Operators with dense images everywhere*

by
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Abstract

In this paper, the authors introduce the dense-image operators T as those with a wild behaviour near of the boundary of a domain G, via certain subsets. The relationship with other kinds of operators with wild behaviour is studied, proving that the new concept generalizes the earlier of omnipresent, but there is no good relationship with the strongly omnipresent operators. We obtain, among other results, that the following kinds of operators are dense-image: onto linear operators; operators with local dense range satisfying soft conditions; Volterra complex integral operators plus infinite order differential operators, multiplication operators. In addition, holomorphic selfmappings and entire functions generating dense-image right or left composition operators are completely characterized.

Key words and phrases: dense-image operator, omnipresent operator, strongly omnipresent operator, differential operator, antidifferential operator, integral operator, holomorphic function, composition operator, left-composition operator, multiplication operator, local dense range, residual set, non-relatively compact set.

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1 Introduction and notation

In what follows $\mathbf{N}$ will be the set of positive integers, $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, $\mathbf{C}$ is the complex plane, $\mathbf{C}_\infty$ is the extended complex plane $\mathbf{C} \cup \{\infty\}$, $B(a, r)$ is the euclidean open ball with center a and radius r ($a \in \mathbf{C}$, $r > 0$). The open unit ball is $\mathbf{D} = B(0, 1)$. G will stand for a nonempty open subset of $\mathbf{C}$ (or a complex domain, that is, a nonempty connected open subset of $\mathbf{C}$), ∂G is the boundary of G in $\mathbf{C}_\infty$. We denote $O(\partial G) = \{V \subset \mathbf{C}_\infty : V$ is open and $V \cap \partial G \neq \emptyset\}$. If $A \subset \mathbf{C}$ then $\overline{A}$ represents the closure of A in $\mathbf{C}_\infty$, and $\|f\|_A := \sup_{z \in A} |f(z)|$, where f is a complex function defined on A, and $\mathrm{LT}(A) := \{$affine linear transformations $\tau(z) = az + b$ such that $\tau(\mathbf{D}) \subset A\}$. $H(G)$ denotes, as usual, the Fréchet space of holomorphic functions on G, endowed with the topology τ of local uniform convergence in G. In particular, $H(G)$ is a Baire space. Let $\mathcal{K}(G)$ be the family of compact subsets of G

and $\mathcal{K}_1(G)$ the family of compact subsets K of G such that each connected component of $\mathbf{C}_\infty \setminus K$ contains at least one connected component of $\mathbf{C}_\infty \setminus G$. It is known that the family

$$\{D(f,K,\varepsilon): \ f \in H(G), \ K \in \mathcal{K}_1(G), \ \varepsilon > 0\},$$

where $D(f,K,\varepsilon) = \{g \in H(G): \ |g(z) - f(z)| < \varepsilon \text{ for all } z \in K\}$, is a basis for τ. An operator on $H(G)$ is a continuous self-mapping $T : H(G) \to H(G)$, not necessarily linear.

As a consequence of a well-known result about interpolation [15, Theorem 15.13], a function $f \in H(G)$ can be found in such a way that $f(A)$ is dense in $\mathbf{C}$, where A is a prefixed non-relatively compact subset of G (see also [7], [8, Chapter IV], [11]). The first author [2] has proved the fact that, given $A \subset G$, the set of functions $f \in H(G)$ such that $f^{(j)}(A)$ is dense in $\mathbf{C}$ for all $f \in \mathbf{N}_0$ is residual in $H(G)$ if and only if A is not relatively compact in G.

In 1998, the second author [6] extended these facts by considering operators each of them being the sum of an infinite order differential operator and an integral operator. Given an operator T on $H(G)$ and a subset $A \subset G$, we denote $M(T,A) = \{g \in H(G): \ (Tg)(A) \text{ is dense in } \mathbf{C}\}$. Then the main result of [6] reads as follows.

Theorem 1.1 *Let $G \subset \mathbf{C}$ be a simply connected domain, $A \subset G$, $b \in G$, $\Phi(z) = \sum_{j=0}^{\infty} a_j z^j$ an entire function of subexponential type if $G \neq \mathbf{C}$ and of exponential type if $G = \mathbf{C}$, and let $\varphi : G \times G \to \mathbf{C}$ be an analytic function with respect to both variables. Consider the linear operator T on $H(G)$ defined by*

$$Tf(z) = \int_b^z \varphi(z,t) f(t)\,dt \qquad (z \in G)$$

where the integral is taken along any rectifiable curve in G joining b to z. Then the following properties are equivalent:

1. *Either*
   - *(a) $\Phi(z) \equiv 0$ on G and for every compact subset L of G there exist $a \in A \setminus L$ and $z \in G$ such that $\varphi(a,z) \neq 0$, or*
   - *(b) $\Phi(z) \not\equiv 0$ on G and A is not relatively compact in G.*
2. *$M(T + \Phi(D), A)$ is residual in G.*
3. *$M(T + \Phi(D), A) \neq \emptyset$.*

We have denoted $\Phi(D) = \sum_{j=0}^{\infty} a_j D^j$, D is the differentiation operator $Df = f'$, $D^0 = I =$ the identity operator and $D^{j+1} = D \circ D^j$. It is easy to see that $\Phi(D)$ is a well-defined operator under the conditions of Theorem 1.1.

