TITLE: HOLOMORPHIC FUNCTIONS HAVING LARGE IMAGES UNDER THE ACTION OF DIFFERENTIAL OPERATORS.

AUTHOR: LUIS BERNAL-GONZÁLEZ.

AFFILIATION: DEPARTAMENTO DE ANÁLISIS MATEMÁTICO. FACULTAD DE MATEMÁTICAS. AVENIDA REINA MERCEDES. APARTADO 1160. 41080 SEVILLA, SPAIN. E-MAIL: lbernal@cica.es.

FOOTNOTES TO THE TITLE: *This work is supported in part by D.G.E.S. grant PB96-1348.

1991 Mathematics Subject Classification: Primary 30E10. Secondary 46E10, 47E05.

Key words and phrases: Entire function of subexponential type, infinite order differential operator, residual set, holomorphic function, relatively compact sequence, linear metric space, large images.
ABBREVIATED TITLE: OPERATORS AND LARGE IMAGES.

NAME AND MAILING ADDRESS OF THE AUTHOR TO WHOM PROOFS SHOULD BE SENT: LUIS BERNAL-GONZÁLEZ. DEPARTAMENTO DE ANÁLISIS MATEMÁTICO. FACULTAD DE MATEMÁTICAS. AVENIDA REINA MERCEDES. APARTADO 1160. 41080 SEVILLA, SPAIN.

E-MAIL: lbernal@cica.es.
HOLOMORPHIC FUNCTIONS HAVING LARGE IMAGES

UNDER THE ACTION OF DIFFERENTIAL OPERATORS

By

LUIS BERNAL-GONZÁLEZ*

Abstract. We prove in this note that, given a simply connected domain $G$ in the complex plane and a sequence of infinite order linear differential operators generated by entire functions of subexponential type satisfying suitable conditions, then there are holomorphic functions $f$ on $G$ such that the image of any open subset under the action of those operators on $f$ is arbitrarily large. This generalizes an earlier result about images of derivatives. A known statement about close orbits is also strengthened.

1. INTRODUCTION AND NOTATION

In this paper we denote, as usual, by $\mathbb{N}$ the set of positive integers, by $\mathbb{C}$ the field of complex numbers and by $D(a, r)$ the open disk $\{z \in \mathbb{C} : |z - a| < r\}$. If $G$ is a nonempty open subset of $\mathbb{C}$, then $H(G)$ will stand for the Fréchet space of holomorphic functions on $G$, endowed with the topology of the uniform convergence on compact subsets. A domain (=nonempty open subset) $G \subset \mathbb{C}$ is said to be simply connected whenever its complement with respect to the extended complex plane is connected. Recall that, by Runge’s theorem [4, pp. 92-97], if $G$ is a simply connected
domain then the set of polynomials is dense in $H(G)$. If $A \subset \mathbb{C}$ is a subset with at least one finite accumulation point and $G$ is simply connected, then the linear manifold

$$H_A = \text{span} \{ e_a : a \in A \}$$

is dense in $H(G)$ (see, for instance, [7, p. 97], [6, pp. 259-260] and [2, Section 4]). We have denoted here $e_a(z) = \exp(az)$ ($z \in \mathbb{C}$). The diameter of a subset $A \subset \mathbb{C}$ is $\text{diam} (A) = \sup \{|z - w| : z, w \in A\}$.

Every entire function $\Phi(z) = \sum_{j=0}^{\infty} c_j z^j$ generates a formal “infinite order linear differential operator” with constant coefficients given by $\Phi(D) = \sum_{j=0}^{\infty} c_j D^j$, where $D$ denotes the differentiation operator $Df = f'$ and $D^0 = I$ = the identity operator.

