Large linear manifolds of non-continuable boundary-regular holomorphic functions

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Abstract

We prove in this paper that if G is a domain in the complex plane satisfying adequate topological or geometrical conditions then there exists a large (dense or closed infinite-dimensional) linear submanifold of boundary-regular holomorphic functions on G all of whose nonzero members are not continuable across any boundary point of G.

Key words and phrases: non-continuable holomorphic function, large linear manifold, boundary-regular function, Faber transform, universal sequence.

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1   Introduction

In 1884 Mittag-Leffler discovered that, given any domain (= nonempty connected open subset) G in the complex plane $\mathbb{C}$, there exists a function $f \in H(G)$ (= the set of holomorphic functions on G) having G as its domain of holomorphy, see [13, Chapter 10]. Recall that G is said to be a domain of holomorphy for f if f is holomorphic exactly on G, that is, $f \in H(G)$ and f is analytically non-continuable across the boundary ∂G of G or, more precisely, for every $a \in G$, the radius of convergence $\rho(f, a)$ of the Taylor series of f with center at a equals the euclidean distance $d(a, \partial G)$ between a and ∂G.

It is well known that $H(G)$ becomes a Fréchet space (= completely metrizable locally convex space) when endowed with the topology of uniform convergence on

compacta; in particular, it is a Baire space. The symbol $H_e(G)$ will stand for the subclass of functions which are holomorphic exactly on G. It is clear that $H_e(G) \subset \{f \in H(G) : \ f$ has no holomorphic extension to any domain containing G strictly$\}$. In general, the last inclusion is strict (take $G := \mathbb{C}\backslash(-\infty, 0]$ and $f :=$ the principal branch of the logarithm at G), but in many cases –for instance, if G is a Jordan domain– both sets are equal. By a Jordan domain we understand as usual the bounded component of the complement of a Jordan curve, and a Jordan curve is a topological image in $\mathbb{C}$ of $\mathbb{T} = \{z \in \mathbb{C} : \ |z| = 1\}$.

In 1933 Kierst and Szpilrajn [19] showed that at least for the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the property discovered by Mittag-Leffler is generic, in the sense that $H_e(\mathbb{D})$ is not only nonempty but even residual –hence dense– in $H(\mathbb{D})$, that is, its complement in $H(\mathbb{D})$ is of first category. There is a rich bibliography or papers studying various extensions and improvements of the Mittag-Leffler and Kierst-Szpilrajn theorems, see for instance [5, Section 4.3], [23], [16] [20, Proposition 5], [6], and further references in [20], [6]. Recently, Kahane and the first author ([17, Theorem 3.1 and following remarks] and [3, Theorem 3.1], see also [14, Proposition 1.7.6]) have observed that Kierst-Szpilrajn's result can be generalized as follows. As usual, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Theorem 1.1. *Let $G \subset \mathbb{C}$ be a domain and X be a Baire topological vector space with $X \subset H(G)$ such that the next conditions hold:*

(i) *For every $a \in G$ and every $r > d(a, \partial G)$ there exists $f \in X$ such that $\rho(f, a) < r$.*

(ii) *All evaluations functionals $f \in X \mapsto f^{(k)}(a) \in \mathbb{C}$ ($a \in G$, $k \in \mathbb{N}_0$) are continuous.*

Then $H_e(G) \cap X$ is residual in X.

Let us insert here some standard terminology. If $S \subset \mathbb{C}$ then S^0, $\overline{S}$ will denote, respectively, its interior and its closure in $\mathbb{C}$. Assume that $G \subset \mathbb{C}$ is a domain. Then $A^\infty(G)$ denotes the space of holomorphic functions in G with "highly boundary-regular behavior", that is, $A^\infty(G) = \{f \in H(G) : \ f^{(j)}$ has a continuous extension to $\overline{G}$ for all $j \in \mathbb{N}_0\}$. It becomes a Fréchet space when it is endowed with the topology of uniform convergence of functions and all their derivatives on each compact set $K \subset \overline{G}$. A family of basic open sets for $A^\infty(G)$ is that of subsets of the form $\{f \in A^\infty(G) : \ |f^{(n)}(z) - g^{(n)}(z)| < \varepsilon$ for all $n = 0, 1, \ldots, N$ and all $z \in K\}$ ($g \in A^\infty(G)$, $N \in \mathbb{N}$, $\varepsilon > 0$, $K \in \{$compact subsets of $\overline{G}\}$). If G is bounded, we can take $K = \overline{G}$ in the last subsets.