By following another point of view, through "modified" cluster sets, operators on $H(G)$ with a wild behaviour near the boundary can be studied. Inspired by the notion of "monster" introduced and developed by W. Luh [12] and K.G. Grosse-Erdmann [10] (see also [13], [14] and [16]), the authors have recently presented the concept of "T-monster" and its associated notion of "strongly omnipresent operator" in the next way.

Given an operator T on $H(G)$, we say that a function $f \in H(G)$ is a T-monster if and only if for each $g \in H(\mathbf{D})$ and each $t \in \partial G$ there exist two sequences $\{a_n\}_n$ and $\{b_n\}_n$ in $\mathbf{C}$ such that $a_n z + b_n \to t$ as $n \to \infty$ uniformly on $\mathbf{D}$, $a_n z + b_n \in G$ (for all $n \in \mathbf{N}$ and all $z \in \mathbf{D}$) and $(Tf)(a_n z + b_n) \to g(z)$ as $n \to \infty$ locally uniformly in $\mathbf{D}$.

We say that T is strongly omnipresent whenever each subset $U(T, g, \varepsilon, r, V) := \{f \in H(G) : \text{there exists } \tau \in \mathrm{LT}(V \cap G) \text{ such that } \|(Tf) \circ \tau - g\|_{r\overline{\mathbf{D}}} < \varepsilon\}$ ($g \in H(\mathbf{D})$, $\varepsilon > 0$, $r \in (0,1)$, and $V \in O(\partial G)$) is dense in $H(G)$.

In [3] the authors proved that:

(1) Every strongly omnipresent operator T is omnipresent [1], i.e., the set $R(T, V, W) := \{f \in H(G) : \text{exists } z \in V \cap G \text{ such that } (Tf)(z) \in W\}$ is dense in $H(G)$, for every $V \in O(\partial G)$ and every non-empty open subset $W \subset \mathbf{C}$.

(2) T is strongly omnipresent if and only if the set $\mathcal{M}(T)$ of T-monsters is residual.

See [3–5] for examples of strongly omnipresent operators.

Our aim in this paper is:

(i) To introduce a new type of mapping, namely, the "dense-image operators", with wild behavior near the boundary of a domain via the image of plane sets, and to study its possible relationship with omnipresent and strongly omnipresent operators.

(ii) To construct new dense-image operators from other ones and to give sufficient conditions for an operator to be dense-image.

(iii) To give several concrete examples of dense-image operators.

We are going to develop each of these points in each of the following three sections.

2 What is a dense-image operator?

Assume that T is an operator on $H(G)$. We say that T is a *dense-image operator (DI-operator)* if and only if $M(T, A)$ is residual in $H(G)$, for every non-relatively compact subset $A \subset G$.

Remark 2.1 In [6] it is proved that the set $M(T,A)$ is always a G_δ-subset, in fact, it can be written as a countable intersection of sets $H_A(T,\delta,w) := \{g \in H(G) : \text{there exists } a \in A \text{ such that } |(Tg)(a) - w| < \delta\}$, where $\delta > 0$ and $w \in \mathbf{C}$. So $M(T,A)$ is residual if and only if it is dense in $H(G)$.

Remark 2.2 We directly demand A to be non-relatively compact because otherwise $M(T,A)$ is empty due to the continuity of T. As we can see in Theorem 1.1 or Example 2.5, the converse is not true.

Remark 2.3 It is obvious that for each subset A of G which is not relatively compact there exist a point $t \in \overline{A} \cap \partial G$ and a sequence $\{a_n\}_n \subset A$ with $a_n \to t$ as $n \to \infty$. Furthermore, $M(T,A) \subset M(T,B)$ whenever $A \subset B \subset G$.

From the last remark, we can easily obtain an alternative definition for DI-operators.

Theorem 2.4 *Let T be an operator on $H(G)$. Then the following two conditions are equivalent:*

(1) T is a DI-operator.

