which does not depend on n, such that

$$|P_{jkm}(a_{1n}, ..., a_{mn})| \leq M \tag{3}$$

for all $k \in \{0, 1, ..., m\}$ and all $j \in \{1, ..., m\}$. Hence a solution of $T_n f = z^m$ is

$$f(z) = f_n(z) = \sum_{k=0}^{m} b_{kn} \frac{z^{k+p(n)}}{(k+p(n))!} \quad (n \in \mathbb{N}),$$

where b_{kn} is given by (2). Let us fix $R > 1$. Then from (3) we obtain for $|z| \leq R$ that

$$|f_n(z)| \leq (m+1) \sum_{j=1}^{m} \frac{MR^m}{|a_{0n}|^{m+1-j}} \cdot \frac{R^{p(n)}}{p(n)!} \to 0 \quad (n \to \infty)$$

since (b) and Stirling's formula leads us to

$$(p(n)!\, |a_{0n}|^{m+1-j})^{1/p(n)} \to \infty \quad (n \to \infty),$$

so the terms ot the latter sequence are eventually greater than, for instance, $1/2R$. Therefore (f_n) tends to zero in $H(G)$. The proof for the case $m = 0$ is easier and left to the reader. Define

$$S_n(z^m) := f_n(z) \quad (m \in \mathbb{N}_0;\ n \in \mathbb{N})$$

and extend S_n to Y_0 by linearity. Then it is clear that $S_n P \to 0\ (n \to \infty)$ and $T_n(S_n P) = P \to P$ as $n \to \infty$. Consequently, conditions (b) and (c) in Lemma 2.1 are also fulfilled, as required.

For instance, there is an entire function f in $\mathbb{C}$ with the property that any entire function can be locally uniformly approximated by functions of the form

$$c_n(f^{(n)} + f^{(n+1)})\ (n \in \mathbb{N}),$$

where $c_n = n^{-n/(\log n)^{1/2}}$. Indeed, the sequence $\{\Phi_n(z) = c_n z^n(1 + z)\}$ satisfies all hypotheses of the latter theorem, because (c_n) is bounded, $p(n) \to \infty$ and $p(n) \cdot n^{-kn/(p(n)\,(\log n)^{1/2})} \to \infty\ (n \to \infty)$ for all $k \in \mathbb{N}$, where $p(n) \equiv n$ here. Note that this example shows that Theorem 2.15 is not included in Theorem 2.4: in fact, $\Phi_n(z) \to 0$ as $n \to \infty$ for all $z \in \mathbb{C}$, hence the $\mathcal{E}$–unicity set B is not available in order to apply the mentioned theorem.

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