Infinite dimensional holomorphic non-extendability and algebraic genericity

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Abstract

In this note, the linear structure of the family $H_e(G)$ of holomorphic functions in a domain G of a complex Banach space that are not holomorphically continuable beyond the boundary of G is analyzed. More particularly, we prove that $H_e(G)$ contains, except for zero, a closed (and a dense) vector space having maximal dimension, as well as a maximally generated free algebra. The results obtained complete a number of previous ones by several authors.

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1 Introduction and preliminaries

In the last decade there has been a generalized trend for the search for algebraic structures inside nonlinear sets. This area of research, called *lineability* ([26, 36]), has attracted the attention of many authors and it has been proven to be quite fruitful, with the appearance of several research papers, surveys (see, e.g. [22]), and even a monograph ([5]).

In this note, we focus on the family of holomorphic functions $G \to \mathbb{C}$ that cannot be holomorphically continued beyond the boundary of G, where G is a domain in a separable infinite dimensional complex Banach space E. Our aim is to contribute to complete the existing knowledge on lineability of the mentioned family.

Our notation will be rather usual. The symbols $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ will stand for the set of positive integers, the field of rational numbers, the real line and the field of complex numbers, respectively. By a domain in a complex Banach space E we mean a nonempty proper connected open subset G of E. We denote by $H(G)$ the space of all holomorphic functions $f : G \to \mathbb{C}$ (see, e.g., [24] for definitions and properties), and by ∂G the boundary of G. We say that a function $f \in H(G)$ is *holomorphically non-extendable beyond* ∂G (or that f is *holomorphic exactly* on G) whenever there do not exist two domains G_1 and G_2 in E and $\widetilde{f} \in H(G_1)$ such that

$$G_2 \subset G \cap G_1, \ G_1 \not\subset G \ \text{ and } \ \widetilde{f} = f \ \text{ on } \ G_2.$$

We denote by $H_e(G)$ the family of all $f \in H(G)$ that are holomorphic exactly on G. A domain G is called a *domain of existence* whenever $H_e(G) \neq \varnothing$. It is well known that *every* domain of $\mathbb{C}$ is a domain of existence (see [30]), but this fails for higher dimensions (see, e.g., [33]).

Now, a number of lineability concepts –that have been recently coined by a number of authors, see [6, 9, 10, 13, 15, 18, 23, 27, 29], the survey [22] and the book [5]– are in order. Namely, if X is a vector space, α is a cardinal number and $A \subset X$, then A is said to be:

- *lineable* if there is an infinite dimensional vector space M such that $M \setminus \{0\} \subset A$,
- *α-lineable* if there exists a vector space M with $\dim(M) = \alpha$ and $M \setminus \{0\} \subset A$ (hence lineability means $\aleph_0$-lineability, where $\aleph_0 = \text{card}\,(\mathbb{N})$, the cardinality of $\mathbb{N}$), and
- *maximal lineable* in X if A is $\dim(X)$-lineable.

If, in addition, X is a topological vector space, then A is said to be:

- *dense-lineable* (*α-dense-lineable*) in X whenever there is a dense vector subspace M of X satisfying $M \setminus \{0\} \subset A$ (and $\dim\,(M) = \alpha$, resp.),
- *maximal dense-lineable* in X if A is $\dim(X)$-dense-lineable,
- *spaceable* (*α-spaceable*) in X if there is a closed infinite dimensional (a closed α-dimensional, resp.) vector subspace M such that $M \setminus \{0\} \subset A$, and
- *maximal spaceable* in X if A is $\dim(X)$-spaceable.

Finally, when X is a topological vector space contained in some (linear) algebra then A is called:

- *algebrable* if there is an algebra M so that $M \setminus \{0\} \subset A$ and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite.

- *densely* (*closely*) *algebrable* in X if, in addition, M can be taken dense (closed, resp.) in X.

- α-*algebrable* if there is an α-generated algebra M with $M \setminus \{0\} \subset A$.