(2) For each point $t \in \partial G$ and each sequence $\{a_n\}_n \subset G$ with $a_n \to t$ as $n \to \infty$ the set $M(T, \{a_n\}_n)$ is dense in $H(G)$.

A first example of DI-operator is, by Theorem 1.1, any nonzero sum of differential and antidifferential infinite order operators and in particular all nonzero operators $\Phi(D)$ and $\Psi(D_b^{-1})$ (hence the identity operator I) are dense-image and strongly omnipresent (see [3]). Here Ψ is any non-zero function that is holomorphic in a neighbourhood of the origin, $b \in G$, $\Psi(D_b^{-1}) = \sum_{j=0}^{\infty} b_j D_b^{-j}$, $D_b^0 = I$ and, for each $j \in \mathbf{N}$, $D_b^{-j} f$ ($f \in H(G)$) denotes the unique j-antiderivative F of f such that $F^{(k)}(b) = 0$ ($k \in \{0, 1, \ldots, j-1\}$). Note that $\Psi(D_b^{-1})$ can be easily expressed as an integral operator T as in Theorem 1.1, for certain $\varphi(z,t)$ depending only on the difference $z - t$. The ways these two properties ("dense-image" and "strongly omnipresent") have been obtained are very similar and one could believe that both properties are the same. This is not true, however, as we are going to see in the next examples.

Example 2.5 If $\{z_n\}_n$ is a sequence in $\mathbf{D} \setminus \{0\}$ such that $\sum_{n=1}^{\infty}(1 - |z_n|) < +\infty$ then its associated Blaschke product is

$$B(z) = \prod_1^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z}.$$

Then $B(z)$ defines a function which is holomorphic in $\mathbf{D}$. In addition, if $E \subset \partial \mathbf{D}$ is the set of accumulation points of $\{z_n\}_n$, then $B(z)$ extends to be holomorphic on $\mathbf{C} \setminus (E \cup \{\overline{z_n}^{-1} : \ n \in \mathbf{N}\})$ (see [9, Theorem 6.1]). If we choose $z_n = 1 - \frac{1}{n^2}$ $(n \geq 2)$ then $E = \{1\}$ and $B \in H(\mathbf{C} \setminus [\{1\} \cup \{1 + \frac{1}{n^2-1} : n \geq 2\}])$. Since each factor has modulus 1 on $\partial \mathbf{D}$ we have that $B(z) \neq 0$ for all $z \in \Gamma := (\partial \mathbf{D}) \setminus \{1\}$ (in fact, $|B| = 1$ on Γ).

Summarizing, we have obtained a function $B(z)$ satisfying:

(1) $B \in H(\mathbf{D})$,

(2) $B(z) = 0$ if and only if $z \in \{1 - \frac{1}{n^2} : \ n \geq 2\}$,

(3) there exists a dense subset $\Gamma \subset \partial \mathbf{D}$ such that for every $t \in \Gamma$ there exists $\lim_{z \to t} B(z) \in \mathbf{C} \setminus \{0\}$.

Consider the mapping

$$\begin{array}{rlcl} T : & H(\mathbf{D}) & \to & H(\mathbf{D}) \\ & f & \mapsto & Tf(z) = B(z)f(z). \end{array}$$

Then T is a strongly omnipresent operator (see [4, Theorem 3.7]) but not a DI-operator. Indeed, if we take $A = \{1 - \frac{1}{n^2} : \ n \geq 2\}$ then A is not relatively compact in $\mathbf{D}$ and $M(T, A) = \emptyset$ because $Tf(A) = \{0\}$ for all $f \in H(\mathbf{D})$.

The latter is a linear example. Let us give an additional one. If we compare Corollary 4.2 of this paper with Corollary 3.2 of [4], we obtain that each composition operator C_φ on $H(\mathbf{C})$ with φ transcendent is also strongly omnipresent but not dense-image.

Example 2.6 In [4], it is shown that for each entire function φ the left-composition operator $L_\varphi : H(G) \to H(G)$ given by $L_\varphi f = \varphi \circ f$ is not strongly omnipresent if φ is not onto, so by Theorem 4.3 (see Section 4) for each entire function φ which is non-constant and non-onto the mapping L_φ is a DI-operator which is not strongly omnipresent. In fact, if φ is not onto then $\mathcal{M}(L_\varphi)$ is empty, so there exist DI-operators without any monster.

The above example is always non-linear, so the question may be asked, is every linear DI-operator strongly omnipresent? We point out here that there are linear omnipresent operators which are not strongly omnipresent [4, Example 3.6].

Let us see now the relationship between DI-operators and omnipresent operators.