2

In view of Theorem 1.1, we can say that for many subspaces X of $H(G)$ –including the full space $X = H(G)$ (see also Nestoridis' paper [22] for this special case and for $X = A(\Omega)$, the Banach algebra of holomorphic functions in Ω that are continuously extendable to $\overline{\Omega}$, where Ω is a domain bounded by a finite number of Jordan curves)– the subset of holomorphically non-continuable functions is *topologically large*.

In spite of the fact that, trivially, $H_e(G)$ is not a linear manifold, we may wonder whether the set of non-continuable functions is, in addition, *algebraically large*, that is, whether $H_e(G) \cap X$ contains, except for zero, linear manifolds which are "large" in some sense. Several results of such nature have been recently obtained in [3] in the case $G = \mathbb{D}$ for a wide class of subspaces X of $H(\mathbb{D})$ (including the space $X = A^\infty(\mathbb{D})$; note also that Theorem 1.1 applies on $A^\infty(\mathbb{D})$), where "large" means either dense or closed infinite-dimensional. In [1] and –independently, with a rather different proof– in [3] it is also proved the existence of a dense linear manifold as before, but only for the full space $H(G)$, where G is an arbitrary domain of $\mathbb{C}$. (In fact, Aron, García and Maestre proved in [1] that, more generally, if $G \subset \mathbb{C}^N$ is a domain of holomorphy then there are a dense subspace $X \subset H(G)$ and an infinite-dimensional closed subspace $Y \subset H(G)$ such that every $f \in (X \cup Y) \setminus \{0\}$ cannot be holomorphically extended across ∂G).

As for the subspaces of $H(G)$, the space $A^\infty(G)$ seems to be the most interesting case, because its members behave "very well" near the boundary and therefore their non-extendability across ∂G is less likely. Consequently, a natural question arises:

> If $G \subset \mathbb{C}$ is a bounded domain, are there "large" manifolds $M \subset A^\infty(G)$ satisfying $M \setminus \{0\} \subset H_e(G)$?

The aim of this paper is to provide positive answers to the last question. Adequate conditions are to be imposed on the domain G. In Section 2 the existence of dense manifolds will be dealt with, while in Section 3 we will be concerned with closed infinite-dimensional manifolds. The way of the proofs –via universality and Faber series– is rather different from that in [3]. Finally, it is proved in Section 4 by an elementary approach that if ∂G does not contain isolated points and X is a subspace of $H(G)$ satisfying mild conditions then there are also infinite-dimensional manifolds of functions in X having G as its domain of holomorphy.

Before going on, we point out that the results contained in this paper could be presented by using the new terminology of "spaceability" and of "algebraic genericity", introduced recently by Gurariy and Quarta [12] and Bayart [2], respectively. Specifically, if A is a subset of a topological vector space X, then A is said to be spaceable (resp., algebraically generic) if A contains, except for zero, some closed infinite-dimensional (resp., some dense) linear submanifold of X.

2   Dense linear manifolds

Before establishing the main result of this section, we need a number of concepts and assertions of topological nature or of universality theory. See the surveys [10] and [11] for a good updated account on universality theory.

Observe first that even in the case of a bounded simply connected domain G the set $A^\infty(G) \cap H_e(G)$ may well be empty; consider for instance $G = \mathbb{D} \setminus [0,1]$ (indeed, a holomorphic function in $\mathbb{D} \setminus [0,1]$ which is continuous on its boundary is automatically holomorphic in $\mathbb{D}$). Consequently, some topological or geometrical conditions should be imposed on G in order to get non-continuability.

Assume that $G \subset \mathbb{C}$ is a domain. Then G is said to be:

(a) *bounded* whenever there is $M \in (0, +\infty)$ such that $|z| \le M$ for all $z \in G$;

(b) *simply connected* whenever $\mathbb{C}_\infty \setminus G$ is connected in the one-point compactification $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$;

(c) *regular* whenever $G = \overline{G}^0$;

(d) *a Carathéodory domain* whenever G is bounded, simply connected and, in addition, $\partial G = \partial G_\infty$, where G_∞ is the unbounded component of $\mathbb{C} \setminus \overline{G}$;

(e) *a finite-length domain* whenever its points can be arc-connected boundedly, that is, there is $M \in (0, +\infty)$ such that for any pair a, b of points of G there exists a curve $\gamma \subset G$ joining a to b for which $\mathrm{length}(\gamma) \le M$;

(f) *a CCC-domain* whenever $\mathbb{C}_\infty \setminus \overline{G}$, the *c*omplement of its *c*losure, is *c*onnected.

