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Universal entire functions for affine endomorphisms of $\mathbb{C}^N$

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Abstract

In this paper the affine endomorphisms of $\mathbb{C}^N$ which support compositionally universal entire functions are completely characterized.

1 Introduction

Throughout this paper $\mathbb{N}$ will denote the set of all positive integers, and if $N \in \mathbb{N}$ then $\mathbb{C}^N$ will stand for the N-dimensional complex space. In particular, $\mathbb{C}^1$ is the complex plane $\mathbb{C}$. The closed polydisk of radius $r \geq 0$ centered at the origin is $D(r) = \{z = (z_1, \ldots, z_N) \in \mathbb{C}^N : \|z\| \leq r\}$, where $\|z\| = \max_{1 \leq j \leq N} |z_j|$. A domain G of $\mathbb{C}^N$ is a nonempty connected open subset of $\mathbb{C}^N$. By $H(G)$ we denote the space of holomorphic functions $f : G \to \mathbb{C}$, endowed with the topology of uniform convergence in compacta. In particular, $H(\mathbb{C}^N)$ is the space of entire functions of N complex variables. It is well known that $H(G)$ becomes a separable Fréchet space under the above topology. The symbol $\mathrm{Aut}\,(G)$ will stand for the group of all automorphisms (= biholomorphic bijective selfmappings) on G.

In 1929 Birkhoff [8] constructed an entire function which is 'universal' for translations. In fact, he proved essentially that given $b \in \mathbb{C} \setminus \{0\}$ there exists

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a function $f \in H(\mathbb{C})$ such that its sequence of translates $\{f(\cdot + nb) : n \in \mathbb{N}\}$ is dense in $H(\mathbb{C})$.

Birkhoff's theorem can be observed under the point of view of the operator theory as a universality result; namely, if $\varphi : \mathbb{C} \to \mathbb{C}$ denotes the translation $z \mapsto z + b$ then the composition operator

$$C_\varphi : f \in H(\mathbb{C}) \mapsto f \circ \varphi \in H(\mathbb{C})$$

is universal. In general, if X is a (necessarily separable) topological vector space and T is an operator (= continuous linear selfmapping) on X then (T_n) is said to be *universal* (or *hypercyclic*) provided that there exists some vector $x \in X$ –called *universal* for T– for which the orbit $\{T^n x : n \in \mathbb{N}\}$ of x under T is dense in X. Here (T^n) represents the sequence of iterates $T^1 = T, T^2 = T \circ T, \dots$ of T. It is easy to see that the set of universal vectors is dense. The operator T is called *hereditarily universal* if and only if given a sequence $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ there is a vector $x \in X$ such that $\{T^{n_k} x : n \in \mathbb{N}\}$ is dense in X. If X is Baire and metrizable and T is universal (hereditarily universal) then the set of universal vectors for T (for each sequence (T^{n_k}), respectively) is residual, that is, its complement is of first category. These notions can be easily extended to a sequence (T_n) of operators. See [16] for a good account about these concepts and their history.

Since 1929 many papers have dealt with the subject of universality through translations in one (complex) variable, but only a few ones in several variables. Let us make a brief report, now in the language of the universality of operators; see also the survey [16] –specially its Section 4a– which contains a rather complete list of references including domains $G \neq \mathbb{C}$ and spaces $X \neq H(G)$. In 1976 Luh [21] proved that for a prescribed unbounded sequence $(b_n) \subset \mathbb{C}$ the sequence (C_{φ_n}) is universal on $H(\mathbb{C})$, where φ_n is the translation $z \mapsto z + b_n$. In 1984 Duyos-Ruis [11] showed by functional analysis methods that C_φ ($\varphi(z) = z + b$, $b \in \mathbb{C} \setminus \{0\}$) is universal on $H(\mathbb{C})$ (hence there is a residual subset of universal functions), while the residuality of the (C_{φ_n})-universal entire functions (where the φ_n are the above translations) was observed by Grosse-Erdmann [15] and Gethner and Shapiro [12]. In 1995 Bernal and Montes [6] were able to show the same result for a sequence $\{\varphi_n(z) = a_n z + b_n : n \in \mathbb{N}\} \subset \text{Aut}(\mathbb{C})$; recall that $\varphi \in \text{Aut}(\mathbb{C})$ if and only if φ is a nonconstant affine endomorphism of $\mathbb{C}$, that is, there are $a, b \in \mathbb{C}$ with $a \neq 0$ and $\varphi(z) = az + b$. Specifically, they proved that (C_{φ_n}) is universal if and only if the sequence $\{\min\{|b_n|, |b_n/a_n|\} : n \in \mathbb{N}\}$ is unbounded, from which they derived that if φ is an affine endomorphism of $\mathbb{C}$ then