The function $\Phi$ is said to be of exponential type if and only if there are constants $K_1, K_2 \in (0, +\infty)$ such that

$$|\Phi(z)| \leq K_1 e^{K_2|z|} \quad (z \in \mathbb{C}).$$

$\Phi$ is said to be of subexponential type if and only if for every $\varepsilon > 0$ there is a constant $K = K(\varepsilon) \in (0, +\infty)$ such that

$$|\Phi(z)| \leq K e^{\varepsilon|z|} \quad (z \in \mathbb{C}).$$

Each function of subexponential type is obviously of exponential type. With essentially the same methods of Valiron [9, p. 35] (see also [3, pp. 58-60] and [2, Theorem 5]), it can be proved that $\Phi(D)$ is a well-defined operator on $H(G)$ as soon as $\Phi$
is of subexponential type (and, in fact, on \( H(C) \) if \( \Phi \) is just of exponential type). The reader is referred to [8] for a systematic study of this kind of operators.

From now on, if \( X \) is a linear metric space, then we denote \( ||x|| = d(x, 0) \) for \( x \in X \), where \( d \) is the (translation-invariant) metric of \( X \). For instance, the translation-invariant metric

\[
d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\sup_{K_n} |f - g|}{1 + \sup_{K_n} |f - g|} \quad (f, g \in H(G))
\]

generates the natural topology of \( G \). Here \( (K_n) \) is an exhaustive nondecreasing sequence of compact subsets of \( G \). In 1987, R. M. Gethner and J. H. Shapiro [5], when studying the existence of universal vectors for certain kinds of operators on spaces of holomorphic functions, proved the following related result [5, Theorem 2.4].

**THEOREM A.** Suppose \( T \) is a continuous linear operator on a separable complete linear metric space \( X \) and \( T^n x \to 0 \) \((n \to \infty)\) for every \( x \in D \), where \( D \) is a dense subset of \( X \) and \( T^n = T \circ T \circ \cdots \circ T \) \((n \text{ times})\). Let \( (x_n) \) be a sequence in \( X \) such that \( x_n \to 0 \) \((n \to \infty)\). Then the set of vectors \( x \in X \) for which

\[
\liminf_{n \to \infty} ||T^n x - T^n x_n|| = 0
\]

is a dense \( G_\delta \) subset of \( X \).

As an application of Theorem A to function theory, it is shown in [5, Theorem 3.7] the next theorem, that establishes the existence of entire functions for which many derivatives have “large images” on prefixed arbitrarily small open subsets.
THEOREM B. Suppose \((\rho_n)\) is an unbounded, increasing sequence of positive numbers for which \(\lim_{n \to \infty} \frac{\rho_n^{1/n}}{n} = 0\). Then there exists a dense \(G_\delta\) subset \(M\) of \(H(G)\) satisfying that, for every \(f \in M\) and every nonempty open set \(V \subset G\), there are infinitely many \(n \in \mathbb{N}\) such that \(f^{(n)}(V) \supset D(0, \rho_n)\).

Our aim in this note is to extend the latter result to certain kinds of infinite order differential operators on simply connected domains. In passing, Theorem A can also be manifestly strengthened.

2. SETS OF POINTS CLOSE TO THE ORBIT OF A RELATIVELY COMPACT SEQUENCE

We start with an elementary lemma. Let \(X\) be a topological space, \((Y, d)\) a metric space and denote, as usual, by \(C(X, Y)\) the space of all continuous mappings from \(X\) into \(Y\). If \(\sigma = (s_n)\) and \(\tau = (T_n)\) are sequences in \(X\) and \(C(X, Y)\) respectively, then we put

\[
M(\sigma, \tau) = \{ x \in X : \liminf_{n \to \infty} d(T_n x, T_n s_n) = 0 \}.
\]

LEMMA 1. If \(X, (Y, d), \sigma = (s_n)\) and \(\tau = (T_n)\) are as before, then \(M(\sigma, \tau)\) is a \(G_\delta\) subset of \(X\).

