STRUCTURAL ASPECTS OF THE NON-UNIFORMLY CONTINUOUS FUNCTIONS AND THE UNBOUNDED FUNCTIONS WITHIN $C(X)$

RAFAEL AYALA-GÓMEZ, LUIS BERNAL-GONZÁLEZ, MARÍA DEL CARMEN CALDERÓN-MORENO, AND JOSÉ ANTONIO VILCHES-ALARCÓN

Dedicated to the loving memory of our colleague Bernardo Cascales (1958-2018)

Abstract. We prove in this paper that if a metric space supports a real continuous function which is not uniformly continuous then, under appropriate mild assumptions, there exists in fact a plethora of such functions, in both topological and algebraical senses. Corresponding results are also obtained concerning unbounded continuous functions on a non-compact metrizable space.

1. Introduction

It is a well known fact that, if (X, d) and (Y, ρ) are metric spaces such that (X, d) is compact, then any continuous mapping $f : X \to Y$ is *uniformly continuous,* that is, it satisfies that, given $\varepsilon > 0$, there is $\delta > 0$ such that $\rho(f(x), f(x')) < \varepsilon$ for every pair of points $x, x' \in X$ with $d(x, x') < \delta$ (see e.g. [40, Chap. 3]). However, the reciprocal assertion is not true, even though we consider $(Y, \rho) = (\mathbb{R}, d_e)$, the real line endowed with the Euclidean distance. Indeed, if X is an infinite set and we endow it with the discrete metric

$$\rho_d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

then (X, d) is not compact but any function (in particular, any continuous function) $X \to \mathbb{R}$ is uniformly continuous. Recall that uniform continuity depends strongly upon the metrics. For instance, if $X = \{1/n : n = 1, 2, \dots\}$ and $f(1/n) := n$ $(n \geq 1)$, then d_e and ρ_d are equivalent metrics on X, but $f : (X, \rho_d) \to (\mathbb{R}, d_e)$ is uniformly continuous while $f : (X, d_e) \to (\mathbb{R}, d_e)$ is not.

Turning to compact spaces, what is true is the following characterization –due to Hewitt [23] in 1948– of this kind of spaces: a metrizable space X is compact if and only if every continuous function $X \to \mathbb{R}$ is bounded.

2010 *Mathematics Subject Classification.* 46E10, 15A03, 08B20, 54C05, 54E35.

Key words and phrases. Atsuji space, non-uniformly continuous function, lineability, spaceability, algebrability.

In 1958 Atsuji [5] gave a fairly comprehensive list of characteristic properties of those metric spaces for which any real continuous function defined on them is uniformly continuous. From now on, these spaces will be called *Atsuji spaces* (they are also frequently called UC spaces in the literature). Specifically, and under this terminology, it was proved in [5] the following result, among others. As usual, if X is a topological space and $A \subset X$, then we denote by A' the set of limit points of A in X. We warn that along this paper the symbol $\subset$ does *not* necessarily mean *strict* inclusion.

Theorem 1.1. *Let (X, d) be a metric space. The following properties are equivalent:*

(a) *(X, d) is an Atsuji space.*
(b) *The subset X' is compact and, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x_1, x_2) \geq \delta$ whenever x_1 and x_2 are distinct points satisfying $d(x_1, x) \geq \varepsilon \leq d(x_2, x)$ for all $x \in X'$.*

Prior to this, spaces on which continuity equals uniform continuity had been studied by Nagata [41], Monteiro and Peixoto [37], and Isiwata [27]. Other characterizations of Atsuji spaces were provided by Rainwater [43], Hadziivanov [22], Mrówka [39], Waterhouse [46], Wong [47], Toader [45], Hueber [25], Beer [9–11], Chaves [18], Borsík [16], and Jain and Kundu [28] (see also [29] for characterizations of those metric spaces whose completions are Atsuji spaces). Moreover, in 1955 Levine [33] was able to give necessary and sufficient conditions for a subset of the Euclidean real line $\mathbb{R}$ to be an Atsuji space, and this characterization was extended in 1960 by Levine and Saunders [34] to subsets of any prescribed metric space (X, d) (corresponding results when the final space is not necessarily $(\mathbb{R}, d_e)$ are also obtained in [34]). The problem of describing those metrizable topological spaces admitting a compatible Atsuji distance has been considered by Nagata [41], Levsenko [35], Rainwater [43], Mrówka [39] and Beer [11]. It is important to mention that any Atsuji distance is complete (see [45]) and, since a metrizable space for which every compatible distance is complete is already compact (Niemytzki- Tychonoff's theorem), one derives that each non-compact metrizable space has a compatible non-Atsuji distance. Moreover, characterizations of compactness of metric spaces in terms of uniform continuity have been provided by Hueber [25] and Snipes [44].

Starting from a *non-Atsuji* metric space (X, d), a natural question is the analysis of the *size* and *structure* of the family of continuous functions on X that are *not* uniformly continuous. Such analysis, which will be carried out in Section 3, is the main aim of this paper. It will be proved that, under appropriate conditions, the mentioned family is very large in both topological and algebraic senses, so as to contain large –that is, dense or closed infinite dimensional– vector subspaces as well as infinitely generated algebras. Corresponding results for size and structure of the family of *unbounded* continuous functions defined on a *non-compact* metrizable topological space will be provided in Section 4. Section 2 is devoted to exhibit the necessary notation and background in order to deal with the above mentioned question.

2. Notation, lineability and preliminary results

Unless explicitly stated, the real line $\mathbb{R}$ will always be endowed with the Euclidean metric $d_e(x,y) = |x-y|$. As usual, $\mathbb{N}$ will denote the set of positive integers, while $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We set $\mathfrak{c} := \text{card}\,(\mathbb{R})$, the cardinality of the continuum. If (X,d) is a metric space, $x \in X$ and $\varepsilon > 0$, then $B_d(x,\varepsilon)$ represents the open ball $\{y \in X : d(x,y) < \varepsilon\}$. If X and Y are topological spaces and $A \subset X$ then $\overline{A}$ and $C(X,Y)$ will stand, respectively, for the closure of A in X and for the set of continuous functions $X \to Y$. Also, we define $C(X) := C(X,\mathbb{R})$, which is a vector space. If X and Y are metric spaces, then by $UC(X,Y)$ we represent the family of all uniformly continuous functions $X \to Y$. Consistently, we denote $UC(X) := UC(X,\mathbb{R})$. We are interested in the set $nUC(X,Y) := C(X,Y) \setminus UC(X.Y)$, and specially in

$$nUC(X) := nUC(X,\mathbb{R}) = C(X) \setminus UC(X),$$

which is nonempty whenever X is a non-Atsuji metric space. For a metrizable topological space X, we shall also consider the set $BC(X)$ of all bounded continuous functions $X \to \mathbb{R}$ as well as its complement

$$nBC(X) := C(X) \setminus BC(X),$$

which by Hewitt's theorem is nonempty if and only if X is non-compact.

A topological space Y is said to be an *absolute extensor* if, for every normal topological space X, every closed subset $A \subset X$ and every $f \in C(A,Y)$, there exists $F \in C(X,Y)$ such that $F|_A = f$. Under this terminology, Tietze's extension theorem (see, e.g., [40, pp. 219–222]) asserts that $\mathbb{R}$ is an absolute extensor. As another example, any product space I^J –where J is a nonempty set and I is an interval of the real line– is an absolute extensor and, in general, any product space $\prod_{j\in J} Y_j$ of absolute extensors is also an absolute extensor: see [36, VII.8.34]. Recall that every metrizable space is normal.