C_φ is universal if and only if φ is a translation,

that is, $\varphi(z) \equiv z+b$ for some $b \in \mathbb{C}\setminus\{0\}$. As for several variables, the history is not too long. In 1941 Seidel and Walsh [23] proved a non-Euclidean version of Birkhoff's theorem for the unit disk, and in 1979 Chee [10] extended this to the unit polidisk and ball of $\mathbb{C}^N$. León [20] has recently characterized the corresponding universal sequences of automorphisms in both domains. In the case of the euclidean translations $\varphi(z) = z + b$ ($z \in \mathbb{C}^N$, $b \in \mathbb{C} \setminus \{0\}$) in several variables –where $z = (z_1, \dots, z_N)$, $b = (b_1, \dots, b_N)$, and these will be the standard representations of z, b along the current paper– the natural extensions of the theorems of Birkhoff and Luh are covered by [13, Section 5] and by the recent papers [1], [3, Section 2] and [7] (see also [2] and [3] for a matricial extension of Zappa's result [24], which in turn is a multiplicative version in $\mathbb{C} \setminus \{0\}$ of Birkhoff's theorem).

In view of the above discoveries, it is natural to pose the problem of characterizing those mappings $\varphi \in \mathrm{Aut}\,(\mathbb{C}^N)$ such that C_φ is universal on $H(\mathbb{C}^N)$. Nevertheless, a complete description of $\mathrm{Aut}\,(\mathbb{C}^N)$ (see for instance [4], [5] and [22] for a study of some subfamilies of it) is unknown up to date. Although there are plenty of automorphisms of $\mathbb{C}^N$, the simplest among them are with no doubt the affine linear mappings (or 'affine endomorphisms') from $\mathbb{C}^N$ into itself which are invertible. Each affine endomorphism $S = S(A, b)$ is biunivocally determined by a pair (A, b), where $A := [a_{ij}]_{i,j=1,\dots,N}$ is a matrix with complex entries and b is a fixed vector of $\mathbb{C}^N$; so S is given by

$$S(z) = Az + b \quad \text{for all} \quad z \in \mathbb{C}^N.$$

Observe that as we make the calculation $S(z) = Az + b$ it is convenient to consider the vectors z and b as 'column' vectors. It is clear that $S \in \mathrm{Aut}(\mathbb{C})$ if and only if S is one-to-one if and only if S is onto if and only if $\det(A) \neq 0$.

Hence the main aim of this paper is to characterize the universality of the composition operator $C_S : H(\mathbb{C}^N) \to H(\mathbb{C}^N)$ generated by an affine endomorphism $S = S(A, b)$ in terms of the matrix A and the vector b. This will be accomplished in Section 3 where we prove, among other things, that *C_S is universal if and only if S is univalent and has no fixed point.* In Section 2 we present a number of statements that will reveal useful for our goal.

2 Several auxiliary results

This section is devoted to background material on N-dimensional complex approximation that will be needed for the work of Section 3.

From now on G will represent a domain in $\mathbb{C}^N$. Let us denote by $H(G,G)$ the set of all holomorphic selfmappings $\varphi = (\varphi_1, \dots, \varphi_n)$ on G, that is, $\varphi(G) \subset G$ and each component $\varphi_j : G \to \mathbb{C}$ $(j = 1, \dots, N)$ is a holomorphic function. Then if $\varphi \in H(G,G)$ the composition operator $C_\varphi : H(G) \to H(G)$ is well defined. Of course, $\mathrm{Aut}\,(G) \subset H(G,G)$. Our first lemma prevents us to use non-injective selfmappings to obtain C_φ-universality.

Lemma 2.1. *If $\varphi \in H(G,G)$ and C_φ is universal on $H(G)$ then φ is one-to-one and has no fixed points.*

Proof. The results contained in this lemma are well known in the one-dimensional context, see for instance [9, pages 3 and 10]. The proof given on page 10 of that monograph for the necessity of univalence works, word for word, in several variables. Indeed, if φ identifies two distinct points a and b of G, then so does the n-th component φ_n of φ, and so does $f \circ \varphi_n$ for each n and each $f \in H(G)$. Thus if g is a limit point of the C_φ-orbit of f, then $g(a) = g(b)$, hence (because some $g \in H(G)$, namely an appropriate coordinate function, takes different values at a and b) no f holomorphic on G can include every function in $H(G)$ in the closure of its orbit. Hence C_φ is not universal.