Proposition 2.7 *Let $t \in \partial G$ and let $\{a_n\}_n$ be a sequence in G with $a_n \to t$ as $n \to \infty$. Then $M(T, \{a_n\}_n) \subset R(T, V, W)$ for every non-empty open subset W of $\mathbf{C}$ and every open set V with $t \in V$.*

Proof. Fix a non-empty open subset $W \subset \mathbf{C}$ and an open set V with $t \in V$. Since $a_n \to t$ $(n \to \infty)$, there exists $n_0 \in \mathbf{N}$ such that $a_n \in V \cap G$ for all $n \geq n_0$. Let f be an element of $M(T, \{a_n\}_n)$. Then

$$W \cap (Tf)\{a_n : \ n \geq n_0\} \neq \emptyset$$

because $(Tf)\{a_n : n \geq n_0\}$ is dense in $\mathbf{C}$ and W is open. So there exists $n_1 \in \mathbf{N}$ such that $a_{n_1} \in G \cap V$ and $Tf(a_{n_1}) \in W$, whence

$$f \in R(T, V, W).$$

The proof is finished. $\diamondsuit$

In particular, if for each point t of a dense subset Γ of ∂G there exists a sequence $\{a_n\} \subset G$ with $a_n \to t$ as $n \to \infty$ and $M(T, \{a_n\}_n)$ dense in $H(G)$ we have that T is omnipresent. From here we get the next corollary:

Corollary 2.8 *Every DI-operator is omnipresent.*

Observe that, by Example 2.5, the converse is not true. In fact, it is easy to check that the example provided in [4, Example 3.6] is not dense-image either. Hence, by assertion (1) in the Introduction, we have that linear omnipresent operators strictly include both dense-image operators and strongly omnipresent operators. However, as in Theorem 2.4, we can characterize the omnipresent operators in terms of sequences near the boundary.

Theorem 2.9 *The following two conditions are equivalent:*

(1) T is an omnipresent operator.

(2) There is a dense subset of $\Gamma \subset \partial G$ such that for every $t \in \Gamma$ there exists a sequence $\{a_n\}_n \subset G$ with $a_n \to t$ as $n \to \infty$ and $M(T, \{a_n\}_n)$ dense in $H(G)$.

Proof. (2) $\Rightarrow$ (1) is done. We need to prove (1) $\Rightarrow$ (2).

Fix $t \in \Gamma$, $w \in \mathbf{C}$ and $\delta > 0$, and for each $n \in \mathbf{N}$ consider the open ball $V_n := B(t, \frac{1}{n})$. Then $R(T, V_n, B(w, \delta)) \cap D(h, K, \varepsilon) \neq \emptyset$ (for all $h \in H(G)$, all $K \in \mathcal{K}(G)$ and all $\varepsilon > 0$), by (1). So, we can find a function $f \in D(h, K, \varepsilon)$ and a point $a_n \in V_n \cap G$ such that $|Tf(a_n) - w| < \delta$. In this manner the sequence $\{a_n\}_n$ is in G, $a_n \to t$ as $n \to \infty$ and $H_{\{a_n\}_n}(T, \delta, w)$ is dense in $H(G)$. Therefore, $M(T, \{a_n\}_n)$ is dense in $H(G)$ by Remark 2.1, and the proof is finished. $\diamondsuit$

To finish this section we provide the next corollary (not known up to now) of Theorem 1.1 and Corollary 2.8. Φ and Ψ denote entire functions, with Φ of exponential type if $G = \mathbf{C}$ and of subexponential type if $G \neq \mathbf{C}$.

Corollary 2.10 *Every operator $T = \Phi(D) + \Psi(D^{-1})$ which is not identically zero is omnipresent on $H(G)$.*

3 General conditions for DI-operators

As promised in the first section, here we are going to furnish sufficient conditions for an operator to be dense-image. Because in the first three results the reasoning is very similar to that in [4, Section 2], we do not include their proofs. Let us only point out that Lemma 3.1 together with the Open Mapping Theorem yields Theorem 3.2.

Lemma 3.1 *Let $T, S : H(G) \to H(G)$ be operators and $A \subset G$. Then we have $S^{-1}(M(T, A)) = M(TS, A)$.*

Theorem 3.2 *Let $T, S : H(G) \to H(G)$ be operators, in such a way that T is a DI-operator and S is linear and onto. Then TS is a DI-operator.*

Corollary 3.3 *If S is an onto linear operator on $H(G)$ then S is DI-operator.*

Before providing additional results we consider it convenient to isolate the following local stability condition:

$$\left.\begin{array}{c}\text{For every } \; K \in \mathcal{K}(G), \text{ there exists a compact set } M \subset G \\ \text{satisfying that for every } a \in G \setminus M, \text{ every } \delta > 0 \text{ and} \\ \text{every } f \in H(G) \text{ there exist a closed ball } B \subset G \setminus K \\ \text{and } \alpha > 0 \text{ such that for every } g \in H(G) \\ \|f - g\|_B < \alpha \Longrightarrow |Tf(a) - Tg(a)| < \delta. \end{array}\right\} (P)$$

Observe that, for instance, the operators $\Phi(D)$ satisfy (P). This is an easy exercise if one employs Cauchy's inequalities.