A linear topological space is called a *Fréchet space* if it is locally convex and completely metrizable. It is well known that the topology of every Fréchet space X is defined by a distance $d(x,y) = \|x-y\|$, where $\|\cdot\|$ is an F-norm on X, that is, a functional $\|\cdot\| : X \to [0,+\infty)$ satisfying the following properties for all $x,y \in X$ and all scalars λ: $\|x+y\| \le \|x\| + \|y\|$, $\|\lambda x\| \le \|x\|$ if $\|\lambda\| \le 1$, $\lim_{\lambda\to 0} \|\lambda x\| = 0$, and $\|x\| = 0$ implies $x = 0$; see [30].

In the following theorem we collect a number of well known results about σ-compactness and the compact-open topology on a space of continuous functions; see e.g. [24, pp. 325–326, 342 and 405–406] or [20]. Recall that a topological space is called *σ-compact* if it is a countable union of compact subspaces, and *second-countable* if it possesses a countable open basis. For background on nets, see, e.g., [32].

Theorem 2.1. *Assume that X is a T_2-locally compact topological space and that (Y,ρ) is a metric space. The following holds:*

(a) *If X is second-countable then it is metrizable and σ-compact.*
(b) *X is σ-compact if and only if there is a sequence $\{U_n\}_{n\geq 1}$ of open sets such that $X=\bigcup_{n\geq 1} U_n$, each $\overline{U_n}$ is compact and $\overline{U_n}\subset U_{n+1}$ for all $n\in\mathbb{N}$.*
(c) *If X is σ-compact then there is a metric D on $C(X,Y)$ generating the compact-open topology τ_K, that is, the metric D satisfies the following property: A net $(f_\alpha)_{\alpha\in J}\subset C(X,Y)$ tends to $f\in C(X,Y)$ uniformly on each compact subset of X if and only if $D(f_\alpha,f)\to 0$ $(\alpha\in J)$.*

In fact, under the assumptions of (c) and the notation in (b), a distance in $C(X,Y)$ generating the compact-open topology is

$$D(f,g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\frac{\sup_{x\in U_n}\rho(f(x),g(x))}{1+\sup_{x\in U_n}\rho(f(x),g(x))}.$$

Moreover, it is easy to prove that if Y is complete then D is complete. Hence (see, e.g., [42]) $(C(X,Y),\tau_K)$ becomes a Baire space provided that Y is complete and X is T_2, locally compact and σ-compact.

Sometimes the problem of "gluing" two uniformly continuous functions on two respective subsets of a topological space arises naturally. Not always this operation yields a uniformly continuous function, even if the global function is continuous. For instance, if $X=\{1,-1,1/2,-1/2,1/3,-1/3,\dots\}$, $A=\{1/n:\ n\in\mathbb{N}\}$ and $B=\{-1/n:\ n\in\mathbb{N}\}$, then $X=A\cup B$, the function $f:(X,d_e)\to(\mathbb{R},d_e)$ given by

$$f(x)=\begin{cases}0 & \text{if } x\in A\\ 1 & \text{if } x\in B\end{cases}$$

is continuous and both restrictions $f|_A$, $f|_B$ are uniformly continuous, but f is not uniformly continuous. In general, we can say that if (X,d) and (Y,ρ) are metric spaces and that A and B are subsets of X such that $X=A\cup B$ and $d(A,B)>0$ then, trivially, every function $f:X\to Y$ with $f|_A$ and $f|_B$ uniformly continuous is uniformly continuous. But this is a rather uncommon situation. More interesting and useful is the following result, which will be used in Section 3 and follows from [26, Corollary 16] as a special case. (Incidently the property of f relative to A in [26, Corollary 16] is called strong uniform continuity of f on A; this variational notion has been studied in depth by Beer and Levi.)

Lemma 2.2. *Assume that* (X,d) *and* (Y,ρ) *are metric spaces and that* $f \in C(X,Y)$. *Let* A, $B \subset X$ *be subsets such that* $X = A \cup B$, A *is compact and* $f|_B$ *is uniformly continuous. Then* $f \in UC(X,Y)$.

Some terminology extracted from the modern theory of lineability (see [3, 14] for concepts and results) will be also needed. Assume that α is a cardinal number, Z is a vector space and $A \subset Z$. Then A is said to be *lineable* if there is an infinite dimensional vector space M such that $M\setminus\{0\} \subset A$, and α*-lineable* if such an M can be found so as to satisfy $\dim(M) = \alpha$. If, in addition, Z is a topological vector space, then the subset A is said to be *spaceable* (α*-dense-lineable*, resp.) in Z whenever there is a closed infinite dimensional (a dense, resp.) vector subspace M of Z such that $M \setminus \{0\} \subset A$ (such that $\dim(M) = \alpha$ and $M \setminus \{0\} \subset A$, resp.).

Now, assume that Z is a vector space contained in some (linear) algebra. Then the subset A is called *algebrable* if there is an infinitely generated algebra M –that is, the cardinality of any system of generators of M is infinite– so that $M \setminus \{0\} \subset A$; and, if α is a cardinal number, then A is said to be *strongly* α*-algebrable* if there exists an α-generated *free* algebra M with $M \setminus \{0\} \subset A$. Recall that if Z is contained in a commutative algebra, then a set $B \subset Z$ is a generating set of some free algebra contained in A if and only if for any $N \in \mathbb{N}$, any nonzero polynomial P in N variables without constant term and any distinct $f_1, \ldots, f_N \in B$, we have $P(f_1, \ldots, f_N) \neq 0$ and $P(f_1, \ldots, f_N) \in A$.

The following auxiliary assertion will be useful in order to achieve dense-lineability from mere lineability. Its proof can be found in [13, Theorem 2.3] (see also [4, Theorem 2.2 and Remark 2.5] and [3, Section 7.3]).

Theorem 2.3. *Assume that* Z *is a metrizable separable topological vector space and that* γ *is an infinite cardinal number. Suppose that* A *and* B *are subsets of* Z *such that* $A + B \subset A$, $A \cap B = \varnothing$, B *is a dense vector subspace of* Z, *and* A *is* γ*-lineable. Then* A *is* γ*-dense-lineable in* Z.

Note that since the space Z is metrizable and separable, its cardinality satisfies $\mathrm{card}(Z) \le \mathfrak{c}$ and hence $\dim(Z) \le \mathfrak{c}$. Then $\mathfrak{c}$-(dense-)lineability is the optimal degree of (dense-)lineability that one can expect for a subset $A \subset Z$.

In order to construct large algebras of functions satisfying special properties, we shall employ the next lemma, that is inspired by the method used in [21] based on the superpositions of a fixed function belonging to the considered class with the representatives of some well chosen algebra of functions.