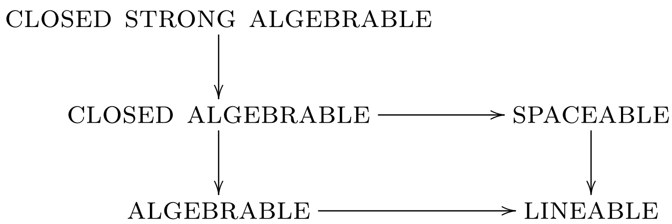
- *strongly* α-*algebrable* if there exists an α-generated *free* algebra M with $M \setminus \{0\} \subset A$ (for $\alpha = \aleph_0$, we simply say *strongly algebrable*).

- *densely* (*closely*) *strongly* α-*algebrable* if, in addition, the free algebra M can be taken dense (closed, resp.) in X.

Observe that if X is contained in a commutative algebra then a set $B \subset X$ is a generating set of some free algebra contained in A if and only if for any $N \in \mathbb{N}$, any nonzero polynomial P in N variables without constant term and any distinct $f_1, \ldots, f_N \in B$, we have $P(f_1, \ldots, f_N) \neq 0$ and $P(f_1, \ldots, f_N) \in A$. Observe that strong α-algebrability $\Longrightarrow$ α-algebrability $\Longrightarrow$ α-lineability, and none of these implications can be reversed, see [22, p. 74].

The reader can found without difficulty further implications, as for instance, dense-lineability $\Longrightarrow$ lineability (as long as $\dim(X) = \infty$), spaceability $\Longrightarrow$ lineability, closed (dense) algebrability $\Longrightarrow$ spaceability (dense-lineability, resp.), and many others.

The links between some of the previous concepts can be made more clear by means of the following diagram, in which the arrows indicate strict inclusions:



For every domain G of a *separable* complex Banach space E, the space $H(G)$ will be endowed with the topology of uniform convergence on compacta. If $E = \mathbb{C}^N$ ($N \in \mathbb{N}$), $H(G)$ becomes a Fréchet space (i.e. a complete metrizable locally convex space), but it is no longer metrizable if E is infinite dimensional, see [1, 4, 24]. Note that $\dim(H(G)) = \mathfrak{c}$ (the cardinality of the continuum), because G is separable and $H(G) \subset \{$continuous funtions $G \to \mathbb{C}\}$. The *topological size* of $H_e(G)$ is well known, at least in the finite dimensional case: if $G \subset \mathbb{C}^N$ is a domain of existence then $H_e(G)$ is a *residual subset* of $H(G)$ (see, e.g., [16, Theorem 3.1] or [32, Theorem 3.1] for one variable and [31, Proposition 1.7.6] for several variables). However, it has been only recently when the *algebraic size* has been considered. The following theorem –whose content has been collected from [7] and [19] (see also [5, Chap. 3])– tries to summarize the existing results for $G \subset \mathbb{C}^N$.

Theorem 1.1. (a) *If G is a domain in $\mathbb{C}$ then $H_e(G)$ is densely strongly $\mathfrak{c}$-algebrable in $H(G)$.*

(b) *Assume that $N \in \mathbb{N}$ and that $G \subset \mathbb{C}^N$ is a domain of existence. Then $H_e(G)$ is maximal dense-lineable and closely algebrable in $H(G)$. In particular, $H_e(G)$ is spaceable.*

Notice that a standard application of Baire's theorem yields that the closed algebra obtained in Theorem 1.1(b) cannot be $\aleph_0$-generated. Hence, for any domain of existence $G \subset \mathbb{C}^N$, the set $H_e(G)$ is $\mathfrak{c}$-*algebrable* and $\mathfrak{c}$-*spaceable.*

Additional properties, as well as growth conditions, can be imposed on the subspaces discovered in Theorem 1.1, see [17, 37]. Concerning subspaces X of $H(G)$ (with $G \subset \mathbb{C}$), assumed to be endowed with natural topologies, a collection of results about lineability of $H_e(G) \cap X$ can be found in [16, 19, 20, 21, 38]; see also [5, Chap. 7].

In the setting of infinite dimensional holomorphy, the credit of starting the analysis of lineability for non-extendable functions must go to Alves, who recently has proved a number of remarkable results (see [2, 3]) that can be summarized as follows.