The next examples illustrate the relationships among the notions defined above. By $\mathrm{clos}_\infty(A)$ we denote the closure of a set $A \subset \mathbb{C}_\infty$ in $\mathbb{C}_\infty$.

Examples 2.1. 1. Evidently, if G is bounded, then G is simply connected (a CCC-domain, resp.) if and only if $\mathbb{C} \setminus G$ is connected ($\mathbb{C} \setminus \overline{G}$ is connected, resp.).

2. Every Carathéodory domain is (bounded, simply connected and) regular. Otherwise, G would be strictly included in $\overline{G}^0$, so $A := (\partial G) \cap \overline{G}^0 \ne \emptyset$. But $A \cap \overline{\mathbb{C} \setminus \overline{G}} = \emptyset$, so $A \cap \partial(\mathbb{C} \setminus \overline{G}) = \emptyset$. Therefore $A \cap \partial G_\infty = \emptyset$, hence $\partial G \ne \partial G_\infty$, a contradiction. The "crescent moon" $G := \{z : |z + 1| < 2\} \setminus \overline{\mathbb{D}}$ is a bounded regular simply connected domain which is neither a Carathéodory domain nor a CCC-domain.

3. The punctured plane $\mathbb{C} \setminus \{0\}$ is a CCC-domain which is not simply connected.

4. Every finite-length domain is bounded. A bounded domain which is either starlike or with rectifiable boundary is a finite-length domain.

5. Due to the Jordan curve theorem, every Jordan domain is a bounded regular CCC-domain. The set $G := \{z = x + iy : 0 < x < 1, |\sin(1/x)| < y < 2\}$ is a bounded regular CCC-domain that is not a Jordan domain.

6. The slit disk $\mathbb{D} \setminus [0,1]$ is a simply connected CCC-domain which is not regular. Nevertheless, every regular CCC-domain G is simply connected. Indeed, $\mathbb{C}_\infty \setminus G = \mathbb{C}_\infty \setminus \overline{G}^0 = \mathbb{C}_\infty \setminus (\mathbb{C} \setminus \overline{\mathbb{C} \setminus G}) = \{\infty\} \cup \overline{\mathbb{C} \setminus \overline{G}} = \mathrm{clos}_\infty(\mathbb{C} \setminus \overline{G}) = \mathrm{clos}_\infty(\mathbb{C}_\infty \setminus \overline{G})$, and the last set is connected in $\mathbb{C}_\infty$ because $\mathbb{C}_\infty \setminus \overline{G}$ is connected.

7. Every bounded regular CCC-domain G is a Carathéodory domain. Indeed, G is simply connected because of Example 6. Moreover, since $\mathbb{C} \setminus \overline{G}$ is connected we have $\partial G_\infty = \partial(\mathbb{C} \setminus \overline{G}) = \partial \overline{G} = \overline{\overline{G}} \cap \overline{\mathbb{C} \setminus \overline{G}} = \overline{G} \cap (\mathbb{C} \setminus \overline{G}^0) = \overline{G} \cap (\mathbb{C} \setminus G) = \overline{G} \cap \overline{\mathbb{C} \setminus G} = \partial G$. An "outer snake" (see [8, pages 17–18]) is an example of a Carathéodory domain which is not a CCC-domain.

The following auxiliary result can be deduced by using arguments similar to those of the last part of the proof of Theorem 4 in [21].

Lemma 2.2. *Assume that $G \subset \mathbb{C}$ is a finite-length CCC-domain. Then the set of polynomials is dense in $A^\infty(G)$.*

Let X, Y be two topological vector spaces and $T_n : X \to Y$ ($n \in \mathbb{N}$) be a sequence of continuous linear mappings. Then the sequence (T_n) is called *universal* (or *hypercyclic*) provided that there is a vector $x_0 \in X$ –called *universal* or *hypercyclic* for (T_n)– whose orbit $\{T_n x_0 : n \in \mathbb{N}\}$ under (T_n) is dense in Y. By $HC((T_n))$ it is denoted the set of hypercyclic vectors for (T_n). If $HC((T_n))$ is dense in X then (T_n) is called *densely hypercyclic*. Finally, (T_n) is said to be *densely hereditarily hypercyclic* whenever (T_{n_k}) is densely hypercyclic for every (strictly increasing) subsequence $(n_k) \subset \mathbb{N}$.