Theorem 3.4 *Let T be an operator on $H(G)$ such that:*

(a) $M(T, A) \neq \emptyset$ for all non-relatively compact subsets $A \subset G$.

(b) T satisfies condition (P).

Then T is a DI-operator.

Proof. Fix a non-relatively compact subset $A \subset G$, $w \in \mathbf{C}$, $\delta > 0$, $K \in \mathcal{K}_1(G)$, $h \in H(G)$ and $\varepsilon > 0$. We have to prove that $H_A(T, \delta, w) \cap D(h, K, \varepsilon)$ is not empty. Let M be the compact subset associated to K furnished by condition (P).

Take a function f in $M(T, A \setminus M)$ (note that $A \setminus M$ is not relatively compact in G). Then $f \in H_{A \setminus M}(T, \delta/2, w)$, and there exists a point $a \in A \setminus M$ such that

$$|Tf(a) - w| < \frac{\delta}{2}. \tag{1}$$

By (P), there exist $\alpha > 0$ and a closed ball $B \subset G \setminus K$ such that for all $\varphi \in H(G)$,

$$\|\varphi - f\|_B < \alpha \Longrightarrow |T\varphi(a) - Tf(a)| < \frac{\delta}{2}. \tag{2}$$

Consider the compact set $L := K \cup B$. Note that $K \cap B = \emptyset$, so each connected component of the complement of L contains at least one component of the complement of G, as K does. Pick open subsets G_1, $G_2 \subset G$ with $G_1 \cap G_2 = \emptyset$ and $K \subset G_1$, $B \subset G_2$. Denote $G_0 = G_1 \cup G_2$. Therefore G_0 is open and $L \subset G_0 \subset G$. Define the function $F : G_0 \to \mathbf{C}$ as

$$F(z) = \begin{cases} h(z) & \text{if } z \in G_1 \\ f(z) & \text{if } z \in G_2. \end{cases}$$

Then $F \in H(G_0)$, and an application of Runge's theorem [15, Chap. 13] yields the existence of a rational function f_1 with poles outside G such that

$$\|F - f_1\|_L < \min\{\varepsilon, \alpha\}.$$

Thus, $f_1 \in H(G)$, and it holds that

$$\|f_1 - h\|_K < \varepsilon \tag{3}$$

and

$$\|f_1 - f\|_B < \alpha.$$

Therefore, by (2),

$$|Tf_1(a) - Tf(a)| < \frac{\delta}{2}. \tag{4}$$

Now, by the triangle inequality, (1) and (4), we have

$$|Tf_1(a) - w| < \delta. \tag{5}$$

And, joining (3) and (5),

$$f_1 \in D(h, K, \varepsilon) \cap H_A(T, \delta, w).$$

Consequently, the proof is finished. $\diamondsuit$

When we have "wild behavior" of T on just one function and "good behavior" on a dense set of functions we are able to show that T is a DI-operator. Here the linearity is needed.

Lemma 3.5 *Let $T : H(G) \to H(G)$ be a linear operator, $t \in \partial G$ and let $\{a_n\}_n$ be a sequence in G with $a_n \to t$ as $n \to \infty$. Suppose that there exists a dense subset $\mathcal{D}$ of $H(G)$ such that for each function $h \in \mathcal{D}$,*

$$\exists \lim_{n\to\infty} (Th)(a_n) \in \mathbf{C}.$$

Then, if $M(T, \{a_n\}_n)$ is not empty, $M(T, \{a_n\}_n)$ is a dense subset of $H(G)$.

Proof. Take $f \in M(T, \{a_n\}_n)$. We have only to prove that $f+h \in M(T, \{a_n\}_n)$ for all $h \in \mathcal{D}$.