Theorem 1.2. *Suppose that G is a domain of existence of a separable complex Banach space E. Then we have:*

(a) *$H_e(G)$ is $\mathfrak{c}$-lineable.*

(b) *$H_e(G) \cup \{0\}$ contains a closed algebra $\mathcal{A}$ as well as a dense algebra $\mathcal{B}$, such that each of them contains an infinite algebraically independent set. In particular, $H_e(G)$ is strongly algebrable, as well as densely algebrable* (*hence dense-lineable*) *and closely algebrable* (*hence spaceable*) *in $H(G)$.*

Remark 1.3. In [3], Alves asserts the dense strong algebrability of $H_e(G)$ (he denotes $H_e(G)$ by $\mathcal{G}(G)$), but in fact he proves the existence of a dense algebra $\mathcal{B}$ satisfying the property given in Theorem 1.2(b). This property is weaker than the one corresponding to the original concept of dense strong algebrability introduced by Bartoszewicz and Głąb in [13], that we have adopted in our definitions above. Namely, the dense algebra in [13] should *be generated* by an infinite algebraically independent set of elements, *not merely contain* such a set, and from the proof in [3] it is not apparent at all that the algebra $\mathcal{B}$ is *freely* generated itself. For the sake of consistency, we have mimicked the paper [13] in order to define the notion of closely strong algebrable family (simply replacing "dense" by "closed"). Under this terminology, analogously to the "dense" case, the property proved in [3] concerning the existence of an algebra as $\mathcal{A}$ in Theorem 1.2(b) is weaker than the closed strong algebrability. Nevertheless, the shape of these algebras (see Lemma 2.1 below) will help us to complete Alves' results.

As a complementary statement, it was shown in [19, Theorem 3.11] that $H_e(G)$ is maximal dense-lineable in $H(G)$ under the assumption that the translated domain $G - x_0$ is balanced for some point $x_0 \in G$.

In this note, we will complete these results, so as to prove that the algebrability –in three senses– of $H_e(G)$ is *maximal* in the infinite dimensional case as well. As a consequence, maximal spaceability and maximal dense-lineability are also obtained. If G is balanced then we will obtain, in addition, the dense maximal strong algebrability of $H_e(G)$.

2 Lineability of $H_e(G)$

In this section, we formulate two results completing the known maximal lineability of $H_e(G)$ in the infinite dimensional case. The reader is referred to [24, 25, 34] for fundamentals of holomorphy on complex Banach spaces. In the case $G \subset \mathbb{C}^N$ for some $N \in \mathbb{N}$, the space $H(G)$ is completely metrizable. Then the existence of a closed infinitely generated algebra inside $H_e(G) \cup \{0\}$ together with the Baire theorem implies automatically $\mathfrak{c}$-spaceability and $\mathfrak{c}$-algebrability of $H_e(G)$. But this is not immediate if E has infinite dimension. Indeed, the dense algebra and the closed algebra constructed by Alves [3] for the proof of Theorem 1.2 are countably generated and, in addition, $H_e(G)$ is *not barrelled* in the infinite dimensional case (see [24, Theorem 16.21]), so it is *not Baire* (see [35, Chapter 2]). This forces us to search for a different approach.

Nevertheless, we can take advantage of some of the (far-reaching) findings by Alves. To be more specific, the content of the following auxiliary result –which will be needed to prove Theorems 2.5 and 2.6– may be found in [3, Proposition 2.1(b) and Proofs of Theorems 4.3-5.2-6.1].

Lemma 2.1. *Assume that E is a separable complex Banach space and that $G \subset E$ is a domain. We have:*

(a) *If G is a domain of existence, then for each sequence $\{x_n\}_{n\geq 1}$ of distinct points of G such that $\lim_{n\to\infty} \operatorname{dist}(x_n, \partial G) = 0$, and each sequence $\{\alpha_n\}_{n\geq 1} \subset \mathbb{C}$, there exists a function $f \in H(G)$ such that $f(x_n) = \alpha_n$ for every $n \in \mathbb{N}$.*

(b) *There is a sequence $\{z_n\}_{n\geq 1} \subset G$ such that the family $\mathcal{A} := \{f \in H(G) : f(z_n) = 0$ for each $n \in \mathbb{N}\}$ does not reduces to 0, the set $\mathcal{B} = \{f \in H(G) :$ exists $N \in \mathbb{N}$ with $f(z_n) = 0$ for all $n \geq N\}$ is dense in $H(G)$ and $\mathcal{B} \setminus \{0\} \subset H_e(G)$.*