A useful characterization of the dense hypercyclicity is the next Birkhoff transitivity property (see [10]).

Lemma 2.3. *Let X and Y be two topological vector spaces such that X is Baire and Y is metrizable and separable. Assume that $L_j : X \to Y$ ($j \in \mathbb{N}$) is a sequence of continuous linear mappings. Then (L_j) is densely hypercyclic if and only if given nonempty open subsets $U \subset X$, $V \subset Y$ there exists $m \in \mathbb{N}$ such that $L_m(U) \cap V \neq \emptyset$.*

The following crucial result can be found in [4, Theorem 3.1].

Lemma 2.4. *Let X and Y be two metrizable topological vector spaces such that X is Baire and separable. Assume that, for each $k \in \mathbb{N}$, $T_n^{(k)} : X \to Y$ $(n \in \mathbb{N})$ is a densely hereditarily hypercyclic sequence of continuous linear mappings. Then there is a dense linear submanifold $M \subset X$ such that*

$$M \setminus \{0\} \subset \bigcap_{k \in \mathbb{N}} HC((T_n^{(k)})).$$

The next elementary statement tells us that a dense subset of a domain is enough to describe a domain of holomorphy.

Lemma 2.5. *Let $G \subset \mathbb{C}$ be a domain and $f \in H(G)$. Assume that S is a dense subset of G. Then $f \in H_e(G)$ if and only if $\rho(f, a) = d(a, \partial G)$ for all $a \in S$.*

Proof. Just take into account that $\rho(f, b) \geq \rho(f, a) - |a - b|$ for all pair of points $a, b \in G$. $\qquad\square$

Now we provide a workable sufficient condition for non-extendability across a point, in the case of a regular domain. By $S(f, n, a)(z)$ we denote the value at z of the partial sum of order n of the Taylor series of f with center at a ($f \in H(G)$, $n \in \mathbb{N}_0$, $a \in G$, $z \in \mathbb{C}$). As usual, $B(a, r)$ ($\overline{B}(a, r)$, resp.) denotes the open (closed, resp.) ball with center a and radius r ($a \in \mathbb{C}$, $r > 0$).

Lemma 2.6. *Assume that $a \in G$, that $f \in H(G)$ and that T is a dense subset of $\mathbb{C} \setminus \overline{G}$, where G is a regular domain of $\mathbb{C}$. Suppose that for each $b \in T$ the set $\{S(f, n, a)(b) : n \in \mathbb{N}\}$ is dense in $\mathbb{C}$. Then $\rho(f, a) = d(a, \partial G)$.*

Proof. Assume, by way of contradiction, that $\rho(f, a) > d(a, \partial G)$. Choose any $r > 0$ with $d(a, \partial G) < r < \rho(f, a)$. Since G is regular, we have that $B(a, r) \cap (\mathbb{C} \setminus \overline{G}) \neq \emptyset$. But the density of T in $\mathbb{C} \setminus \overline{G}$ yields the existence of a point $b \in T \cap B(a, r)$. Therefore the sequence $(S(f, n, a)(b))$ must be convergent, which contradicts the hypothesis. $\qquad\square$

We first establish that the set of boundary-regular non-continuable functions is topologically large.

Proposition 2.7. *If $G \subset \mathbb{C}$ is a regular domain then $H_e(G) \cap A^\infty(G)$ is residual in $A^\infty(G)$.*

Proof. If $G = \mathbb{C}$ the statement is trivial, so we can assume $G \neq \mathbb{C}$. Fix a point $a \in G$ and a number $r > d(a, \partial G)$. Since G is regular, the set $B(a, r) \setminus \overline{G}$ is not

empty, so it contains some point b. Take $f(z) := (z - b)^{-1}$. Then $f \in X := A^\infty(G)$ and $\rho(f, a) = |a - b| < r$. Therefore X satisfies condition (i) of Theorem 1.1 and, since compact convergence implies pointwise convergence, it also satisfies (ii), which concludes the proof. $\qquad\square$

Remark 2.8. The statement of the last proposition has been obtained by the first author in [3, Remark 5.2.2] but only for Jordan domains, as a consequence of Theorem 1.1 and of a strong result due to J. Siciak [26] about non-continuability in an N-dimensional setting. Hence our result is more general and its proof is easier. It must be also pointed out that in 1980 J. Chmielowski [6, Proposition 6] had already discovered –again as a consequence of an N-dimensional result– that $H_e(G) \cap A^\infty(G) \neq \emptyset$ for any regular domain $G \subset \mathbb{C}$. Finally, Nestoridis [22, Theorem 5.4], by a nice proof using universal Taylor series, has recently shown the same conclusion as Chmielowski, at least for domains in $\mathbb{C}$ bounded by a finite number of disjoint Jordan curves.