Fix $w \in \mathbf{C}$ and $\delta > 0$. Then there exists $N \in \mathbf{N}$ such that

$$|Th(a_n) - l_h| < \frac{\delta}{2} \qquad (\forall n \geq N), \tag{1}$$

where $l_h = \lim_{n\to\infty} Th(a_n)$ $(\in \mathbf{C})$. Since $\overline{(Tf)(\{a_n\}_n)} = \mathbf{C}_\infty$, we also have $\overline{(Tf)(\{a_n : n \geq N\})} = \mathbf{C}_\infty$ so we can find a natural number $m \in \mathbf{N}$ with $m \geq N$ and

$$|Tf(a_m) - (w - l_h)| < \frac{\delta}{2}. \tag{2}$$

Therefore, by (1), (2) and the triangle inequality, we have

$$|T(f+h)(a_m) - w| = |Tf(a_m) + Th(a_m) - w| \leq$$

$$|Tf(a_m) - (w - l_h)| + |Th(a_m) - l_h| < \delta.$$

So, $\overline{(T(f+h))(\{a_n\}_n)} = \mathbf{C}_\infty$ and this completes the proof. $\diamondsuit$

Theorem 3.6 *Let T be a linear operator on $H(G)$ satisfying*

(1) There exists a dense subset $\mathcal{D}$ in $H(G)$ with the property that for each $h \in \mathcal{D}$ and each $t \in \partial G$

$$\exists \lim_{z\to t} (Th) \in \mathbf{C}.$$

(2) $M(T, A)$ is not empty for every non-relatively compact subset $A \subset G$.

Then T is a DI-operator.

Proof. By Remark 2.1 we need only prove that for each $t \in \partial G$ and each sequence $\{a_n\}_n \subset G$ with $a_n \to \infty$, as $n \to \infty$, we have

$$(1) + [M(T, \{a_n\}_n) \neq \emptyset] \quad \text{implies} \quad [M(T, \{a_n\}_n) \text{ is dense}].$$

But this is clear, because (1) gives us the hypothesis of Lemma 3.5. So, the proof is finished. $\diamondsuit$

For instance, condition (1) in the above theorem is satisfied by a differential operator $\Phi(D)$ and by a finite-order antidifferential operator $\Psi(D^{-1})$, whenever G is a bounded, simply connected domain in $\mathbf{C}$: just take $\mathcal{D} = \{\text{polynomials}\}$.

In our next statement we furnish a sufficient condition for an operator "pointwise dense range near the boundary" to be dense-image. We meet again property (P).

Theorem 3.7 *Let T be an operator on $H(G)$ such that the following two properties are satisfied:*

(A) T satisfies condition (P).

(B) For each point $t \in \partial G$ there exists an open subset $V \subset \mathbf{C}_\infty$ with $t \in V$ satisfying that the set $\{(Tf)(a) : \; a \in E, \; f \in H(G)\}$ is dense in $\mathbf{C}$ for each infinite subset $E \subset V \cap G$.

Then T is a DI-operator.

Proof. Fix a point $t \in \partial G$, a sequence $\{a_n\}_n \subset G$ with $a_n \to t$ as $n \to \infty$, two positive numbers δ and ε, a complex number w, a subset $K \in \mathcal{K}_1(G)$ and a function $h \in H(G)$. We have to show that

$$H_{\{a_n\}_n}(T, \delta, w) \cap D(h, K, \varepsilon) \neq \emptyset.$$

Again, there is a compact set $M \subset G$ associated to K satisfying the property stated in (P).

Since $a_n \to t$ as $n \to \infty$, $\{a_n\} \subset G$ and $t \in \partial G$, there exists $N \in \mathbf{N}$ such that the set $E := \{a_n : \; n \geq N\}$ is infinite and is contained in $(V \cap G) \setminus M$.

So, by (B), we obtain a function $f \in H(G)$ and a natural number $m \geq N$ with

$$|Tf(a_m) - w| < \frac{\delta}{2}.$$

But, since by (A), T satisfies (P) and $a_m \in (V \cap G) \setminus M \subset G \setminus M$, there exist a closed ball $B \subset G \setminus K$ and a positive number α such that if $\varphi \in H(G)$ and $\|f - \varphi\|_B < \alpha$ then $|T\varphi(a_m) - Tf(a_m)| < \delta/2$.

Now, we have only to continue the proof in the same way as in Theorem 3.4. $\diamondsuit$

Note that, as a consequence, we obtain again, independently, that I is a DI-operator.

It is evident that the sum $T + S$ of two DI-operators need not be a DI-operator: take, for instance, $T = I$, $S = -I$. In the next result it is shown that if T is DI-operator and it is possible to exert a control on S near the boundary, then we can sum T and S without loss of wild behavior.

Theorem 3.8 *Let $T, S : H(G) \to H(G)$ be operators. Assume that:*

(a) T is a DI-operator.

(b) For every $f \in H(G)$ and every $t \in \partial G$,

$$\exists \lim_{z \to t} (Sf)(z) \in \mathbf{C}.$$

Then, $T + S$ is a DI-operator.