Firstly, we undertake the question of strong maximal algebrability. We will need a criterion that was originally given in [11, Proposition 7] by Balcerzak et al. (see also [12, Theorem 1.5] and [14, 28]) for a family $\mathcal{F}$ of functions $[0,1] \to \mathbb{R}$, and then in [19, Proposition 2.3] for the general functions $\Omega \to \mathbb{C}$. In fact, we shall use a variant of [19, Proposition 2.3] stated in Lemma 2.2 below. By $\mathcal{E}_+$ we denote the algebra of functions $\mathbb{C} \to \mathbb{C}$ of the form

$$\varphi(z) = \sum_{j=1}^{m} a_j e^{b_j z}$$

for some $m \in \mathbb{N}$, some $a_1, \dots, a_m \in \mathbb{C} \setminus \{0\}$ and some distinct $b_1, \dots, b_m \in (0, +\infty)$.

Lemma 2.2. *Let Ω be a nonempty set and $\mathcal{F}$ be a family of functions $\Omega \to \mathbb{C}$. Assume that there exists a function $f : \Omega \to \mathbb{C}$ such that $f(\Omega)$ is uncountable and $\varphi \circ f \in \mathcal{F}$ for every $\varphi \in \mathcal{E}_+$. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable. More precisely, if $H \subset (0, +\infty)$ is a set with $\operatorname{card}(H) = \mathfrak{c}$ and linearly independent over $\mathbb{Q}$, then*

$$\{\, e^{rf} : r \in H\}$$

is a free system of generators of an algebra contained in $\mathcal{F} \cup \{0\}$.

Remark 2.3. In [11, Proposition 7] and [19, Proposition 2.3] the family $\mathcal{E}_+$ is replaced by the bigger family $\mathcal{E}$ of exponential-like functions, in which the coefficients $b_1, \dots, b_m$ appearing in its functions φ, being distinct, may move along the whole set $\mathbb{R} \setminus \{0\}$ or $\mathbb{C} \setminus \{0\}$ (resp.), not only along $(0, +\infty)$. But this makes no influence in the proof, which can be mimicked from [11, Proposition 7] or [19, Proposition 2.3], and so it is omitted.

The content of the following auxiliary result is well known; see for instance [24, Theorem 19.14].

Lemma 2.4. *Assume that E is a separable complex Banach space and that G is a domain in E. Let $f \in H(G)$ satisfying that, for every open ball B intersecting ∂G, f is unbounded on each connected component of $G \cap B$. Then $f \in H_e(G)$.*

We are now ready to establish strong maximal algebrability.

Theorem 2.5. *Let E be a separable infinite dimensional complex Banach space and G be a domain of existence in E. Then the set $H_e(G)$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Firstly, we need a sequence $\{z_n\} \subset G$ of pairwise different points with the property that, for every open ball B intersecting ∂G, the intersection of $\{z_n\}$ with every connected component of $B \cap G$ is infinite and, in addition, $\lim_{n\to\infty} \operatorname{dist}(z_n, \partial G) = 0$. An example of the required sequence may be defined as follows. Let $\{\alpha_n\}$ be a dense countable subset of G (recall that E is separable and metrizable, hence any subset of E is separable as well). Denote $B(x, r) := \{y \in E : \|y - x\| < r\}$, the open ball with center $x \in E$ and radius $r > 0$. For each $n \in \mathbb{N}$, consider the ball $B_n = B(\alpha_n, \operatorname{dist}(\alpha_n, \partial G))$ and take a sequence $\{\beta_{n,k}\}_{k\geq 1} \subset B_n$ such that $\operatorname{dist}(\beta_{n,k}, \partial G) < \frac{1}{n+k}$ $(n, k \in \mathbb{N})$. Then each one-fold sequence $\{z_n\} \subset G$ (without repetitions) consisting of all distinct points of the set $\{\beta_{n,k} : n, k \geq 1\}$ has the required properties.