Finally, if $a \in G$ were a fixed point for φ and $f \in H(G)$ were C_φ-universal then by considering the compact set $K = \{a\}$ the closure in $\mathbb{C}^N$ of the set $\{f(\varphi^n(a)) : n \in \mathbb{N}\} = \{f(a)\}$ would be dense in $\mathbb{C}$, which is absurd. □

A set $B \subset \mathbb{C}^N$ is said to be $H(\mathbb{C}^N)$-*convex* (see [17] or [19]) whenever $\widetilde{B} = B$, where

$$\widetilde{B} := \{z \in \mathbb{C}^N : |f(z)| \le \sup_{t \in B} |f(t)| \text{ for all } f \in H(\mathbb{C}^N)\}.$$

The next generalization of Runge's approximation theorem for several complex variables is a special case of a statement that can be found in [17, Theorem 4.3.2 and following note].

Proposition 2.2. *Let f be a holomorphic function in a neighborhood of an $H(\mathbb{C}^N)$-convex compact subset K of $\mathbb{C}^N$. Then there is a sequence $(f_j) \subset H(\mathbb{C}^N)$ such that $f_j \to f$ $(j \to \infty)$ uniformly on K.*

In 1965 Kallin [18] proved an important separation lemma in several variables, from which the following proposition –that will be crucial for our approximation problem– is a particular case. The word "convex" means "geometrically convex", that is, a set $B \subset \mathbb{C}^N$ is convex whenever $z, w \in B$ implies $\lambda z + (1-\lambda)w \in B$ for all $\lambda \in [0,1]$.

Proposition 2.3. *If K and L are disjoint convex compact sets in $\mathbb{C}^N$ then $K \cup L$ is $H(\mathbb{C}^N)$-convex.*

In connection with the last proposition we point out that it is not known yet whether the disjoint union of 4 closed balls in $\mathbb{C}^N$ is $H(\mathbb{C}^N)$-convex.

The following lemma settles the question of which sequences of automorphisms are adequate to generate universality. Following [6], we say that a sequence $(\varphi_n) \subset H(G,G)$ is *run-away* if and only if given a compact set $K \subset G$ there exists $n_0 = n_0(K) \in \mathbb{N}$ such that $K \cap \varphi_{n_0}(K) = \emptyset$; and we say that a function $\varphi \in H(G,G)$ is *non-recurrent* whenever its sequence (φ^n) is run-away.

Lemma 2.4. *Suppose that φ is an affine automorphism of $\mathbb{C}^N$. Then C_φ is universal on $H(\mathbb{C}^N)$ if and only if φ is non-recurrent.*

Proof. Let us suppose that C_φ is universal and that, by way of contradiction, φ is not non-recurrent. Then there is a compact set K such that $K \cap \varphi^n(K) \neq \emptyset$ for all $n \in \mathbb{N}$. Choose a sequence $(z_n) \subset K$ with $(\varphi_n(z_n)) \subset K$ and a C_φ-universal function $f \in H(G)$. Consider the constant function $g(z) = 1 + \max_K |f|$. We have that, for every $n \in \mathbb{N}$,

$$\max_{z \in K} |g(z) - f(\varphi^n(z))| \geq ||g(z)| - |f(\varphi^n(z_n))|| = 1 + \max_K |f| - |f(\varphi^n(z_n))| \geq 1,$$

which is a contradiction.

Conversely, assume that φ is non-recurrent. Our final goal is to show that the set M of universal functions for C_φ is residual (so nonempty). Since $H(G)$ is a second-countable Baire space, Birkhoff's transitivity theorem (see for instance [14, 9.20]) asserts that M is a dense G_δ-subset (hence residual) if and only if for every pair of nonempty open subsets A and B of $H(\mathbb{C}^N)$ there exists some $m \in \mathbb{N}$ with