Lemma 2.4. *Let* Ω *be a nonempty set,* α *be a cardinal number and* $\mathcal{F}$ *be a family of functions* $\Omega \to \mathbb{R}$. *Assume that there exist an* α*-generated free algebra* Φ *consisting of analytic functions* $\mathbb{R} \to \mathbb{R}$ *and a function* $f : \Omega \to \mathbb{R}$

satisfying that $f(\Omega)$ *has an accumulation point in* $\mathbb{R}$ *and* $\varphi \circ f \in \mathcal{F}$ *for every* $\varphi \in \Phi \setminus \{0\}$. *Then* $\mathcal{F}$ *is strongly* α*-algebrable.*

Proof. Consider the set $\mathcal{A} := \{\varphi \circ f : \varphi \in \Phi\}$, that is an algebra contained in $\mathcal{F} \cup \{0\}$. Assume that $\mathcal{B}$ is a free generator system for Φ with card $(\mathcal{B}) = \alpha$. Trivially, $\widetilde{\mathcal{B}} := \{\varphi \circ f : \varphi \in \mathcal{B}\}$ is a generator system for $\mathcal{A}$. Moreover, card $(\widetilde{\mathcal{B}}) = \alpha =$ card $(\mathcal{B})$ because $\varphi \circ f \neq \psi \circ f$ as soon as $\varphi \neq \psi$ $(\varphi, \psi \in \Phi)$. Indeed, if $\varphi \circ f = \psi \circ f$ then $\varphi = \psi$ on the set $f(\Omega)$, which has an accumulation point in $\mathbb{R}$, and so $\varphi = \psi$ on the whole $\mathbb{R}$ by the Identity Principle for analytic functions (see, e.g., [1]). Finally, $\widetilde{\mathcal{B}}$ is a free generator system. To prove this, assume that $N \in \mathbb{N}$, that P is a polynomial of N real variables and that $\widetilde{\varphi_1}, \ldots, \widetilde{\varphi_N}$ are distinct functions in $\widetilde{\mathcal{B}}$ satisfying

$$P(\widetilde{\varphi_1}, \ldots, \widetilde{\varphi_N}) = 0 \text{ on } \Omega.$$

There are mutually different functions $\varphi_1, \ldots, \varphi_N \in \mathcal{B}$ with $\widetilde{\varphi}_i = \varphi_i \circ f$ $(i = 1, \ldots, N)$. Therefore $Q \circ f = 0$ on Ω, where $Q := P(\varphi_1, \ldots, \varphi_N)$. It follows that $Q = 0$ on $f(\Omega)$, and again the Identity Principle tells us that $Q = 0$ on $\mathbb{R}$, which implies $P = 0$ because $\mathcal{B}$ is algebraically free. Thus, $\mathcal{A}$ is an α-generated free algebra, which concludes the proof. □

Remark 2.5. Explicit examples in which the technique of the last lemma (with $\alpha = \mathfrak{c}$) is applied are provided in [2, 6–8, 21]. In [12] a complex version of such technique is used. We need for our goals the family Φ considered in [6]. In the next lemma we list the properties of Φ that will be used in Section 3.

Lemma 2.6. *There exists an algebra* Φ *of functions* $\mathbb{R} \to \mathbb{R}$ *satisfying the following properties:*

(a) Φ *is freely* $\mathfrak{c}$*-generated.*
(b) *Every* $\varphi \in \Phi$ *is analytic on* $\mathbb{R}$.
(c) *For every* $\varphi \in \Phi \setminus \{0\}$*, we have* $\lim_{x \to +\infty} |\varphi(x)| = +\infty$.
(d) Φ *is stable under derivations, that is, if* $\varphi \in \Phi$ *then* $\varphi' \in \Phi$.

Proof. Let $H \subset (0, +\infty)$ be a linearly $\mathbb{Q}$-independent set satisfying card $(H) = \mathfrak{c}$. For each $c \in H$, we denote $\varphi_c(x) := e^{cx}$. Let Φ be the algebra generated by $\{\varphi_c\}_{c \in H}$.

On the one hand, (b) follows from the facts that every φ_c is analytic on $\mathbb{R}$ and the analytic functions on $\mathbb{R}$ form an algebra. On the other hand, (a) is a consequence of (c). And (d) is easily deduced from the fact that $((\varphi_{c_1})^{j_1} \cdots (\varphi_{c_N})^{j_N})' = (c_1 j_1 + \cdots + c_N j_N) \varphi_{c_1 j_1 + \cdots + c_N j_N}$ for all reals $c_1, \ldots, c_N$ and all natural numbers $j_1, \ldots, j_N$. Therefore, we only have to prove (b).

With this aim, let us represent each N-tuple $(r_1, \ldots, r_N) \in \mathbb{R}^N$ by $\mathbf{r}$, and set $|\mathbf{r}| := r_1 + \cdots + r_N$ and $\mathbf{r} \cdot \mathbf{s} := r_1 s_1 + \cdots + r_N s_N$. Assume that $\varphi \in \Phi \setminus \{0\}$. Then there exist an $N \in \mathbb{N}$, mutually different $c_1, \ldots, c_N \in H$ and a nonzero polynomial P in N real variables such that P does not possess constant term and $\varphi = P(\varphi_{c_1}, \dots, \varphi_{c_N})$. Specifically, there exists a nonempty finite set $J \subset \mathbb{N}_0^N \setminus \{(0,0,\dots,0)\}$ and scalars $\alpha_{\mathbf{j}} \in \mathbb{R} \setminus \{0\}$ such that $P(x_1,\dots,x_N) = \sum_{\mathbf{j}\in J} \alpha_{\mathbf{j}}\, x_1^{j_1} \cdots x_N^{j_N}$. Hence

$$\varphi(x) = \sum_{\mathbf{j}\in J} \alpha_{\mathbf{j}}\, e^{\mathbf{c}\cdot\mathbf{j}\,x} \quad \text{for all} \quad x \in \mathbb{R}.$$

Let $\mathbf{k}$ be the *unique* element in J such that $\mathbf{c}\cdot\mathbf{k} = \max\{\mathbf{c}\cdot\mathbf{j} : \mathbf{j} \in J\}$. Note that $\mathbf{k}$ is unique due to the linear $\mathbb{Q}$-independence of the c_j's. Then $\mathbf{c}\cdot\mathbf{j} - \mathbf{k}\cdot\mathbf{c} < 0$ for all $\mathbf{j} \in J \setminus \{\mathbf{k}\}$. Of course, $\mathbf{j}\cdot\mathbf{c} > 0$ for all $\mathbf{j} \in J$. Finally, the conclusion of (b) is obtained from the inequality

$$|\varphi(x)| \ge e^{\mathbf{c}\cdot\mathbf{k}\,x} \cdot \Big(|\alpha_{\mathbf{k}}| - \sum_{\mathbf{j}\in J\setminus\{\mathbf{k}\}} |\alpha_{\mathbf{j}}|\, e^{(\mathbf{c}\cdot\mathbf{j} - \mathbf{c}\cdot\mathbf{k})\,x} \Big)$$

and the facts that $\lim_{x\to+\infty} e^{\mathbf{c}\cdot\mathbf{k}\,x} = +\infty$ and $\lim_{x\to+\infty} e^{(\mathbf{c}\cdot\mathbf{j} - \mathbf{k}\cdot\mathbf{c})\,x} = 0$ for all $\mathbf{j} \in J \setminus \{\mathbf{k}\}$. □

3. Topological and algebraic properties of the set of non-uniformly continuous functions

In this section, we shall be able to find a rich algebraic structure inside the subset of non-uniformly continuous functions, assuming that the space of departure is non-Atsuji and satisfies appropriate conditions. Concerning topological size, the next theorem shows the denseness of the mentioned subset and the existence of some "reasonable" structure in it.