Proof. Fix a point \(x \in X\). Observe that \(x \in M(\sigma, \tau)\) if and only if for each
pair $N, k \in \mathbb{N}$ there is $n > N$ such that $d(T_n x, T_n s_n) < 1/k$, that is,

$$M(\sigma, \tau) = \bigcap_{N \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \bigcup_{n > N} T_n^{-1}(G_{n, k}),$$

where $G_{n, k} = \{ y \in Y : d(y, T_n s_n) < 1/k \}$, which is an open ball in $X$. Since $T_n$ is continuous, $T_n^{-1}(G_{n, k})$ is open in $X$ and so $M(\sigma, \tau)$ is a $G_\delta$ subset.  

For every sequence $\sigma = (s_n)$ in $X$, denote

$$LP(\sigma) = \{ \alpha \in X : \alpha \text{ is a limit point for } (s_n) \}.$$

$LP(\sigma)$ may well be empty. If $X$ and $Y$ are topological vector spaces, then $L(X, Y)$ will stand for the subspace of $C(X, Y)$ of all linear mappings from $X$ into $Y$. Recall that a subset in a Baire space is residual if and only if it contains a dense $G_\delta$ subset.

The next result generalizes Theorem A.

**THEOREM 1.** Assume that $X$ and $Y$ are linear metric spaces, in such a way that $X$ is a Baire space. Let $\sigma = (s_n)$ and $\tau = (T_n)$ be sequences in $X$ and $L(X, Y)$, respectively. Suppose that the following three conditions are satisfied:

(a) $\sigma$ is relatively compact.

(b) $\lim_{n \to \infty} T_n \alpha = 0$ for every $\alpha \in LP(\sigma)$.

(c) There exists a dense subset $\mathcal{D} \subset X$ such that $\liminf_{n \to \infty} ||T_n x|| = 0$ for all $x \in \mathcal{D}$.

Then $M(\sigma, \tau)$ is residual in $X$.  

7
Proof. Note that, by (a), $LP(\sigma)$ is not empty. Recall that $\|x\| = d_1(x, 0)$ for every $x \in X$ and $\|y\| = d(y, 0)$ for every $y \in Y$, where $d_1, d$ are the metrics on $X, Y$ (resp.), which are translation-invariant. By Lemma 1, $M(\sigma, \tau)$ is a $G_\delta$ subset of $X$.

Let us keep in mind the notation of the proof of that lemma. Since $X$ is Baire and the sets $S(N, k) := \bigcup_{n>N} T_n^{-1}(G_n, k)$ ($N, k \in \mathbb{N}$) are open, it suffices to show that every $S(N, k)$ is dense in $X$. For this, fix $N, k \in \mathbb{N}$, a point $x_0 \in D$ and $\varepsilon > 0$. By (c), there is a sequence $n_1 < n_2 < \ldots < n_j < \ldots$ of positive integers such that

$$\lim_{j \to \infty} \|T_{n_j}x_0\| = 0.$$ 

But $\sigma$ is relatively compact, so there is a point $\alpha \in X$ and a subsequence $m_1 < m_2 < \ldots < m_j < \ldots$ of $(n_j)$ such that

$$\lim_{j \to \infty} \|s_{m_j} - \alpha\| = 0.$$ 

From (b), we have that

$$\lim_{j \to \infty} \|T_{m_j} \alpha\| = 0.$$ 

In particular, there exists $n > N$ such that $\|T_n x_0\| < \frac{1}{2k}$, $\|T_n \alpha\| < \frac{1}{2k}$ and $\|s_n - \alpha\| < \varepsilon$. Define the point

$$x = x_0 + s_n - \alpha.$$ 

Then $\|x - x_0\| = \|s_n - \alpha\| < \varepsilon$ and, by linearity, $d(T_n x, T_n s_n) = \|T_n x - T_n s_n\| = \|T_n x_0 - T_n \alpha + T_n s_n - T_n s_n\| \leq \|T_n x_0\| + \|T_n \alpha\| < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}$. 

8
Thus, $x \in S(N, k) \cap \{z \in X : d_1(z, x_0) < \varepsilon\}$, that is, every point $x_0 \in \mathcal{D}$ is in the closure of $S(N, k)$. But $\mathcal{D}$ is dense in $X$. Consequently, $S(N, k)$ is also dense in $X$, as required.