It follows from Lemma 2.1(a) that there exists a function $f \in H(G)$ such that $f(z_n) = n$ for all $n \in \mathbb{N}$. Fix $\varphi \in \mathcal{E}_+$. Then there are nonzero complex numbers $a_1, \dots, a_m$ and distinct $b_1, \dots, b_m \in (0, +\infty)$ with $\varphi(z) = \sum_{j=1}^m a_j e^{b_j z}$. Since we can assume $b_1 > \cdots > b_m$, we get

$$|(\varphi \circ f)(z_n)| \geq |a_1| e^{b_1 n} \big(1 - \sum_{j=2}^m |a_j/a_1| e^{(b_j - b_1)n}\big) \text{ for all } n \in \mathbb{N}.$$

Therefore $\lim_{n\to\infty} |(\varphi \circ f)(z_n)| = +\infty$. But, for every open ball B with $B \cap \partial G \neq \varnothing$, any component V of $B \cap G$ contains infinitely many points z_n. Thus, $\varphi \circ f$ is unbounded on V. According to Lemma 2.4, $\varphi \circ f \in H_e(G)$. An application of Lemma 2.2 with $\mathcal{F} := H_e(G)$ yields the desired conclusion. □

Finally, we consider the problem of the existence of closed or dense large subspaces.

Theorem 2.6. *Let E be a separable infinite dimensional complex Banach space and G be a domain of existence in E. Then the family $H_e(G)$ is closely $\mathfrak{c}$-algebrable and densely $\mathfrak{c}$-algebrable in $H(G)$. In particular, $H_e(G)$ is $\mathfrak{c}$-spaceable and $\mathfrak{c}$-dense-lineable in $H(G)$.*

Proof. Since $\dim(E) = \infty$, the topological dual space E^* is also infinite dimensional thanks to the Hahn–Banach theorem. It is known that $E^* \subset H(E)$, so the vector space E^*_G of restrictions to G of the members of E^* is contained in $H(G)$. When endowed with the dual norm, E^* is an infinite dimensional Banach space, so an application of Baire's theorem yields $\dim(E^*) = \mathfrak{c}$. If $f_0 \in H(G) \setminus \{0\}$ then from the Identity Principle it follows that the set Z of zeros of f_0 is a closed subset of G with empty interior. Consider the family

$$f_0 E^*_G = \{fh : \, h \in E^*_G\},$$

which is obviously a vector subspace of $H(G)$. Then $\dim(f_0 E^*_G) = \mathfrak{c}$ too. Indeed, let $\{g_i\}_{i \in I}$ (with $\mathrm{card}(I) = \mathfrak{c}$) be an algebraic basis of E^*, and set $h_i = g_i|_G$ $(i \in I)$. It is evident that the functions $f_0 h_i$'s generate the space $f_0 E^*_G$ but, in addition, they are linearly independent because, given $p \in \mathbb{N}$, $a_1, \dots, a_p \in \mathbb{C}$ and distinct elements $i(1), \dots, i(p)$ of I satisfying

$$a_1 f_0 \, h_{i(1)} + \cdots + a_p f_0 \, h_{i(p)} = 0 \ \text{ on } \ G,$$

we have $a_1 g_{i(1)} + \cdots + a_p g_{i(p)} = 0$ on the nonempty open set $G \setminus Z$. The Identity Principle implies $a_1 g_{i(1)} + \cdots + a_p g_{i(p)} = 0$ on the whole E, so $a_1 = \cdots = a_p = 0$. This entails $\dim(f_0 E^*_G) = \mathfrak{c}$, as required.

Now, take the families $\mathcal{A}, \mathcal{B}$ furnished by Lemma 2.1(b). Choose any $f_0 \in \mathcal{A} \setminus \{0\}$. On the one hand, it is evident that both $\mathcal{A}$ and $\mathcal{B}$ are algebras, that $\mathcal{A}$ is closed (for compact convergence implies pointwise convergence) and that $\mathcal{A} \subset \mathcal{B}$. On the other hand, $\mathcal{A} \setminus \{0\} \subset \mathcal{B} \setminus \{0\} \subset H_e(G)$ and $f_0 E^*_G \subset \mathcal{A} \subset \mathcal{B}$. Recall also that $\mathcal{B}$ is dense. Finally, the algebra $\mathcal{A}$ (and, analogously, the algebra $\mathcal{B}$) is $\mathfrak{c}$-generated. Indeed, the cardinal of any system of generators of $\mathcal{A}$ cannot exceed $\dim(H(G)) = \mathfrak{c}$. But such a system cannot be countable because, otherwise, the family $\mathcal{A}$, considered as a *vector* space, would be generated by a countable system of vectors (namely, by the finite products of powers of the elements of the first system), which is impossible by the preceding paragraph. Therefore $\mathcal{A}$ and $\mathcal{B}$ are $\mathfrak{c}$-generated. This concludes the proof. □