Proof. We need only check that $M(T, \{a_n\}_n) \subset M(T + S, \{a_n\}_n)$ for each sequence $\{a_n\}_n \subset G$ with $a_n \to t \in \partial G$ as $n \to \infty$. The way to obtain this is analogous to that used in Lemma 3.5. $\diamondsuit$

To finish this section we want to say something about the product of mappings. We have that a non-zero multiple of a DI-operator is trivially a DI-operator. In general, we can make sure that the product of T and S is a DI-operator if T is and S keeps a control.

Lemma 3.9 *Let T, S be operators on $H(G)$, $t \in \partial G$ and $\{a_n\}_n \subset G$ with $a_n \to t$ as $n \to \infty$. Then, for each function $f \in M(T, \{a_n\}_n)$ such that there exists $\lim_{n\to\infty} Sf(a_n) \in \mathbf{C} \setminus \{0\}$, we have $f \in M(T \cdot S, \{a_n\}_n)$.*

Here $(T{\cdot}S)f := (Tf){\cdot}(Sf)$. The proof of the latter lemma is straightforward and is left to the reader. As a consequence of Lemma 3.9, we obtain

Theorem 3.10 *Let T, S be operators on $H(G)$ satisfying:*

(a) T is DI-operator.

(b) For all $t \in \partial G$ and all $f \in H(G)$,

$$\exists \lim_{z\to t} Sf(z) \in \mathbf{C} \setminus \{0\}.$$

Then, $T \cdot S$ is a DI-operator.

4 Examples of DI-operators

Up to now we have, by Theorem 1.1, only one specific type of DI-operator, which includes differential, antidifferential, and some integral operators.

In this section we are going to give some examples which are essentially different from the previous ones, such as composition operators.

Recall that if $\varphi \in H(G, G) := \{f \in H(G) : \ f(G) \subset G\}$ then its associated composition operator is the mapping defined as

$$C_\varphi : f \in H(G) \mapsto f \circ \varphi \in H(G).$$

It is easy to see that C_φ is in fact an operator on $H(G)$.

Theorem 4.1 *Let $G \subset \mathbf{C}$ be a domain, $A \subset G$, and $\varphi \in H(G, G)$. Then the following three properties are equivalent:*

(a) $\varphi(A)$ is not relatively compact in G.

(b) $M(C_\varphi, A)$ *is residual in* $H(G)$*.*

(c) $M(C_\varphi, A)$ *is not empty.*

As a consequence, C_φ *is a DI-operator if and only if* φ *takes non-relatively compact subsets to non-relatively compact subsets.*

Proof. $(c) \Rightarrow (a)$ is an immediate consequence of two facts, namely, Remark 2.2 and $M(C_\varphi, A) = M(I, \varphi(A))$. $(b) \Rightarrow (c)$ is trivial. To get $(a) \Rightarrow (b)$ we have only to take into account the above equality and that, by Theorem 1.1, I is a DI-operator. So the proof is complete. $\diamondsuit$

Observe that the condition on φ in the second part of the last theorem is equivalent to φ is proper, that is, the preimage $\varphi^{-1}(K)$ of each compact set $K \subset G$ is again a compact set.

Corollary 4.2 *Assume that* φ *is an entire function and consider the composition operator* C_φ *on* $H(\mathbf{C})$*. Then the following properties are equivalent:*

(a) C_φ *is a DI-operator.*

(b) $M(C_\varphi, A)$ *is not empty for every non-relatively compact subset* $A \subset \mathbf{C}$*.*

(c) φ *is a non-constant polynomial.*

Proof. It is clear that non-constant polynomials are the unique entire functions preserving the property of being not relatively compact for subsets. $\diamondsuit$

As a second example, we can consider the left-composition operators. If φ is an entire function, the left-composition operator associated to φ is the (generally non-linear) continuous self-mapping given by

$$L_\varphi : f \in H(G) \mapsto \varphi \circ f \in H(G).$$

It happens that L_φ is omnipresent if and only if φ is non-constant: take $T = I$ and $g = \varphi$ in Theorem 1(c) of [1]. Therefore, by Corollary 2.8, it is a necessary condition. Let us see that it is also a sufficient condition for L_φ to be a DI-operator.

Theorem 4.3 *Let* φ *be an entire function. Then the following properties are equivalent:*

(a) L_φ *is a DI-operator.*

(b) $M(L_\varphi, A)$ *is not empty for every non-relatively compact subset* $A \subset G$*.*

(c) φ *is non-constant.*

Proof. It is obvious that $(a) \Rightarrow (b) \Rightarrow (c)$. So, we need only get $(c) \Rightarrow (a)$.