3. DIFFERENTIAL OPERATORS AND LARGE IMAGES

In this section we extend Theorem B to differential operators. Recall that if $\Phi(z) = \sum_{j=0}^{\infty} c_j z^j$ is a nonconstant entire function, then its multiplicity for the zero at the origin is $m = \min\{j \in \{0, 1, 2, \ldots\} : c_j \neq 0\}$. We start with the following easy lemma, whose proof is omitted since it is a simple calculation.

**Lemma 2.** If $a, b$ are complex numbers with $a \neq 0$, and $m \in \mathbb{N}$, then

$$ (aD^m + bD^{m+1})(\frac{1}{a} \frac{z^{m+1}}{(m+1)!} - \frac{b}{a^2 m!}) = z \quad (z \in \mathbb{C}). $$

**Theorem 2.** Let $G$ be a simply connected domain of $\mathbb{C}$, $\Phi_n(z) = \sum_{j=0}^{\infty} c_{n,j} z^j$ ($n \in \mathbb{N}$) nonconstant entire functions of subexponential type and $(\rho_n)$ an unbounded sequence of positive numbers. Denote by $m(n)$ the multiplicity of $\Phi_n$ for the zero at the origin. Suppose that the following conditions are fulfilled:

1. The sequence $(m(n))$ is unbounded.
2. $\max\{\limsup_{n \to \infty} \frac{1}{1+m(n)} (\frac{\rho_n}{|c_{m(n)}^{(n)}|})^{\frac{1}{1+m(n)}}), \limsup_{n \to \infty} \frac{1}{1+m(n)} (\frac{|c_{m(n)}^{(n)}|}{|c_{m(n)}^{(n)}|})^{\frac{1}{1+m(n)}} \}
   \leq \frac{1}{\varepsilon \cdot \text{diam}(G)}.$

Then there exists a residual subset $M$ of $H(G)$ satisfying that, for every $f \in M$
and every nonempty open subset $V \subset G$, there are infinitely many $n \in \mathbb{N}$ for which 
$$(\Phi_n(D)f)(V) \supset D(0, \rho_n).$$

**Proof.** Let us try to apply Theorem 1. Firstly, observe that every $\Phi_n(D)$ is a well-defined linear operator on $H(G)$. We start in a way that is very similar to the proof of [5, Theorem 3.7]. By the definition of multiplicity, $c_j^{(n)} = 0$ whenever $j < m(n)$ ($n \in \mathbb{N}$). Since $(m(n))$ is unbounded, we can assume without loss of generality that $1 \leq m(n) \uparrow \infty$. Indeed, there exists a subsequence $(m(n_k))$ such that $1 \leq m(n_k) \uparrow \infty$. Then (2) obviously holds if we change $(m(n))$ to $(m(n_k))$. If the statement of the theorem is true when $(m(n))$ is nondecreasing and unbounded (as we will prove immediately), then one would get the existence of a residual subset $M \subset H(G)$ with the property that, for every $f \in M$ and every nonempty open subset $V \subset G$, $(\Phi_{n_k}(D)f)(V) \supset D(0, \rho_{n_k})$ for infinitely many $k \in \mathbb{N}$. But then, trivially, the same conclusion holds by changing $(n_k)$ to the entire sequence of positive integers.

Consequently, we can start with the following hypotheses: $1 \leq m(n) \uparrow \infty$ and

$$\max\{\limsup_{n \to \infty} \frac{1}{m(n)} \left( \frac{\rho_n}{|c^{(n)}_{m(n)}|} \right)^{\frac{1}{m(n)}}, \limsup_{n \to \infty} \frac{1}{m(n)} \left( \frac{\rho_n |c^{(n)}_{m(n)+1}|}{|c^{(n)}_{m(n)}|^2} \right)^{\frac{1}{m(n)}}\} \leq \frac{1}{e \cdot \text{diam}(G)},$$