We are now ready to state our main result. This is achieved in the next theorem, where "many" –in an algebraic sense– non-continuable boundary-regular functions are obtained, just by imposing adequate topological or geometrical hypotheses on the domain.

Theorem 2.9. *Assume that $G \subset \mathbb{C}$ is a regular finite-length CCC-domain. Then there exists a dense linear manifold M in $A^\infty(G)$ such that $M \setminus \{0\} \subset H_e(G)$.*

Proof. Since $A^\infty(G)$ is complete, it is a Baire metrizable space. From Lemma 2.2 we have that the set $\mathcal{P} := \{\text{polynomials}\}$ is dense in $A^\infty(G)$. In turn, the polynomials whose coefficients have rational real and imaginary parts are dense in $\mathcal{P}$, hence $A^\infty(G)$ and, of course, $\mathbb{C}$ are Baire metrizable separable spaces.

Let us choose countable dense subsets S, T of G, $\mathbb{C} \setminus \overline{G}$ respectively. For each pair $(a, b) \in S \times T$, let us consider the sequence of mappings

$$T_n^{(a,b)} : f \in A^\infty(G) \mapsto S(f, n, a)(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b - a)^k \in \mathbb{C} \quad (n \in \mathbb{N}).$$

It is clear that each $T_n^{(a,b)}$ is linear and continuous.

Fix a pair $(a, b) \in S \times T$ as well as a sequence $\{n_1 < n_2 < \cdots < n_j < \cdots\} \subset \mathbb{N}$. Consider the sequence $L_j : A^\infty(G) \to \mathbb{C}$ $(j \in \mathbb{N})$ defined as $L_j = T_{n_j}^{(a,b)}$. Fix also respective nonempty open subsets $U \subset A^\infty(G)$ and $V \subset \mathbb{C}$. Note that G is bounded. Then there exist $\varepsilon > 0$, $N \in \mathbb{N}$, $c \in \mathbb{C}$, $g \in A^\infty(G)$ for which

$$\{f \in A^\infty(G) : |f^{(n)}(z) - g^{(n)}(z)| < \varepsilon \quad \text{for all } z \in \overline{G} \text{ and all } n = 0, 1, \ldots, N\} \subset U \tag{1}$$

and
$$B(c, \varepsilon) \subset V. \tag{2}$$

Recall that G is CCC and finite-length. By the denseness of $\mathcal{P}$ in $A^\infty(G)$, there is a polynomial h such that

$$|h^{(n)}(z) - g^{(n)}(z)| < \frac{\varepsilon}{2} \quad \text{for all } z \in \overline{G} \text{ and all } n = 0, 1, \dots, N. \tag{3}$$

Consider a simply connected domain G_1 and an $r > 0$ satisfying $G_1 \supset \overline{G}$ (this is possible because $\overline{G}$ has no holes) and $G_1 \cap B(b, r) = \emptyset$. Let $\Omega := G_1 \cup B(b, r)$, which is a simply connected open set. Define the function $F : \Omega \to \mathbb{C}$ as

$$F(z) = \begin{cases} h(z) & \text{if} \quad z \in G_1 \\ c & \text{if} \quad z \in B(b, r). \end{cases}$$