Recall that if Z is a topological space and $A \subset Z$, then A is called a G_δ set if A is a countable intersection of open subsets of Z, an F_σ set if it is a countable union of closed subsets of Z, a $G_{\delta\sigma}$ set if it is a countable union of G_δ subsets, and an $F_{\sigma\delta}$ set if it is a countable intersection of F_σ subsets. Recall that τ_K denotes compact-open topology. The first three parts of the following theorem are surely known, but since we have not been able to find a reference we provide a proof of them.

Theorem 3.1. *Assume that (X,d) and (Y,ρ) are metric spaces, and that X is locally compact and σ-compact. We have:*

(a) *$UC(X,Y)$ is an $F_{\sigma\delta}$ subset of $(C(X,Y),\tau_K)$.*
(b) *$nUC(X,Y)$ is a $G_{\delta\sigma}$ subset of $(C(X,Y),\tau_K)$.*
(c) *If Y is either an absolute extensor or a Fréchet space over $\mathbb{R}$ then $UC(X,Y)$ is dense in $(C(X,Y),\tau_K)$. In particular, $UC(X)$ is dense in $(C(X),\tau_K)$.*
(d) *If (X,d) is a non-Atsuji space and Y is a Fréchet space over $\mathbb{R}$ whose metric ρ is induced by an F-norm defining the topology of Y, then $nUC(X,Y)$ is a dense subset of $(C(X,Y),\tau_K)$. In particular, $nUC(X)$ is dense in $(C(X),\tau_K)$.*

Proof. (a) Note first that the set $UC(X,Y)$ can be written as

$$UC(X,Y) = \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} F_{n,k}, \tag{1}$$

where $F_{n,k} := \{f \in C(X,Y) : \rho(f(x), f(y)) \le \frac{1}{n}$ for all $(x,y) \in X \times X$ with $d(x,y) \le \frac{1}{k}\}$. If $(f_j) \subset F_{n,k}$ is a sequence tending to some f of the (metrizable, by Theorem 2.1) space $C(X,Y)$, then from the continuity of the map $\rho : Y \times Y \to \mathbb{R}$ and the fact that τ_K-convergence implies pointwise convergence it follows that $f \in F_{n,k}$. Hence each set $F_{n,k}$ is closed, which together (1) tells us that $UC(X,Y)$ is an $F_{\sigma\delta}$ set.

(b) This is a consequence of (a) because the complement of an $F_{\sigma\delta}$ subset is a $G_{\delta\sigma}$ subset.

(c) Recall that the sets

$$V = V(f, L, \varepsilon) := \{g \in C(X,Y) : \rho(f(x), g(x)) < \varepsilon \text{ for all } x \in L\}$$

($f \in C(X,Y)$, $\varepsilon > 0$, K a compact subset of X) form a basis for the τ_K-topology of $C(X,Y)$. Fix such a subset V. In order to prove the denseness of $UC(X,Y)$, we search for some $g \in U \cap UC(X,Y)$. With this aim, consider the sequence $\{U_n\}_{n\ge1}$ of open sets such that $X = \bigcup_{n\ge1} U_n$, each $\overline{U_n}$ is compact and $\overline{U_n} \subset U_{n+1}$ ($n \in \mathbb{N}$) provided by Theorem 2.1(b). Since $\{U_n\}_{n\ge1}$ is in particular increasing, we can select an $m \in \mathbb{N}$ with $K \subset U_m$. Now, X is normal because it is metrizable. Assume that Y is an absolute extensor. Fix any point $y_0 \in Y$. Since the closed sets $\overline{U_m}$, $X \setminus U_{m+1}$ are disjoint, the function $\overline{U_m} \cup (X \setminus U_{m+1}) \to \mathbb{R}$ given by f on $\overline{U_m}$ and y_0 on $X \setminus U_{m+1}$ is well defined and continuous. Then there exists a function $g \in C(X,Y)$ such that

$$g|_{\overline{U_m}} = f \quad \text{and} \quad g|_{X\setminus U_{m+1}} = y_0.$$

The same conclusion holds if Y is a real Fréchet space because any metrizable space is paracompact (see, e.g., [40, Theorem 41.4]), and any continuous function on a closed subset (with the restricted topology) of a Hausdorff paracompact space Z with values in a real Fréchet space extends to a continuous function on Z (see [36, X.3.33]). To summarize, we have that g is continuous, $\overline{U_{m+1}}$ is compact, $g|_{X\setminus U_{m+1}}$ is constant (so $g|_{X\setminus U_{m+1}}$ is uniformly continuous) and $X = \overline{U_{m+1}} \cup (X \setminus U_{m+1})$. Therefore Lemma 2.2 applies and we conclude that $g \in UC(X,Y)$. But $g = f$ on K because $\overline{U_m} \supset U_m \supset K$. Thus, trivially, $g \in V$ and $g \in V \cap UC(X,Y)$, as required.

(d) By assumption, we have that $\rho(x,y) = \|x - y\|$, where $\|\cdot\|$ is an F-norm on Y defining its topology. From the hypothesis we can select a continuous function $f_0 : X \to \mathbb{R}$ that is not uniformly continuous. Fix any nonzero vector $y_0 \in Y$, and consider the mapping $f : X \to Y$ given by $f(x) := f_0(x)y_0$. It is evident that $f \in C(X,Y)$. Then $f \in nUC(X,Y)$: indeed, there are sequences (u_n) and (v_n) in X and an $\alpha > 0$ such that

$$d(u_n, v_n) \to 0 \text{ as } n \to \infty \quad \text{but} \quad |f_0(u_n) - f_0(v_n)| \ge \alpha \text{ for all } n \in \mathbb{N}.$$

Since $\|\cdot\|$ is an F-norm and $|\alpha/(f_0(u_n)-f_0(v_n))|\le 1$, it follows that

$$\rho(f(u_n),f(v_n))=\|f(u_n)-f(v_n)\|=\|(f_0(u_n)-f_0(v_n))\,y_0\|\ge\|\alpha\,y_0\|>0$$

for all $n\in\mathbb{N}$, which proves the claim. Now, from the statement (c) we have that $UC(X,Y)$ is dense in $(C(X,Y),\tau_K)$. But $C(X,Y)$ is a vector space and $UC(X,Y)$ is a vector subspace of $C(X,Y)$, which implies that the set of translates

$$f+UC(X,Y):=\{f+g:\, g\in UC(X,Y)\}$$

satisfies $f+UC(X,Y)\subset nUC(X,Y)$. Since $(C(X,Y),\tau_K)$ is a topological vector space, we conclude that the translation

$$g\in C(X,Y)\longmapsto f+g\in C(X,Y)$$

is a homeomorphism, from which it follows that $f+UC(X,Y)$ is dense in $(C(X,Y),\tau_K)$, and so its superset $nUC(X,Y)$ is also dense. □

The next two theorems assert algebrability, dense-lineability and spaceability for the set $nUC(X)$ in a high degree. No extra condition will be needed to guarantee algebrability. Recall that a subset A of a topological space X is said to be *discrete* whenever $A'=\varnothing$ or, that is equivalent, the restricted topology on A is the discrete one.