because of (2) and the fact $\frac{m(n)}{m(n)+1} \to 1$ ($n \to \infty$). By Stirling’s formula, $\frac{1}{m(n)} \sim \frac{1}{e \cdot m(n)!^{1/m(n)}}$ ($n \to \infty$), so the two numbers $\limsup_{n \to \infty} \left( \frac{\rho_n}{(m(n)+1)! |c^{(n)}_{m(n)}|} \right)^{\frac{1}{m(n)+1}}$ and

$$\limsup_{n \to \infty} \left( \frac{\rho_n |c^{(n)}_{m(n)+1}|}{(m(n))! |c^{(n)}_{m(n)}|^2} \right)^{\frac{1}{m(n)}}$$

are $\leq \frac{1}{\text{diam}(G)}$.

Fix a disk $D(w, \varepsilon) \subset G$ and associate to it the sequence of degree one monomials
\begin{align*}
(f_n) \text{ defined as } \\
\quad f_n(z) = \frac{2\rho_n}{\varepsilon} \cdot (z - w) \quad (z \in \mathbb{C}).
\end{align*}

Consider the sequence of functions \((g_n)\) given by

\begin{align*}
\quad g_n(z) = \frac{2\rho_n}{\varepsilon} \cdot \left[ \frac{1}{c_{m(n)}^{(n)}} \cdot \frac{(z - w)^{m(n)+1}}{(m(n) + 1)!} - \frac{c_{m(n)+1}^{(n)}}{(c_{m(n)}^{(n)})^2} \cdot \frac{(z - w)^{m(n)}}{m(n)!} \right] \quad (z \in \mathbb{C}).
\end{align*}

Observe that \(D^j((z - w)^k) = 0\) whenever \(j > k\). This and Lemma 2 yield

\begin{align*}
\quad (\Phi_n(D)g_n)(z) &= (c_{m(n)}^{(n)} D^{m(n)} + c_{m(n)+1}^{(n)} D^{m(n)+1})g_n(z) \\
\quad &= \frac{2\rho_n}{\varepsilon} (z - w) = f_n(z) \quad (z \in \mathbb{C}).
\end{align*}

Now, fix a compact subset \(K \subset G\). Then there are positive numbers \(r, R\) such that

\begin{align*}
\sup_{z \in K} |z - w| < r < R < \text{diam}(G), \text{ so } \frac{1}{R} > \frac{1}{\text{diam}(G)}. \text{ Hence there exists } n_0 \in \mathbb{N}
\end{align*}

such that \(\frac{\rho_n}{(m(n)+1)!|c_{m(n)}^{(n)}|} \frac{1}{m(n)+1} \quad \text{and} \quad \frac{\rho_n |c_{m(n)+1}^{(n)}|}{m(n)!|c_{m(n)}^{(n)}|^2} \frac{1}{m(n)}\) are less that \(1/R\) for all \(n \geq n_0\). If \(z \in K\) and \(n \geq n_0\), we obtain

\begin{align*}
|g_n(z)| &\leq \frac{2}{\varepsilon} \cdot \frac{\rho_n}{|c_{m(n)}^{(n)}|} \cdot \frac{|z - w|^{m(n)+1}}{(m(n) + 1)!} + 2 \cdot \frac{\rho_n |c_{m(n)+1}^{(n)}|}{|c_{m(n)}^{(n)}|^2} \cdot \frac{|z - w|^{m(n)}}{m(n)!} \\
&< \frac{2}{\varepsilon} \cdot \left( \frac{r}{R} \right)^{1+m(n)} + \frac{2}{\varepsilon} \cdot \left( \frac{r}{R} \right)^{m(n)} \to 0 \quad (n \to \infty).
\end{align*}

Thus, \(g_n \to 0 \quad (n \to \infty)\) uniformly on compact subsets of \(G\). In particular, the null function is the unique limit point of \((g_n)\), i.e., \(LP((g_n)) = \{0\}\). If \(P\) is a fixed polynomial, then \(D^j P = 0\) for all \(j > \text{degree}(P)\). But there is \(n_1 \in \mathbb{N}\) with \(m(n) > \text{degree}(P)\) for all \(n > n_1\), so \(\Phi_n(D)P = \sum_{j=m(n)}^{\infty} c_j^{(n)} D^j P = 0\).
for all \( n > n_1 \), which implies, trivially, that \( \Phi_n(D)P \to 0 \) \((n \to \infty)\) in \( H(G) \).