We finish the present section by posing an obvious problem which, as far as we know, remains unsolved:

> *Assume that E is a separable infinite dimensional complex Banach space and that $G \subset E$ is a domain. Is $H_e(G)$ densely/closely strongly ($\mathfrak{c}$-)algebrable?*

3 Balanced domains

In the special case of domains that are balanced with respect to some point, we will solve partially the last problem in the affirmative; see Theorem 3.3 below. Recall that a subset A of a vector space X is called balanced if $\lambda x \in A$ whenever $x \in A$ and λ is a scalar with $|\lambda| \leq 1$. We need the following lemma, whose content can be extracted from [25, Proposition 3.36].

Lemma 3.1. *Suppose that G is a balanced domain of a complex Banach space E. Then the Taylor series centered at 0 of each $f \in H(G)$ converges to f uniformly on compacta in G. Consequently, the set $\mathcal{P}$ of (continuous) polynomials is dense in $H(G)$.*

Remark 3.2. The last result also holds for holomorphic mappings between complex locally convex spaces and for some topologies different from the one of the uniform convergence on compacta; see [25].

Theorem 3.3. *Let E be a separable infinite dimensional complex Banach space and G be a domain of existence in E satisfying that $G - x_0$ is balanced for some $x_0 \in G$. Then the set $H_e(G)$ is densely strongly $\mathfrak{c}$-algebrable.*

Proof. Since $f \in H_e(G)$ if and only if $f(\cdot + x_0) \in H_e(G - x_0)$, we can suppose that $x_0 = 0 \in G$ and G is balanced. By Lemma 3.1, the set $\mathcal{P}$ is dense in $H(G)$. On the one hand, it follows from the separability of E that $\text{card}(C(E)) = \mathfrak{c}$ (the cardinality of the family of continuous functions $E \to \mathbb{C}$), so $\text{card}(\mathcal{P}) = \mathfrak{c}$ as well. On the other hand, again the separability of E implies that the cardinality of the family of open subsets of E is $\mathfrak{c}$; indeed, E is second-countable because it is metrizable, hence it possesses a countable open basis, so that each open subset is a countable union of sets extracted from a fixed countable family. Therefore there are also $\mathfrak{c}$ closed subsets of E and, consequently, the cardinality of the family $\mathcal{K}$ of compact subsets of G is also $\mathfrak{c}$. This entails that the family $\mathcal{O}$ consisting of all sets

$$O(P, K, \varepsilon) := \{f \in H(G) : |f(x) - P(x)| < \varepsilon\} \quad (P \in \mathcal{P},\ K \in \mathcal{K},\ \varepsilon > 0)$$

satisfies $\text{card}(\mathcal{O}) = \mathfrak{c}$ as well. Thanks to the density of $\mathcal{P}$ in $H(G)$, the family $\mathcal{O}$ is an open basis for the topology of $H(G)$.

Fix a sequence $\{z_n\} \subset G$ and a function $f \in H(G)$ as in the proof of Theorem 2.5, so that $f(z_n) = n$ for all $n \in \mathbb{N}$ and, for every open ball B intersecting ∂G, each connected component of $B \cap G$ contains infinitely many points z_n. Consider the algebraically free system $\mathcal{S} := \{F_r : r \in H\}$ constructed according to Lemma 2.2, where $F_r := e^{r\,f}$. Since $\text{card}(H) = \mathfrak{c}$, we get $\mathrm{card}(\mathcal{S}) = \mathfrak{c}$ and, as $\mathrm{card}(\mathcal{O}) = \mathfrak{c}$, we are allowed to use H as index set for $\mathcal{O}$, so that we may write $\mathcal{O} = \{O_r = O(P_r, K_r, \varepsilon_r)\}_{r\in H}$ (with the O_r's pairwise different). For given $r \in H$, let

$$\delta_r := \frac{\varepsilon_r}{2(1+\sup_{x\in K_r}|F_r(x)|)} \quad \text{and} \quad \Phi_r := P_r + \delta_r F_r.$$

Then $\Phi_r \in O_r$. Therefore the set $\{\Phi_r : r \in H\}$ is dense in $H(G)$. It then suffices to show that the algebra $\mathcal{C}$ generated by $\{\Phi_r : r \in H\}$ is freely generated and that each nonzero member of $\mathcal{C}$ belongs to $H_e(G)$. Since every member of $H_e(G)$ is nonzero, it is enough to prove that, for every $p \in \mathbb{N}$, every nonzero polynomial P of p complex variables without constant term and every set $r_1, \dots, r_p$ of different elements of H, the function $F := P(\Phi_{r_1}, \dots, \Phi_{r_p})$ belongs to $H_e(G)$.