Theorem 3.2. *Let (X,d) be a non-Atsuji metric space. Then the set $nUC(X)$ is strongly $\mathfrak{c}$-algebrable.*

Proof. We are going to built a non-uniformly continuous function $f:X\to\mathbb{R}$ satisfying that there exist three mutually disjoint sequences (x_n), (y_n), $(z_n)\subset X$ such that $f(x_n)=n$, $f(y_n)=n+1, f(z_n)=\frac{1}{n}$ for all $n\in\mathbb{N}$ and $d(x_n,y_n)\to 0$ as $n\to\infty$. Assume that this has already been done. Let Φ be the family furnished in Lemma 2.6, and take $\varphi\in\Phi\setminus\{0\}$. According to assertions (c) and (d) of the mentioned lemma, we have that $\lim_{x\to+\infty}|\varphi'(x)|=+\infty$. Then there exists $x_0\in\mathbb{R}$ with $|\varphi'(x)|\ge 1$ for all $x\ge x_0$. It follows from the mean value theorem that

$$|(\varphi\circ f)(x_n)-(\varphi\circ f)(y_n)|=|\varphi(n)-\varphi(n+1)|\ge 1\cdot|n-(n+1)|=1$$

provided that $n\ge x_0$. Therefore

$$d(x_n,y_n)\to 0 \;\text{ but }\; d_e((\varphi\circ f)(x_n),(\varphi\circ f)(y_n))\not\longrightarrow 0 \;\text{ as }\; n\to\infty.$$

This tells us that $\varphi\circ f\in nUC(X)$. Observe that, since $f(z_n)=\frac{1}{n}$ $(n\in\mathbb{N})$, the set $f(X)$ has 0 as a limit point. It now suffices to apply Lemma 2.4 with $\Omega:=X$, $\alpha:=\mathfrak{c}$ and $\mathcal{F}:=nUC(X)$.

Thus, our unique task is to construct a function f as claimed at the beginning of the proof. According to Theorem 1.1, we have that at least one of the following assertions holds:

(i) The set X' is not compact.
(ii) There exists $\alpha>0$ such that, for every $\delta>0$, there are $x,y\in X$ with $d(x,X')\ge\alpha\le d(y,X')$ and $0<d(x,y)<\delta$.

Assume first that (i) is true. Then there exists an infinite set $S \subset X'$ without limit points in X', so without limit points in X, because X' is always closed. Then we can select a countable set $A = \{u_n : n \in \mathbb{N}\} \subset S$ with the u_n's mutually different. Let us set $x_n := u_{2n-1}$, $z_n := u_{2n}$ $(n \in \mathbb{N})$, $B := \{x_n\}_{n \geq 1}$ and $C := \{z_n\}_{n \geq 1}$. Then A, B and C are closed sets with $A' = B' = \varnothing = C'$ and $B \cap C = \varnothing$. Define, for each $n \in \mathbb{N}$, the number

$$\gamma_n := \min\Big\{\frac{1}{n}, \frac{1}{3}\, d\big(x_n, A \setminus \{x_n\}\big)\Big\}.$$

Note that $\gamma_n > 0$. Since $x_n \in X'$, there exists $y_n \in X$ such that $y_n \neq x_n$ and $d(x_n, y_n) < \gamma_n$. The triangle inequality implies that $d(y_n, x) \geq \gamma_n$ for all $x \in (A \setminus \{x_n\}) \cup \big\{y_k : k \in \mathbb{N} \setminus \{n\}\big\}$. In particular, the y_n's are pairwise distinct, the set $D := \{y_n : n \in \mathbb{N}\}$ is closed, $D' = \varnothing$ and $D \cap A = \varnothing$. Then the set $E := A \cup D$ is closed and $E' = \varnothing$. Hence any function defined on E (with the restricted topology) is continuous. Now, it is enough to extend –via Tietze's extension theorem– continuously to the whole X the function $f : E \to \mathbb{R}$ given by $f(x_n) := n$, $f(y_n) := n + 1, f(z_n) := \frac{1}{n}$ $(n \in \mathbb{N})$.

Finally, suppose that (ii) holds. Consider the number $\alpha > 0$ furnished by (ii). On the one hand, the condition on α entails that the set $F := \{x \in X : d(x, X') \geq \alpha\}$ (where it is understood that $F = X$ if $X' = \varnothing$) is infinite: otherwise, just take $\delta := \min\{d(x, y) : x, y \in F, x \neq y\}$ to arrive at a contradiction. The continuity of the mapping $x \in X \mapsto d(x, X') \in \mathbb{R}$ shows that F is closed. On the other hand, the set F is discrete because it is closed (so $F' \subset F$) and therefore $F' \subset F \cap X' = \varnothing$. Observe that every subset of F is discrete and hence closed. We begin by choosing points $x_1, y_1 \in F$ with $0 < d(x_1, y_1) < 1$. Since F is infinite, we can pick $z_1 \in F \setminus \{x_1, y_1\}$. Now, we proceed by induction. Assume that $n \in \mathbb{N}$ and that mutually different points $x_1, y_1, z_1, \ldots, x_n, y_n, z_n$ have been selected in F so as to satisfy $d(x_k, y_k) < \frac{1}{k}$ for all $k \in \{1, 2, \ldots, n\}$. Let us set $G := \{x_1, y_1, z_1, \ldots, x_n, y_n, z_n\}$ and

$$\delta_j := \min\Big\{\frac{1}{(n+1)j}, \min\{d(x, y) : x, y \in G, x \neq y\}\Big\} > 0 \quad (j \in \mathbb{N}).$$

According to (ii), for each $j \in \mathbb{N}$ we can find points $u_j, v_j \in F$ such that $0 < d(u_j, v_j) < \delta_j$. Then, by the choice of δ_j, the set $\{u_j, v_j\} \cap G$ is either empty or a singleton. Assume, by way of contradiction, that $\{u_j, v_j\} \cap G$ is a singleton for all $j \in \mathbb{N}$, say, $\{u_j, v_j\} \cap G = \{u_j\}$. Since G is finite, at least one $w \in G$ equals u_j for infinitely many $j \in \mathbb{N}$. That is, there is a sequence $\{j_1 < j_2 < \cdots < j_k < \cdots\} \subset \mathbb{N}$ such that $v_{j_k} \in B_d(w, \delta_{j_k}) \setminus \{w\}$. As $\delta_{j_k} \to 0$, we conclude that $w \in F'$, which is absurd. Consequently, there must be an $m \in \mathbb{N}$ such that $\{u_m, v_m\} \cap G = \varnothing$. If we define $x_{n+1} := u_m$, $y_{m+1} := v_m$ and then we choose any $z_{n+1} \in F \setminus (G \cup \{x_{n+1}, y_{m+1}\})$, we get $3(n + 1)$ mutually different points $x_1, y_1, z_1 \ldots, x_{n+1}, y_{n+1}, z_{n+1} \in F$ with $d(x_k, y_k) < \frac{1}{k}$ for all $k \in \{1, 2, \ldots, n, n + 1\}$. Hence, we obtain three countably infinite, mutually disjoint sets

$$H := \{x_n : n \in \mathbb{N}\}, \ I := \{y_n : n \in \mathbb{N}\} \ \text{ and } \ J := \{z_n : n \in \mathbb{N}\}$$

satisfying $d(x_n, y_n) \to 0$ as $n \to \infty$. Moreover, all of them are discrete, which implies that the set $K := H \cup J \cup I$ is (closed and) discrete. Finally, the claimed function f can be obtained as in the end of case (i). □

The following result studies the algebraic-topological size of $nUC(X)$ as a subset of $C(X)$.