Then conditions (a), (b), (c) of Theorem 1 are fulfilled if we take \( X = H(G) = Y \), 
\( T_n = \Phi_n(D) \) \((n \in \mathbb{N})\), \( s_n = g_n \) \((n \in \mathbb{N})\) and \( D = \{\text{polynomials}\} \) (note that, trivially, 
\( T_n \alpha \to 0 \) as \( n \to \infty \) if \( \alpha = 0 \)). Therefore the set \( M(\sigma, \tau) \) is residual in \( H(G) \) for 
\( \sigma = (s_n) \), \( \tau = (T_n) \). We now relabel \( M(\sigma, \tau) = H(w, \varepsilon) \) because it depends upon
the disk \( D(w, \varepsilon) \).

Next, let us observe that \( H(w, \varepsilon) \) is precisely the set of holomorphic functions 
\( f \) on \( G \) for which some subsequence of \((\Phi_n(D)f - f_n)\) tends to zero uniformly on
compact subsets of \( G \). Therefore, for every \( f \in H(w, \varepsilon) \), there are infinitely many
positive integers \( n \) satisfying \( |(\Phi_n(D)f)(z) - f_n(z)| < 1 \) for all \( z \in \partial D(w, \varepsilon) \) and 
\( \rho_n > 1 \), since \((\rho_n)\) is unbounded. Fix one of these \( n \), a point \( a \in D(0, \rho_n) \) and a
point \( z \in \partial D(w, \varepsilon) \). We have that

\[
|((\Phi_n(D)f)(z) - a) - (f_n(z) - a)| < 1 < \rho_n = 2\rho_n - \rho_n
\]

\[
< \frac{2}{\varepsilon} \rho_n \cdot \varepsilon - |a| = |f_n(z)| - |a| \leq |f_n(z) - a|
\]

so by Rouché’s Theorem [1, p. 153], \( \Phi_n(D)f \) takes the value \( a \) in \( D(w, \varepsilon) \). Then
\( D(0, \rho_n) \subset (\Phi_n(D)f)(D(w, \varepsilon)) \).

Denote \( M = \{f \in H(G) : \text{given any nonempty open subset } V \subset G \text{ there are}
infinitely many } n \in \mathbb{N} \text{ with } D(0, \rho_n) \subset (\Phi_n(D)f)(V)\} \). If \( \{D(w_j, \varepsilon_j) : j \in \mathbb{N}\} \)
is the set of all open disks contained in \( G \) having rational radii and centers with
rational coordinates, then it is an open basis for $G$ and

$$M = \bigcap_{j \in \mathbb{N}} H(w_j, \varepsilon_j).$$

Since $H(G)$ is a Baire space and each $H(w_j, \varepsilon_j)$ is residual, we conclude that $M$ is residual, as required.

// // //

**Remarks.**

1. Observe that the “high order” coefficients of the functions $\Phi_n$ do not appear in condition (2) of Theorem 2.

2. We have put $m(n) + 1$ instead of $m(n)$ on some denominators in condition (2) in order to avoid that such denominators can be zero.

3. The statement of Theorem 2 holds for $G = \mathbb{C}$ if the word “subexponential” is replaced to “exponential” (see Section 1). The same is true for Theorems 3,4 below.