Since h is a polynomial, $F \in H(\Omega)$. Then the Runge approximation theorem (see [8]) together with the Weierstrass convergence theorem yield the existence of a polynomial f (extracted from a suitable sequence of polynomials converging to F compactly in Ω) such that

$$|f^{(n)}(z) - F^{(n)}(z)| < \frac{\varepsilon}{2} \quad \text{for all } z \in \widetilde{K} \text{ and all } n = 0, 1, \dots, N,$$

where $\widetilde{K} := \overline{G} \cup \{b\}$ is a compact set contained in Ω. In particular,

$$|f^{(n)}(z) - h^{(n)}(z)| < \frac{\varepsilon}{2} \quad \text{for all } z \in \overline{G} \text{ and all } n = 0, 1, \dots, N \tag{4}$$

and

$$|f(b) - c| < \varepsilon. \tag{5}$$

From (3), (4) and the triangle inequality we obtain that $|f^{(n)}(z) - g^{(n)}(z)| < \varepsilon$ ($z \in \overline{G}$, $n \in \{0, 1, \dots, N\}$). Hence, by (1), $f \in U$. Furthermore, one gets from (5) that $|S(f, n_j, a)(b) - c| < \varepsilon$ for every $j \geq m$, where $m \in \mathbb{N}$ is such that $n_m \geq \text{degree}(f)$. Therefore $L_m f \in V$ by (2). Consequently, $L_m f \in L_m(U) \cap V$ and an application of Lemma 2.3 yields that the sequence (L_j) is densely hypercyclic. Then by Lemma 2.4 –as applied to $X := A^\infty(G)$ and $Y := \mathbb{C}$– there exists a dense linear submanifold $M \subset A^\infty(G)$ satisfying

$$M \setminus \{0\} \subset \bigcap_{(a,b) \in S \times T} HC((T_n^{(a,b)})). \tag{6}$$

But according to Lemma 2.6 if a function f belongs to any set $HC((T_n^{(a,b)}))$ for all $b \in T$ (for fixed $a \in G$) then $\rho(f, a) = d(a, \partial G)$. And by Lemma 2.5 if this last property holds for any point a of the dense set S (in G) then $f \in H_e(G)$. Thus, the intersection that appears in (6) is included in $H_e(G)$, which concludes the proof. $\square$

3 Closed linear manifolds

In order to obtain large closed linear manifolds of non-continuable boundary-regular functions some background about Faber expansions is needed. For the basic results on Faber series and Faber transforms we refer the reader to [7], [8], [9], [15], [25], [27] and, more recently, [18].

Assume that G is a Jordan domain. We use g to denote the unique one-to-one function $g \in H(\{w : |w| > 1\})$ such that $g(\{w : |w| > 1\}) = \mathbb{C} \setminus \overline{G}$ and has expansion

$$g(w) = cw + c_0 + \frac{c_1}{w} + \frac{c_2}{w^2} + \cdots \qquad (c > 0)$$

in a neighborhood of ∞. The constant c is called the logarithmic capacity of Ω. The Faber polynomials associated with G are the polynomials Φ_n $(n \in \mathbb{N}_0)$ determined by the following generating function relationship:

$$\frac{g'(w)}{g(w) - z} = \sum_{k=0}^{\infty} \frac{\Phi_k(z)}{w^{k+1}}.$$

The operator $\mathcal{F}$ that takes a function $f(w) := \sum_{k=0}^{\infty} c_k w^k \in H(\mathbb{D})$ and maps it to the (formal) Faber series $(\mathcal{F}f)(z) := \sum_{k=0}^{\infty} c_k \Phi_k(z)$ $(z \in G)$ is called the Faber transform.

It is well known (see [25]) that if the boundary of G is an analytic curve then the series $\mathcal{F}f$ converges uniformly on compact subsets of G to a function $F \in H(G)$ and, in addition, the function g can be holomorphically and univalently continued to some domain $\{|w| > r_0\}$ for some $r_0 \in (0, 1)$. In this case, the Faber transform $\mathcal{F} : f \in H(\mathbb{D}) \mapsto F \in H(G)$ is (linear and) continuous.

The proof of the main statement is based upon the three following auxiliary results, which can be found in [15, Theorem 1], [18, Section 3] and [3, Theorem 4.3], respectively.

Lemma 3.1. *Let G be a Jordan domain with analytic boundary and J be a subarc of ∂G. Let $f \in H(\mathbb{D})$ and consider its Faber transform $F = \mathcal{F}f \in H(G)$. Then F has an analytic continuation across J if and only if f has an analytic continuation across $g^{-1}(J)$.*

Lemma 3.2. *Let G be a Jordan domain with analytic boundary. Then the Faber transform $\mathcal{F} : H(\mathbb{D}) \to H(G)$ is a topological isomorphism such that $\mathcal{F}(A^{\infty}(\mathbb{D})) = A^{\infty}(G)$ and the restriction map $\mathcal{F} : A^{\infty}(\mathbb{D}) \to A^{\infty}(G)$ is also a topological isomorphism.*