Theorem 3.3. *Assume that (X, d) is a locally compact σ-compact non-Atsuji metric space, and endow the space $C(X)$ with the compact-open topology τ_K. Then the following holds:*

(a) *The set $nUC(X)$ is $\mathfrak{c}$-dense-lineable in $C(X)$.*
(b) *The set $nUC(X)$ is spaceable in $C(X)$.*

Proof. (a) According to Theorem 3.1(c), the set $UC(X)$ is dense in $C(X)$. It follows from Theorem 3.2 that $nUC(X)$ is strongly $\mathfrak{c}$-algebrable, hence $\mathfrak{c}$-lineable. It is well known (see, e.g., [17, p. 198]) that if (K, d) is a compact metric space then $C(K)$ is separable under the uniform metric. From this fact and Tietze's extension theorem (as applied to an exhaustive sequence of compact subsets $K \subset X$), it is easily derived that the metrizable space $(C(X), \tau_K)$ is separable as well. Since $UC(X)$ is a vector space, we get $nUC(X) + UC(X) \subset nUC(X)$. The conclusion follows from Theorem 2.3 as soon as we take $Z = C(X)$, $A = nUC(X)$, $B = UC(X)$ and $\gamma = \mathfrak{c}$.

(b) As in the proof of Theorem 3.2, we can construct two disjoint sequences (x_n) and (y_n) of mutually distinct points of X such that $S := \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$ is a discrete subset and $d(x_n, y_n) \to 0$ as $n \to \infty$. Then we can select pairwise disjoint open sets W_n such that $x_n, y_n \in W_n$ $(n \in \mathbb{N})$. Choose a countable family $\{N_k\}_{k \geq 1}$ of infinite, mutually disjoint subsets of $\mathbb{N}$. For each $k \in \mathbb{N}$ consider the closed sets $S_k := \{x_n : n \in N_k\} \cup \{y_n : n \in N_k\}$, $C_k := X \setminus \bigcup_{n \in N_k} W_n$ and $F_k := S_k \cup C_k$. Since $S_k \cap C_k = \varnothing$ and any function on a discrete set is continuous, we have that the function $g_k : F_k \to \mathbb{R}$ given by

$$g_k(x) = \begin{cases} n & \text{if } x = x_n, \ n \in N_k \\ n + 1 & \text{if } x = y_n, \ n \in N_k \\ 0 & \text{if } x \in C_k \end{cases}$$

is continuous.

By Tietze's extension theorem, there is a function $f_k \in C(X)$ such that $f_k|_{F_k} = g_k$. For each function $f : X \to \mathbb{R}$ we consider its (set-theoretical) support $\text{supp}(f) := \{x \in X : f(x) \neq 0\}$. Since $f_k = g_k = 0$ on C_k, it follows that $\text{supp}(f_k) \subset \bigcup_{n \in N_k} W_n$. The fact that the sets W_n's are pairwise disjoint entails that the $\text{supp}(f_k)$'s are also mutually disjoint. Fix a nonempty compact set $K \subset X$. Let $p < q$ be natural numbers and $c_1, c_2, \ldots, c_q$ be prescribed real scalars. Fix $x \in K$. Then either $f_j(x) = 0$

for all $j \in \{1,\dots,p\}$ or there is exactly one $m \in \{1,\dots,p\}$ such that $f_m(x) \neq 0$ (in which case $f_j(x) = 0$ for all $j \in \{p+1,\cdots,q\}$). In both cases, we deduce $|\sum_{j=1}^p c_j f_j(x)| \le |\sum_{j=1}^q c_j f_j(x)|$. It follows that there exists a constant $C_K \in (0,+\infty)$ (in fact, $C_K = 1$) such that $\|\sum_{j=1}^p c_j f_j\|_K \le C_K \cdot \|\sum_{j=1}^q c_j f_j\|_K$, where $\|h\|_K := \sup_{z\in K} |h(x)|$. Since the collection of seminorms $\{\|\cdot\|_K : K \subset X \text{ compact}\}$ defines the Fréchet topology of $C(X)$, we obtain via Nikolskii's theorem for Fréchet spaces (see, e.g., [31, Theorem 5.1.8, p. 67]) that $\{f_k\}_{k\in\mathbb{N}}$ is a basic sequence in $C(X)$. Therefore $M := \overline{\operatorname{span}}\{f_k\}_{k\in\mathbb{N}}$ –its closed linear span– equals the set of functions of the form

$$\sum_{k=1}^{\infty} \alpha_k f_k \quad \text{with} \quad \alpha_k \in \mathbb{R} \quad \text{for all} \quad k \in \mathbb{N} \tag{2}$$

that are τ_K-convergent in $C(X)$. Note that τ_K-convergence implies pointwise convergence and that, given $x \in X$, each series as in (2) reduces to at most one nonzero term when evaluated at x. Note also that M is a closed vector space and, since the f_k's are linearly independent (this follows from the disjointness of their supports), we get that M is infinite dimensional.

It remains to prove that $M \setminus \{0\} \subset nUC(X)$. For this, fix $F \in M \setminus \{0\}$, so that $F = \sum_{k=1}^\infty \alpha_k f_k$, a series as in (2) being τ_K-convergent. Let $m := \min\{k \in \mathbb{N} : \alpha_k \neq 0\}$, $N_m = \{n_1 < n_2 < \cdots < n_j < \cdots\}$, $z_j := x_{n_j}$ and $u_j := y_{n_j}$. Since $u_j, v_j \notin \operatorname{supp}(f_k)$ ($j \in \mathbb{N}$, $k \neq m$) and $d(x_n, y_n) \to 0$ as $n \to \infty$, we derive $d(z_j, u_j) \to 0$ and

$$|F(z_j) - F(u_j)| = |c_k||f_k(x_{n_j}) - f_k(y_{n_j})| = |c_k||n_j + 1 - n_j| = |c_k| \not\longrightarrow 0$$

as $j \to \infty$, from which we conclude that $F \in nUC(X)$. □

4. Topological and algebraic properties of the set of unbounded continuous functions

As in the case of non-uniformly continuous functions on non-Atsuji spaces, we can find, under appropriate conditions, many unbounded continuous functions, in both topological and algebraical senses. Our results are collected in the following two theorems. Sometimes, the proofs will follow patterns that are similar to those of the corresponding theorems on non-uniform continuity.

The first result concerns algebraic size, in which case not any special condition on the departure space X –apart from its non-compactness– is needed.

Theorem 4.1. *Let X be a non-compact metrizable space. Then the set $nBC(X)$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Since X is not compact, Hewitt's theorem [23] guarantees the existence of a function in $nBC(X)$. However, we need a modification of the original proof of [23] in order to construct such a function with additional properties. If fact, our proof has some items similar to the one of Theorem

3.2, but somewhat less involved. Since X is metrizable and noncompact, there is an infinite subset $A \subset X$ with $A' = \varnothing$. Then we can extract from A two disjoint countably infinite subsets $B = \{x_n : n \in \mathbb{N}\}$ and $C = \{y_n : n \in \mathbb{N}\}$ (with the x_n's pairwise different, and the same for the y_n's). Therefore $B' = \varnothing = C'$, and so B and C are closed sets. Hence $B \cup C$ is a closed set carrying the discrete topology. Then any function on $B \cup C$ is continuous. In particular, the function $g : B \cup C \to \mathbb{R}$ given by

$$g(x) = \begin{cases} n & \text{if } x = x_n \\ \frac{1}{n} & \text{if } x = y_n \end{cases}$$

is continuous. Then Tietze's extension theorem guarantees the existence of a function $f \in C(X)$ such that $f|_{B \cup C} = g$. Consequently, for such a function f we have got sequences $(x_n), (y_n) \subset X$ such that $f(x_n) \to +\infty$ and $f(y_n) = \frac{1}{n} \to 0$ as $n \to \infty$. Take the algebra Φ furnished by Lemma 2.6. It follows from the statement (c) of this lemma that $|\varphi(f(x_n))| \to +\infty$ as $n \to \infty$ for every $\varphi \in \Phi \setminus \{0\}$. Therefore $\varphi \circ f \in nBC(X)$ for such functions φ. Moreover, the set $f(X)$ has 0 as a limit point because $f(X) \supset \{\frac{1}{n} : n \in \mathbb{N}\}$. Finally, it is enough to apply Lemma 2.4 with the characters $\Omega := X$, $\alpha := \mathfrak{c}$ and $\mathcal{F} := nBC(X)$. □

Recall that a subset of a Baire space is called *residual* whenever its complement is of the Baire first category or, equivalently, whenever it contains some dense G_δ subset.