4. Theorem B is a special case of Theorem 2: Take $G = \mathbb{C}$, $\Phi_n(z) = z^n$ $(n \in \mathbb{N})$. Note that here $\text{diam}(G) = \infty$, $m(n) = n$, $c_{m(n)}^{(n)} = 1$ and $c_{m(n)+1}^{(n)} = 0$ $(n \in \mathbb{N})$. Note also that, trivially, the hypothesis $\lim_{n \to \infty} \frac{n^{1/n}}{n} = 0$ in Theorem B is equivalent to

$$\max\{\limsup_{n \to \infty} \frac{n^{1/(n+1)}}{n^{1/n}} - 0\} \leq 0.$$ 

Next, we state an additional result in order to have large images under differential operators. This time, the result does not contain Theorem B as a special case. However, the sequence of multiplicities $(m(n))$ need not be bounded. Since the proof is parallel to that of Theorem 2, we will abridge it. Just one observation before giving the promise assertion: If $a \in \mathbb{C}$, then $\Phi(D)e_a = \Phi(a)e_a$, because
\[ D^j e_a = a^j e_a \quad (j \geq 0). \]

**THEOREM 3.** Let \( G \) be a simply connected domain of \( \mathbb{C} \), \( \Phi_n(z) = \sum_{j=0}^{\infty} c_j^{(n)} z^j \) \((n \in \mathbb{N})\) nonconstant entire functions of subexponential type and \((\rho_n)\) an unbounded sequence of positive numbers. Denote by \( m(n) \) the multiplicity of \( \Phi_n \) for the zero at the origin. Suppose that the following conditions are fulfilled:

1. There exists a point \( c \in \mathbb{C} \setminus \{0\} \) with \( \Phi_n(c) \neq 0 \) for every \( n \in \mathbb{N} \) and
   \[
   \lim_{n \to \infty} \frac{\rho_n}{\Phi_n(c)} = 0.
   \]
2. \[
   \lim_{n \to \infty} \frac{\rho_n R^{m(n)}}{|c_m(n)| \cdot m(n)!} = 0 \quad \text{for every } R \in (0, \text{diam}(G)).
   \]
3. There exists a subset \( A \subset \mathbb{C} \) with at least one finite accumulation point such that \( \lim_{n \to \infty} \Phi_n(a) = 0 \) for all \( a \in A \).

Then the same conclusion of Theorem 2 holds.

**Proof.** Fix \( \varepsilon_0 > 0 \) with the property that \( |\frac{e^t - 1}{t}| > |c|/2 \) for all \( t \in D(0, \varepsilon_0) \), where \( c \) is the point provided by condition (1). This time we associate to every disk \( D(w, \varepsilon) \subset G \) (with \( 0 < \varepsilon < \varepsilon_0 \)) the sequence of functions \( (f_n) \) given by

\[
 f_n(z) = \frac{4 \rho_n}{c \varepsilon \cdot \Phi_n(c)} \cdot e^{c(z-w)} - 1 \quad (z \in \mathbb{C}).
\]

Then \( \Phi_n(D) g_n = f_n \), where we have denoted

\[
 g_n(z) = \frac{4 \rho_n}{c \varepsilon \cdot \Phi_n(c)} \cdot e^{c(z-w)} - \frac{4 \rho_n}{c \varepsilon \cdot c_m(n) \cdot m(n)!} \cdot (z - w)^{m(n)} \quad (z \in \mathbb{C}).
\]

Again the three conditions (a), (b), (c) in Theorem 1 are satisfied if we take \( X = H(G) = Y, T_n = \Phi_n(D), s_n = g_n \) \((n \in \mathbb{N})\) and \( D = \text{span}\{e_a : a \in A\} \). Indeed,
\( g_n \to 0 \ (n \to \infty) \) in \( H(G) \) by (1) and (2). On the other hand, if \( \varphi \in \mathcal{D} \), then there are finitely many complex constants \( c_1, \ldots, c_p \) and points \( a_1, \ldots, a_p \) in \( A \) such that \( \varphi = \sum_{j=1}^{p} c_j e_{a_j} \). Therefore \( T_n \varphi = \sum_{j=1}^{p} c_j T_n e_{a_j} = \sum_{j=1}^{p} c_j \Phi_n(a_j) e_{a_j} \to 0 \ (n \to \infty) \) by (3).