$$(C_\varphi)^m(A) \cap B = \emptyset. \tag{1}$$

Note that $(C_\varphi)^n = C_{\varphi^n}$ $(n \in \mathbb{N})$. Fix A, B as before. Then there exists $\varepsilon > 0$, $R > 0$ and $f, h \in H(\mathbb{C}^N)$ such that $A \supset A_1 := \{g \in H(\mathbb{C}^N) : \max_{z \in D(R)} |g(z) - f(z)| < \varepsilon\}$ and $B \supset B_1 := \{g \in H(\mathbb{C}^N) : \max_{z \in D(R)} |g(z) - h(z)| < \varepsilon\}$. Since φ is non-recurrent, there exists $m \in \mathbb{N}$ such that $D(R) \cap \varphi^m(D(R)) = \emptyset$. But $\varphi_m(D(R))$ is a convex compact set because φ^m is continuous and convex-preserving (that is, $\varphi^m(C)$ is convex whenever C is convex; this is true because φ is affine). Then $L := D(R) \cup \varphi^m(D(R))$ is $H(\mathbb{C}^N)$-convex by Proposition 2.3. Fix U and V open subsets in $\mathbb{C}^N$ such that $D(R) \subset U$, $\varphi^m(D(R)) \subset V$ (so $U \cup V \supset L$) and $U \cap V = \emptyset$. Define the function $F : U \cap V \to \mathbb{C}$ as

$$F(z) = \begin{cases} f(z) & \text{if } z \in U \\ h(\varphi^{-m}(z)) & \text{if } z \in V, \end{cases}$$

which is holomorphic on $U \cup V$. Hence by Proposition 2.2 there exists an entire function g satisfying

$$|F(z) - g(z)| < \varepsilon \ \text{ for all } z \in L.$$

But from the definition of F we get

$$|g(z) - f(z)| < \varepsilon \ \text{ for all } z \in D(R).$$

and

$$|g(z) - h(\varphi^{-m}(z))| < \varepsilon \ \text{ for all } z \in \varphi^m(D(R)).$$

The last display is clearly equivalent to

$$|g(\varphi^m(z)) - h(z)| < \varepsilon \ \text{ for all } z \in D(R).$$

In other words, $g \in A_1$ and $(C_\varphi)^m g \in B_1$. Thus, $g \in A$ and $(C_\varphi)^m g \in B$, so (1) holds. □

Remark 2.5. The proof of Lemma 2.4 can be easily modified to obtain the following extension: Suppose that G is a convex domain of $\mathbb{C}^N$ and that (φ_n) is a sequence in $\mathrm{Aut}(G)$ of convex-preserving mappings. Then (C_{φ_n}) is universal if and only if (φ_n) is run-away. Moreover, in this case there exists a residual subset of universal functions. Suffice to say that if G is convex then the polidisks $D(R)$ of the proof of Lemma 2.4 can be replaced to convex compact subsets of G and that Birkhoff's transitivity theorem also works with a sequence (T_n) of mappings from a Baire space into a second-countable space, see [16, Theorem 1]. It might be interesting to investigate whether the last lemma can be extended to non-convex domains or to sequences of automorphisms that do not preserve convexity.

3 Universal functions for endomorphisms

From now on A will represent an $(N \times N)$-matrix with complex entries and b will be a fixed vector in $\mathbb{C}^N$.

Lemma 2.4 focuses attention on the dynamics of affine mappings of $\mathbb{C}^N$. In order to characterize when they generate universal composition operators, we need to know which of such mappings are non-recurrent. We are now ready to state our main result.

Theorem 3.1. *Assume that $S : \mathbb{C}^N \to \mathbb{C}^N$ is an affine endomorphism, say $Sz = Az + b$ $(z \in \mathbb{C}^N)$. Consider the composition operator C_S generated by S. Then the following properties are equivalent:*

(a) *S has no fixed point in $\mathbb{C}^N$ and* $\det(A) \neq 0$.

(b) *The vector b is not in* $\operatorname{ran}(A - I)$ *and* $\det(A) \neq 0$.

(c) *C_S is universal.*

(d) *C_S is hereditarily universal.*

Proof. (a) $\iff$ (b): Simply observe that $b \in \operatorname{ran}(A - I)$ if and only if there exists $z_0 \in \mathbb{C}^N$ such that $(A - I)z_0 = b$ if and only if $Az_0 + b = z_0$ for some $z_0 \in \mathbb{C}^N$ if and only if $Sz_0 = z_0$ for some $z_0 \in \mathbb{C}^N$.

(d) $\Longrightarrow$ (c): This is trivial.

(c) $\Longrightarrow$ (a): If C_S is universal then S is one-to-one (hence $\det(A) \neq 0$) and has no fixed point by Lemma 2.1.