Theorem 4.2. *Assume that X is a locally compact σ-compact non-compact metrizable space, and endow the space $C(X)$ with the compact-open topology τ_K. Then the following holds:*

(a) *The set $nBC(X)$ is residual in $C(X)$.*
(b) *The set $nBC(X)$ is $\mathfrak{c}$-dense-lineable in $C(X)$.*
(c) *The set $nBC(X)$ is spaceable in $C(X)$.*

Proof. (a) Recall that $(C(X), \tau_K)$ is a completely metrizable space, hence a Baire space. Observe that $BC(X) = \bigcup_{n \in \mathbb{N}} F_n$, where $F_n := \{f \in C(X) : |f(x)| \le n \text{ for all } x \in X\}$. Since τ_K-convergence implies pointwise convergence, we derive that each set F_n is closed. Therefore

$$nBC(X) = \bigcap_{n \in \mathbb{N}} G_n,$$

where every $G_n := F_n^c$ is open. This proves that $nBC(X)$ is a G_δ subset. In order to see that $nBC(X)$ is residual, it is enough to show that it is dense. To this aim, we shall prove that $BC(X)$ is dense. This can be done in the same was as the proof of part (c) in Theorem 3.1, just by replacing Y, ρ and y_0 by $\mathbb{R}$, d_e and 0, respectively. Indeed, the continuous function $g : X \to \mathbb{R}$ resulting with this approach is 0 outside a compact set (so bounded) and satisfies $g \in U$, where U is a prefixed basic open set for τ_K. On the one hand, since X is not compact, Hewitt's theorem [23] guarantees the existence of a function $h \in nBC(X)$. On the other hand, $BC(X)$ is a vector subspace of $C(X)$, which implies

$$h + BC(X) \subset nBC(X,Y).$$

Since the translation mapping $g \in C(X) \mapsto h + g \in C(X)$ is a homeomorphism from the topological vector space $C(X)$ into itself, it follows that the set $h + BC(X)$ (and so $nBC(X)$) is dense in $C(X)$.

(b) According to Theorem 4.1, $nBC(X)$ is strongly $\mathfrak{c}$-algebrable, hence $\mathfrak{c}$-lineable. Now, the proof of (a) shows that the set $BC(X)$ is dense in $C(X)$. Moreover it is a vector space, hence $f + g \in nBC(X)$ if $f \in nBC(X)$ and $g \in BC(X)$. Of course, $BC(X) \cap nBC(X) = \varnothing$. Then it suffices to apply Theorem 2.3 with $Z = C(X)$, $A = nUC(X)$, $B = UC(X)$ and $\gamma = \mathfrak{c}$.

(c) According to Theorem 2.1(b), there is a sequence $\{O_n\}_{n\geq 1}$ of open sets such that $X = \bigcup_{n\geq 1} O_n$, each $\overline{O_n}$ is compact and $\overline{O_n} \subset O_{n+1}$ for all $n \in \mathbb{N}$. Since X is not compact, there exists an infinite set $A \subset X$ such that $A' = \varnothing$. In particular, A is closed. Let $m(1)$ be the first $m \in \mathbb{N}$ with $A \cap O_m \neq \varnothing$, and choose $x_1 \in A \cap O_{m(1)}$. Since $\overline{O_{m(1)}}$ is compact, the set $(A \setminus \{x_1\}) \cap \overline{O_{m(1)}}$ is finite, so that we can take $m(2) \in \mathbb{N}$ satisfying

$$m(2) = \min \big\{ m \in \mathbb{N} : \, m > m(1) \text{ and } (A \setminus \{x_1\}) \cap O_m \setminus \overline{O(m_1)} \neq \varnothing \big\}.$$

Choose $x_2 \in (A \setminus \{x_1\}) \cap (O_{m(2)} \setminus \overline{O(m_1)})$. By this procedure, we can obtain recursively a strictly increasing sequence $\{m(n)\}_{n\geq 1} \subset \mathbb{N}$ as well as a countable set $B := \{x_n : \, n \in \mathbb{N}\} \subset A$ such that the x_n's are pairwise different and $x_n \in V_n$ for all $n \in \mathbb{N}$, where we have set $V_n := U_n \setminus \overline{U_{n-1}}$ and $U_n := O_{m(n)}$ (with the convention $U_0 := \emptyset$). Note that $B' = \varnothing$ and, in particular, B (and any subset of B) is closed. Note also that (U_n) is a sequence of open sets such that $X = \bigcup_{n\geq 1} U_n$, each $\overline{U_n}$ is compact and $\overline{U_n} \subset U_{n+1}$ for all $n \in \mathbb{N}$. Observe that the open sets V_n satisfy $V_m \cap V_n = \varnothing$ if $m \neq n$. Since every metrizable space is regular, we can select, for each $n \in \mathbb{N}$, an open set W_n satisfying

$$x_n \in W_n \subset \overline{W_n} \subset V_n.$$

Choose a countable family $\{N_k\}_{k\geq 1}$ of infinite, mutually disjoint subsets of $\mathbb{N}$. For each $k \in \mathbb{N}$, we set $F_k := \{x_n : \, n \in N_k\} \cup \big(X \setminus \bigcup_{n \in N_k} W_n\big)$, which is closed. Let us define the function $g_k : F_k \to \mathbb{R}$ by

$$g_k(x) = \begin{cases} n & \text{if } x = x_n \text{ with } n \in N_k \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{x_n : \, n \in N_k\}$ and $X \setminus \bigcup_{n \in N_k} W_n$ are disjoint closed sets and g_k is continuous on each of them, we deduce that $g_k \in C(F_k)$. Once again, Tietze's extension theorem comes in our help, so as to produce a function $f_k \in C(X)$ such that $f_k|_{F_k} = g_k$. It follows that $f_k = 0$ on $X \setminus \bigcup_{n \in N_k} W_n$, and so $\operatorname{supp}(f_k) \subset \bigcup_{n \in N_k} W_n$. The fact that the sets W_n's are pairwise disjoint (as $W_n \subset V_n$) entails that the supports $\operatorname{supp}(f_k)$ ($k \in \mathbb{N}$) are also mutually disjoint. From Nikolskii's theorem for Fréchet spaces, we derive as in the proof of Theorem 3.3(b) that $\{f_k\}_{k\in\mathbb{N}}$ is a basic sequence in $C(X)$. Therefore, its closed linear span $M := \overline{\text{span}}\,\{f_k\}_{k\in\mathbb{N}}$ equals the set of functions of the form

$$\sum_{k=1}^{\infty} \alpha_k f_k \quad \text{with} \quad \alpha_k \in \mathbb{R} \quad \text{for all} \quad k \in \mathbb{N} \tag{3}$$

that are τ_K-convergent in $C(X)$. Then M is a closed vector space and the disjointness of the supp(f_k)'s shows once again that M is infinite dimensional. It is then enough to prove that every $F \in M \setminus \{0\}$ belongs to $nBC(X)$. This is easy, because such an F has the form (3) with some $\alpha_m \neq 0$. But $F(x_n) = \alpha_m \cdot f_m(x_n) = \alpha_m \cdot n$ for $n \in N_m$. The set N_m being infinite, one derives that F is unbounded. □