The remainder of the proof is similar to that of Theorem 2 as soon as one realizes that

\[
|f_n(z)| = \frac{4}{|c|\varepsilon} \rho_n |e^{z-w} - 1| \geq \frac{4}{\varepsilon} \rho_n \frac{|z-w|}{2} = 2\rho_n
\]

for every \( n \in \mathbb{N} \) and every \( z \in \partial D(w, \varepsilon) \).

We finish with a consequence of Theorems 2,3 for the iterates of a single differential operator \( \Phi(D) \). Observe that the operator generated by a punctual product \( \Phi(z) \ldots \Phi(z) \) (\( n \) times) is the compositional product \( \Phi(D) \circ \ldots \circ \Phi(D) \) (\( n \) times), and that every \( \Phi^n \) is of subexponential type whenever \( \Phi \) is. The proof, which is left to the reader, is based upon the following three elementary facts about a non-constant entire function \( \Phi(z) = \sum_{j=0}^{\infty} c_j z^j \) with multiplicity \( m \) for the zero at the origin:

1) For every \( n \in \mathbb{N} \), \( mn \) is the multiplicity of \( \Phi^n \).

2) If \( \{c_j^{(n)} : j \geq 0\} \) is the sequence of Taylor coefficients of \( \Phi^n \), then \( c_{mn}^{(n)} = c_m^n \) and \( c_{mn+1}^{(n)} = nc_{m-1}^{n-1} c_{m+1} \).

3) For each \( r > 0 \), the sets \( A(r) = \{z \in \mathbb{C} : |\Phi(z)| < r\} \) and \( B(r) = \{z \in \mathbb{C} : |\Phi(z)| > r\} \) are nonempty and open. In particular, \( A(1) \) has at least one finite
acumulation point and $B(1 + |c_0|) \setminus \{0\}$ is not empty.

Observe again that $c_m$ is the only coefficient relevant to the conclusion of the next result. Theorem 4 also contains Theorem B as a special case.

**THEOREM 4.** Let $G$ be a simply connected domain of the complex plane, $\Phi(z) = \sum_{j=0}^{\infty} c_j z^j$ a nonconstant entire function of subexponential type with multiplicity $m$ and $(\rho_n)$ an unbounded sequence of positive numbers. Suppose that one of the following properties is satisfied:

(1) $c_0 \neq 0$ and $\limsup_{n \to \infty} \frac{\rho_n}{|c_0|^m} = 0$.

(2) $c_0 = 0$ and $\limsup_{n \to \infty} \frac{\rho_n^{1/n}}{n^m} \leq \left( \frac{m \cdot \text{diam}(G)}{e \cdot \text{diam}(G)} \right)^m \cdot |c_m|$.

Then the same conclusion of Theorem 2 holds.

Only a remark before the end. Fix $N \in \mathbb{N}$. By considering sequences of functions of the form $f_n(z) = \alpha(n, \varepsilon)(z - w)^N$ or of the form $f_n(z) = \alpha(n, \varepsilon)[e^{c(z-w)} - \sum_{j=0}^{N-1} \frac{c^j(z-w)^j}{j!}]$ ($n \in \mathbb{N}$), where $c$ and $\alpha(n, \varepsilon)$ are appropriate constants, the interested reader (if any) could try to show that, under suitable conditions on the Taylor coefficients $c_j^{(n)}$ of the entire functions $\Phi_n$ (or on the coefficients $c_j$ of a single $\Phi$), there exists a residual subset $M \subset H(G)$ with the following property: for each member $f \in M$ and each nonempty open subset $V \subset G$, there are infinitely many $n \in \mathbb{N}$ for which the equation $f^{(n)}(z) = w$ has at least $N$ solutions in $V$ for every $w \in D(0, \rho_n)$. 

16
REFERENCES


Luis Bernal-González
Departamento de Análisis Matemático
Facultad de Matemáticas
Avenida Reina Mercedes. Apartado 1160
41080 Sevilla (Spain)
E-mail: lbernal@cica.es