Before going on, a little more notation will be needed. We set $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. If $p \in \mathbb{N}$ then each element of $[0,+\infty)^p$ (in particular, each element of $\mathbb{N}_0^p$ or H^p) has the form $\mathbf{c} = (c_1,\dots,c_p)$ with $c_j \geq 0$ for all j. We let $|\mathbf{c}| = c_1 + \cdots + c_p$. For $\mathbf{u},\mathbf{v} \in [0,+\infty)^p$, we adopt the notation $\mathbf{uv} = (u_1v_1,\dots,u_pv_p)$, so that $|\mathbf{uv}| = u_1v_1 + \cdots + u_pv_p$.

Fix a function $F = P(\Phi_{r_1},\dots,\Phi_{r_p})$ as above. Then there is a nonempty subset $J \subset \mathbb{N}_0^p \setminus \{(0,\dots,0)\}$ as well as nonzero reals $\alpha_{\mathbf{m}}$ $(\mathbf{m}\in J)$ such that

$$F(x) = \sum_{\mathbf{m}\in J} \alpha_{\mathbf{m}} \prod_{j=1}^{p} (P_{r_j}(x) + \delta_{r_j}F_{r_j}(x))^{m_j} \quad \text{for all } x \in G.$$

Due to the linear $\mathbb{Q}$-independence of $r_1,\dots,r_p$, the numbers $|\mathbf{mr}|$ $(\mathbf{m}\in J)$ are pairwise different, where $\mathbf{r} = (r_1,\dots,r_p)$. Therefore, there exists a unique $\mathbf{n}\in J$ such that $|\mathbf{nr}| = \max\{|\mathbf{mr}| : \mathbf{m}\in J\}$.

Finally, fix an open ball B with $B\cap\partial G \neq \varnothing$. Then take any connected component V of $B\cap G$. There exists a sequence $\{\nu_1 < \cdots < \nu_k < \cdots\} \subset \mathbb{N}$ such that $z_{\nu_k} \in V$ for all $k \geq 1$. Since any polynomial is bounded on bounded sets, there is $M \in (0,+\infty)$ satisfying $|P_{r_j}(x)| \leq M$ $(x\in V;\ j = 1,\dots,p)$. Now, with the help of the triangle inequality we obtain

$$\begin{aligned}|F(z_{\nu_k})| &\geq |\alpha_{\mathbf{n}}| \prod_{j=1}^{p} (\delta_{r_j}|F_{r_j}(z_{\nu_k})| - M)^{n_j} - \sum_{\mathbf{m}\in J\setminus\{\mathbf{n}\}} |\alpha_{\mathbf{m}}| \prod_{j=1}^{p} (\delta_{r_j}|F_{r_j}(z_{\nu_k})| + M)^{m_j} \\ &= |\alpha_{\mathbf{n}}| \prod_{j=1}^{p} (\delta_{r_j}e^{r_j\nu_k} - M)^{n_j} - \sum_{\mathbf{m}\in J\setminus\{\mathbf{n}\}} |\alpha_{\mathbf{m}}| \prod_{j=1}^{p} (\delta_{r_j}e^{r_j\nu_k} + M)^{m_j} \\ &=: A_k + B_k \longrightarrow +\infty \quad \text{as} \quad k\to\infty\end{aligned}$$

because A_k behaves like $e^{|\mathbf{nr}|\nu_k}$, while B_k is a finite sum of terms each of them behaving like $e^{|\mathbf{mr}|\nu_k}$ (with $|\mathbf{nr}| > |\mathbf{mr}|$). Consequently, $\lim_{k\to\infty} |F(z_{\nu_k})| = +\infty$, which shows that F is unbounded on V. An application of Lemma 2.4 concludes the proof. □

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