5. Final remarks

1. Recall that if a metric space (X, d) is non-Atsuji then it is non-compact. So, according to Theorem 4.1, $nBC(X)$ is strongly $\mathfrak{c}$-algebrable. Moreover, $nUC(X)$ is also strongly $\mathfrak{c}$-algebrable due to Theorem 3.2. But a glance to the proof of this theorem shows that, in fact, the set

$$nBC(X) \cap nUC(X)$$

of *unbounded, non-uniformly continuous* functions $X \to \mathbb{R}$ is strongly $\mathfrak{c}$-algebrable. Similar conclusions follow from Theorems 3.3 and 4.2 regarding dense-lineability and spaceability. Concerning this double property of unboundedness plus nonuniform continuity, and in the specific case in which $(X, d) = (\mathbb{R}, d_e)$, Moothathu [38, Theorem 3.7] has recently proved that, given an unbounded uniformly continuous function $g : \mathbb{R} \to \mathbb{R}$, the collection of all bounded, uniformly continuous functions $f : \mathbb{R} \to \mathbb{R}$ having the property that $f \cdot g \in nBC(\mathbb{R}) \cap nUC(\mathbb{R})$ is $\mathfrak{c}$-lineable.

2. In the case $(X, d) = (\mathbb{R}, d_e)$, we can even assert a little more about the algebrability of the set of unbounded non-uniformly continuous functions. Namely, the set $nBC(\mathbb{R}) \cap nUC(\mathbb{R})$ is *densely strongly* $\mathfrak{c}$*-algebrable*. This means that there exists a *dense* freely $\mathfrak{c}$-generated algebra all of whose nonzero members belong to $nBC(\mathbb{R}) \cap nUC(\mathbb{R})$. Indeed, consider the freely $\mathfrak{c}$-generated Φ constructed in Lemma 2.6. Recall that Φ was generated by the functions $\varphi_c(x) = e^{cx}$ $(c \in H)$, where H is an appropriate subset of $(0, +\infty)$. The set of these functions is separating, that is, given $x \neq y$ in $\mathbb{R}$, at least one of these functions f satisfies $f(x) \neq f(y)$: simply take $f = \varphi_c$ with c any point in H. Then the Stone–Weierstrass theorem (see, e.g., [24, p. 425]) guarantees that the algebra Φ is dense in $C([a, b])$ (endowed with the topology of uniform convergence) for every interval $[a, b] \subset \mathbb{R}$. Since convergence in $C(\mathbb{R})$ is equivalent to convergence in each $C([a, b])$, we obtain the desired result.

3. Finally, and turning back to a general locally compact σ-compact non-Atsuji metric space (X, d), we also can find large algebraic structures in the family of *bounded* non-uniformly continuous functions $BC(X) \cap nUC(X)$. Namely, this set is *spaceable* in the (Banach) space $BC(X)$ endowed with the supremum norm $\|\cdot\|_\infty$, and $\mathfrak{c}$-*dense-lineable* in $(C(X), \tau_K)$. In order to see this, notice that if in the proof of Theorem 3.3(b) we had take the functions $g_k : F_k \to \mathbb{R}$ $(k \in \mathbb{N})$ given by

$$g_k(x) = \begin{cases} 0 & \text{if } x = x_n,\ n \in N_k \\ 1 & \text{if } x = y_n,\ n \in N_k \\ 0 & \text{if } x \in C_k \end{cases}$$

instead of the ones provided in the mentioned proof, then we would have obtained extensions $f_k \in C(X)$ that are bounded (in fact, with $\|f_k\|_\infty = \sup\{|g_k(x)| : x \in F_k\} = 1$) thanks to Tietze's extension theorem for bounded functions (see, e.g., [15, p. 91]). An application of Nikolskii's theorem for Banach spaces (see, e.g., [19, pp. 36–38]) leads easily to the conclusion that (f_k) is a basic sequence. Therefore its closed linear span M in $(BC(X), \|\cdot\|_\infty)$ happens to be an infinite dimensional closed vector subspace, and arguing as in the ending part of the proof of Theorem 3.3(b) one can conclude that $M \setminus \{0\} \subset nUC(X)$. This shows the claimed spaceability. Now, a well-known consequence of the Baire category theorem tells us that $\dim(M) = \mathfrak{c}$. Hence $A := BC(X) \cap nUC(X)$ is γ-lineable, where $\gamma := \mathfrak{c}$. By Theorem 3.1(c), the set $UC(X)$ is dense in $(C(X), \tau_K)$, but an analysis of its proof reveals –again via Tietze's extension theorem for bounded functions– that the approximating function $g \in V \cap UC(X, Y)$ (with $Y = \mathbb{R}$) can be taken bounded. In other words, the set $B := BC(X) \cap UC(X)$ is dense in $C(X)$. Moreover, since $BC(X)$ and $UC(X)$ are vector spaces, we obtain that B is a dense vector subspace of $Z := C(X)$ and $A + B \subset A$. Of course, $A \cap B = \varnothing$. The claimed $\mathfrak{c}$-dense-lineability of A follows after an application of Lemma 2.3.

We want to pose here the following question:

Is $BC(X) \cap nUC(X)$ *(strongly) algebrable?*

Acknowledgements. The authors are indebted to the referee for helpful comments and suggestions. The first and fourth authors have been supported by the Plan Andaluz de Investigación de la Junta de Andalucía FQM-189 and by MEC Grant MTM2015-65397-P. The second and third authors have been supported by the Plan Andaluz de Investigación de la Junta de Andalucía FQM-127 Grant P08-FQM-03543 and by MEC Grant MTM2015-65242-C2-1-P.

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Departamento de Geometría y Topología
Facultad de Matemáticas
Universidad de Sevilla
Avenida Reina Mercedes, Sevilla, 41012, Spain
E-mail address: rdayala@us.es

Departamento de Análisis Matemático
Facultad de Matemáticas
Instituto de Matemáticas Antonio de Castro Brzezicki (IMUS)
Universidad de Sevilla
Avenida Reina Mercedes, Sevilla, 41012, Spain
E-mail address: lbernal@us.es

Departamento de Análisis Matemático
Facultad de Matemáticas
Instituto de Matemáticas Antonio de Castro Brzezicki (IMUS)
Universidad de Sevilla
Avenida Reina Mercedes, Sevilla, 41012, Spain
E-mail address: mccm@us.es

Departamento de Geometría y Topología
Facultad de Matemáticas
Universidad de Sevilla
Avenida Reina Mercedes, Sevilla, 41012, Spain
E-mail address: vilches@us.es