Let us try to apply Theorem 3.7 under the assumption that φ is non-constant. Note that the continuity of φ guarantees that (P) is satisfied. Furthermore, if $\{w_n\}_n$ is a dense sequence in $\mathbf{C}$ and f_n is the constant function $f_n(z) := w_n$ $(n \in \mathbf{N})$ then, for any fixed $a \in G$, the set $\{(L_\varphi f_n)(a) : n \in \mathbf{N}\} = \varphi(\{w_n : n \in \mathbf{N}\})$ is dense in $\mathbf{C}$ because $\varphi(\mathbf{C})$ is. Therefore (A) and (B) in Theorem 3.7 are fulfilled and L_φ is a DI-operator. $\diamondsuit$

To finish this section (and the paper) we want to study the multiplication operator. Let T be an operator on $H(G)$ and let $\psi \in H(G)$. Then the multiplication operator $M_\psi T : H(G) \to H(G)$ is defined as $(M_\psi T)f(z) = \psi(z)Tf(z)$. By Theorem 3.10 we know that if ψ extends to ∂G as a continuous function, always non-zero in the boundary, and T is a DI-operator then $M_\psi T$ is also a DI-operator. However, in several cases we can ask for a weaker condition for ψ. Note that because of the Analytic Continuation Principle, $\psi(z) \neq 0$ near the boundary of G (i.e., $\psi(z) \neq 0$ for all $z \in G \setminus L$, for some compact set $L \subset G$) if and only if the set $Z(\psi) = \{z \in G : \psi(z) = 0\}$ of zeros of ψ is finite.

Theorem 4.4 *Suppose that $\psi \in H(G)$ and that T is an operator on $H(G)$ satisfying the following properties:*

(a) The zero set $Z(\psi)$ is finite.

(b) There exists $L \in \mathcal{K}(G)$ such that for each $a \in G \setminus L$ the set $\{Tf(a) : f \in H(G)\}$ is dense in $\mathbf{C}$.

(c) T satisfies condition (P).

Then $M_\psi T$ is a DI-operator.

Proof. Fix a non-relatively compact subset $A \subset G$, $w \in \mathbf{C}$, $\delta > 0$, $h \in H(G)$, $K \in \mathcal{K}_1(G)$, and $\varepsilon > 0$. Let us associate to K the compact set M given by condition (P).

Observe that $A \setminus (M \cup Z(\psi) \cup L) \neq \emptyset$. Pick $a \in A \setminus (M \cup Z(\psi) \cup L)$ $(\subset G \setminus M)$, whence $\psi(a) \neq 0$.

From (b), there is a function $f \in H(G)$ such that

$$|(Tf)(a) - \frac{w}{\psi(a)}| < \frac{\delta}{2|\psi(a)|}. \tag{1}$$

Consider G_1, G_2, and G_0 as in the proof of Theorem 3.4 and the function $F : G_0 \to \mathbf{C}$ defined as

$$F(z) = \begin{cases} h(z) & \text{if } z \in G_1 \\ f(z) & \text{if } z \in G_2. \end{cases}$$

Again $F \in H(G_0)$, and an application of Runge's theorem gives us a function $g \in H(G)$ with

$$\|g - h\|_K < \varepsilon \tag{2}$$

and

$$\|g - f\|_B < \alpha,$$

where the positive number α and the closed ball $B \subset G \setminus K$ are provided by (c), but with δ replaced by $\frac{\delta}{2|\psi(a)|}$ when (P) is used. Therefore,

$$|Tg(a) - Tf(a)| < \frac{\delta}{2|\psi(a)|}. \tag{3}$$

Now (1), (3) and the triangle inequality lead us to

$$|(M_\psi Tg)(a) - w| = |\psi(a)Tg(a) - w| \leq \delta. \tag{4}$$

Finally, by joining (2) and (4),

$$g \in H_A(M_\psi T, \delta, w) \cap D(h, K, \varepsilon),$$

as needed. $\diamondsuit$

Since $T := I$ trivially satisfies conditions (b) and (c) above, we obtain the next corollary. Its "only if" part follows from the fact that if $Z(\psi)$ is infinite then it is not relatively compact in G, but $M(M_\psi, Z(\psi)) = \emptyset$.

Corollary 4.5 *If $\psi \in H(G)$ then the mapping $M_\psi : H(G) \to H(G)$ defined as $M_\psi f(z) = \psi(z)f(z)$ is a DI-operator if and only if $Z(\psi)$ is finite.*

To finish, observe that if we let $\psi \equiv 1$ in Theorem 4.4 then one gets that (b) plus (c) imply that T is dense-image. But this criterion is weaker than Theorem 3.7 because (B) of Theorem 3.7 is weaker than (b) of Theorem 4.4.

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