(b) $\Longrightarrow$ (d): Since $\det(A) \neq 0$, S is an affine automorphism of $\mathbb{C}^N$. Let us prove that S is non-recurrent. Denote by J the Jordan matrix of A. Then there is an invertible $(N \times N)$-matrix Q such that $A = QJQ^{-1}$. Define

$$c := Q^{-1}b \quad \text{and} \quad Mz := Jz + c \quad (z \in \mathbb{C}^N).$$

It is easy to see that $b \notin \operatorname{ran}(A - I)$ if and only if $c \notin \operatorname{ran}(J - I)$. On the other hand, non-recurrence is preserved by similarities; more precisely, if $\varphi \in H(\mathbb{C}^N, \mathbb{C}^N)$ and $\psi \in \operatorname{Aut}(\mathbb{C}^N)$, then φ is non-recurrent if and only $\psi \circ \varphi \circ \psi^{-1}$ is non-recurrent. Since $S = Q \circ M \circ Q^{-1}$ we obtain that S is non-recurrent if and only if M is.

The matrix J is a direct sum of Jordan blocks J_j. The vector c has components corresponding to each of these blocks and to say $c \notin \operatorname{ran}(J - I)$ is to say that at least one of these component, say c_j is not in $\operatorname{ran}(J_j - I_j)$. Here I_j is the identity matrix having the same dimension as J_j. Let M_j be the restriction of M to the direct summand of $\mathbb{C}^N$ on which J_j acts, i.e., $M_j z_j = J_j z_j + c_j$. It suffices to prove that S_j is non-recurrent (this follows from the fact that if $\mathbb{C}^N$ is represented as a product space, then each compact subset of $\mathbb{C}^N$ is contained in a product of compact subsets corresponding to the factors of $\mathbb{C}^N$). Hence we may drop the subscripts and assume that J itself is a Jordan block with $c \notin \operatorname{ran}(J - I)$. In particualr, $J - I$ is not invertible. Thus the spectrum of J is the singleton $\{1\}$, so J itself has just 1's on the main diagonal and the first superdiagonal, and zeros elsewhere. But, by induction, one obtains

$$M^n z = J^n z + \sum_{k=0}^{n-1} J^k c \quad (z \in \mathbb{C}^N,\, n \in \mathbb{N}).$$

The "1" in the (N, N) position is crucial here; it is also the (N, N) entry of any power of M, and in all these powers the rest of the N-th row consists of zeros. Thus $(M^n z)_N$, the N-th component of $M^n z$, is $z_N + nc_N$. Now $c \notin \operatorname{ran}(J - I)$ means $c_N \neq 0$, hence as $n \to \infty$ the N-th component of $M^n z$ goes to infinity uniformly on each compact subset of $\mathbb{C}^N$. Hence $||S^n z|| \to +\infty$ $(n \to \infty)$ in the same way. It is easy to see that because of this M is recurrent.

Consequently, S is non-recurrent and, due to Lemma 2.4, C_S is universal. Finally, observe that if we fix a sequence $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ then the same reasoning above –replacing n to n_j– shows that (M^{n_j}) (hence (S^{n_j})) is run-away, which by Remark 2.5 proves that $(C_{S^{n_j}})$ is universal. In other words, C_S is hereditarily universal. □

Remarks 3.2. 1. The proof of Theorem 3.1 reveals that, even in the case that S is not invertible, we have: S has no fixed point if and only if $b \notin$

ran$(A-I)$ if and only if S is non-recurrent. Observe also that, by Rouché–Frobenius' theorem, the middle condition can be expressed as $\text{rank}(A-I) < \text{rank}(A-I|b)$, where $A-I|b$ represents the $(N \times (N+1))$-matrix obtained from $A-I$ by adding the column b to $A-I$. In addition, note that if S is non-recurrent then 1 is an eigenvalue of A.

2. There are other popular automorphisms of $\mathbb{C}^N$ which are not affine. For example there are the "shears" introduced by Rosay and Rudin [22], and defined by

$$\sigma(z) = z + f(\Lambda(z))u \quad (z \in \mathbb{C}^N),$$

where Λ is a linear functional on $\mathbb{C}^N$ with $\Lambda(u) = 0$, u is a fixed non-zero and f is an entire function of one complex variable that is never zero. Since $\sigma^n(z) = z + nf(\Lambda(z))$, it is clear that σ is non-recurrent, hence if σ preserved convexity then the induced composition operator would be universal. Shears are of interest because the group they generate is known to be dense in the full automorphism group of $\mathbb{C}^N$ [5]. The example of Rosay–Rudin can be generalized: Let Λ be a linear map on $\mathbb{C}^N$ onto a proper subspace, and let f be an entire function non-zero on $\text{ran}\,\Lambda$. Then $\sigma(z) := z + f(\Lambda(z))$ defines an automorphism of $\mathbb{C}^N$ with the same iteration properties as the one above.

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