Lemma 3.3. *Let X be a Baire topological vector space with $X \subset H(\mathbb{D})$ satisfying the following conditions:*

(i) *For every $a \in \mathbb{D}$ and every $r > d(a, \partial\mathbb{D})$ there exists $f \in X$ such that $\rho(f, a) < r$.*

(ii) *All evaluations functionals $f \in X \mapsto f^{(k)}(a) \in \mathbb{C}$ $(a \in G;\ k \in \mathbb{N}_0)$ are continuous.*

(iii) *X is stable under projections, that is, given $Q \subset \mathbb{N}_0$ and $f(z) := \sum_{n=0}^{\infty} a_n z^n \in X$, the function $P_Q f(z) := \sum_{n \in Q} a_n z^n$ also belongs to X.*

Then there is an infinite-dimensional closed linear manifold $M_0 \subset X$ such that $M_0 \setminus \{0\} \subset H_e(\mathbb{D})$.

Finally, we state our main result in this section.

Theorem 3.4. *Assume that $G \subset \mathbb{C}$ is a Jordan domain with analytic boundary. Then there exists a closed infinite-dimensional linear manifold $M \subset A^\infty(G)$ such that $M \setminus \{0\} \subset H_e(G)$.*

Proof. Since uniform convergence is stronger than pointwise convergence, the Fréchet (so Baire) space $X := A^\infty(\mathbb{D})$ satisfies the condition (ii) of Lemma 3.3. As for (iii), it is also fulfilled because a function $f(z) := \sum_{n=0}^{\infty} a_n z^n$ is in $A^\infty(\mathbb{D})$ if and only if each sequence $(n^k a_n)$ $(k \in \mathbb{N})$ is bounded. The condition (i) is true for $A^\infty(\mathbb{D})$ by the same argument given in the proof of Proposition 2.7. Consequently, there exists an infinite-dimensional closed linear manifold $M_0 \subset A^\infty(\mathbb{D})$ for which $M_0 \setminus \{0\} \subset H_e(\mathbb{D})$.

Define $M := \mathcal{F}(M_0)$. According to Lemma 3.2, the set M is an infinite-dimensional closed linear submanifold of $A^\infty(G)$. Finally, consider a function $F \in M \setminus \{0\}$. Then there is a (unique) function $f \in M_0 \setminus \{0\}$ such that $F = \mathcal{F}f$. Suppose, by way of contradiction, that $F \notin H_e(G)$. Therefore there would exist a subarc $J \subset \partial G$ with the property that F has an analytic continuation across J. Hence, by Lemma 3.1, the function f would have an analytic continuation across the subarc $g^{-1}(J) \subset \partial\mathbb{D}$, which is absurd since $M_0 \setminus \{0\} \subset H_e(\mathbb{D})$. $\qquad\square$

Remark 3.5. Alternatively, we can see that condition (i) of Lemma 3.3 holds for $A^\infty(\mathbb{D})$ by showing that $A^\infty(\mathbb{D}) \cap H_e(\mathbb{D}) \neq \emptyset$. And this is true since the function $f(z) := \sum_{n=0}^{\infty} a_n \exp(-\sqrt{n}) z^n$, where $a_n = \begin{cases} 1 & \text{if } n \text{ is a power of } 2 \\ 0 & \text{otherwise,} \end{cases}$ belongs to that intersection, see [24, Chapter 16].

4 Large manifolds in vector spaces

In view of Theorems 2.9 and 3.4, one may wonder whether given a vector subspace X (not necessarily topologized) of $H(G)$, there exists a large submanifold $M \subset X$ consisting, except for zero, of functions which are holomorphically non-extendable across ∂G, where this time "large" carries a purely algebraic sense, that is, it means "of infinite dimension". With a rather elementary proof, we have found a positive answer for rather general domains G. By $H(\overline{G})$ we denote the class of functions f which are holomorphic in some domain $\Omega = \Omega_f \supset \overline{G}$.

Theorem 4.1. *Let $G \subset \mathbb{C}$ be a domain whose boundary does not contain isolated points and X be a vector space over $\mathbb{C}$ with $X \subset H(G)$ satisfying the following conditions:*

(a) $X \cap H_e(G) \neq \emptyset$.

(b) *There is a nonconstant function $\varphi \in H(\overline{G})$ such that $\varphi X \subset X$.*

Then there exists an infinite-dimensional linear manifold $M \subset X$ such that $M \setminus \{0\} \subset H_e(G)$.

Proof. Choose $f \in X \cap H_e(G)$ and consider the function φ provided by (b). Then there is a domain $\Omega \supset \overline{G}$ such that $\varphi \in H(\Omega)$. Moreover, $\varphi^n f \in X$ ($n \in \mathbb{N}_0$), so $(P \circ \varphi)f \in X$ for every (holomorphic) polynomial P because X is a vector space. Let us define

$$M := \{(P \circ \varphi)f : \ P \text{ is a polynomial}\}.$$

It is clear that M is a linear submanifold of X. Let us show that M has infinite dimension. For this, since M is the linear span of the functions $\varphi^n f$ ($n \in \mathbb{N}_0$), it is enough to prove that such functions are linearly independent. Assume, by way of contradiction, that this is not the case. Then there exists a nonzero polynomial P with $(P \circ \varphi)f \equiv 0$ on G. Since $H(G)$ is an integral ring –and, clearly, $f \not\equiv 0$– we get $P \circ \varphi \equiv 0$ on G. Therefore P vanishes on the nonempty open (due to the Open Mapping Theorem, because φ is not constant) set $\varphi(G)$. Hence, from the Analytic Continuation Principle, $P \equiv 0$, which is absurd.

To conclude the proof, it must be shown that each function $F \in M \setminus \{0\}$ is in $H_e(G)$. Indeed, for such function F there exists a nonzero polynomial P with $F = (P \circ \varphi)f$. Suppose, by way of contradiction, that $F \notin H_e(G)$. Let us denote by S_{z_0} the sum of the Taylor series of F with center at z_0. Then there are a point $a \in G$ and a number $r > d(a, \partial G)$ such that $S_a \in H(B(a, r))$. Of course, $S_a = (P \circ \varphi)f$ in

$B(a, |a - b|)$, where b is a point on ∂G such that $|a - b| = d(a, \partial G)$. Therefore there are a point $c \in \partial G$ and a number $\varepsilon > 0$ with $B(c, \varepsilon) \subset \Omega \cap B(a, r)$ and $P(\varphi(z)) \neq 0$ for all $z \in B(c, \varepsilon)$; indeed, $\Omega \cap B(a, r)$ is a neighborhood of b, the point b is not isolated in ∂G (by (a)), and the set of zeros of $P \circ \varphi$ in Ω is discrete in Ω. Now, take a point $\zeta \in B(c, \varepsilon/2) \cap G$. Then $B(\zeta, \varepsilon/2) \subset B(c, \varepsilon) \subset B(a, r)$ and $P(\varphi(z)) \neq 0$ for all $z \in B(\zeta, \varepsilon/2)$. The function S_ζ equals F in a neighborhood of ζ, whence $S_\zeta/(P \circ \varphi)$ equals f in a neighborhood of ζ. But $S_\zeta \in H(B(\zeta, \varepsilon/2))$, hence also $S_\zeta/(P \circ \varphi) \in H(B(\zeta, \varepsilon/2))$. Finally, we get from the non-extendability of f that

$$\frac{\varepsilon}{2} > d(\zeta, c) \geq d(\zeta, \partial G) = \rho(f, \zeta) = \rho(\frac{S_\zeta}{P \circ \varphi}, \zeta) \geq \frac{\varepsilon}{2},$$

which is the sought-after contradiction. □

Observe that property (a) of the last theorem is fulfilled if, for instance, condition (i) of Theorem 1.1 holds (this condition is purely algebraic in relation to X) and if, in addition, X can be endowed with a vector topology for which X is Baire and condition (ii) of Theorem 1.1 is satisfied.

We finish this paper with the following consequence of Theorem 4.1, that complements Theorem 2.9.

Corollary 4.2. *If $G \subset \mathbb{C}$ is a regular domain, then $A^\infty(G) \cap H_e(G)$ contains, except for zero, an infinite-dimensional linear manifold.*

Proof. Since G is regular, one gets that ∂G does not contain isolated points and, moreover, the condition (a) of Theorem 4.1 is satisfied by Proposition 2.7. Finally, for the space $A^\infty(G)$ and for every domain G it holds the "multiplier condition" (b) of Theorem 4.1: simply choose $\varphi(z) \equiv z$. □

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