TITLE: HOLOMORPHIC FUNCTIONS HAVING LARGE IMAGES UNDER THE ACTION OF DIFFERENTIAL OPERATORS.

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HOLOMORPHIC FUNCTIONS HAVING LARGE IMAGES UNDER THE ACTION OF DIFFERENTIAL OPERATORS

By

LUIS BERNAL–GONZÁLEZ*

Abstract. We prove in this note that, given a simply connected domain G in the complex plane and a sequence of infinite order linear differential operators generated by entire functions of subexponential type satisfying suitable conditions, then there are holomorphic functions f on G such that the image of any open subset under the action of those operators on f is arbitrarily large. This generalizes an earlier result about images of derivatives. A known statement about close orbits is also strengthened.

1. INTRODUCTION AND NOTATION

In this paper we denote, as usual, by $\mathbf{N}$ the set of positive integers, by $\mathbf{C}$ the field of complex numbers and by $D(a,r)$ the open disk $\{z \in \mathbf{C} : |z-a| < r\}$. If G is a nonempty open subset of $\mathbf{C}$, then $H(G)$ will stand for the Fréchet space of holomorphic functions on G, endowed with the topology of the uniform convergence on compact subsets. A domain (=nonempty open subset) $G \subset \mathbf{C}$ is said to be simply connected whenever its complement with respect to the extended complex plane is connected. Recall that, by Runge's theorem [4, pp. 92-97], if G is a simply connected domain then the set of polynomials is dense in $H(G)$. If $A \subset \mathbf{C}$ is a subset with at least one finite accumulation point and G is simply connected, then the linear manifold

$$H_A = \text{span}\,\{e_a \,:\, a \in A\}$$

is dense in $H(G)$ (see, for instance, [7, p. 97], [6, pp. 259-260] and [2, Section 4]). We have denoted here $e_a(z) = \exp(az)$ $(z \in \mathbf{C})$. The diameter of a subset $A \subset \mathbf{C}$ is $\text{diam}\,(A) = \sup\{|z - w| \,:\, z, w \in A\}$.

Every entire function $\Phi(z) = \sum_{j=0}^{\infty} c_j z^j$ generates a formal "infinite order linear differential operator" with constant coefficients given by $\Phi(D) = \sum_{j=0}^{\infty} c_j D^j$, where D denotes the differentiation operator $Df = f'$ and $D^0 = I =$ the identity operator. The function Φ is said to be of exponential type if and only if there are constants $K_1, K_2 \in (0, +\infty)$ such that

$$|\Phi(z)| \leq K_1 e^{K_2|z|} \quad (z \in \mathbf{C}).$$

Φ is said to be of subexponential type if and only if for every $\varepsilon > 0$ there is a constant $K = K(\varepsilon) \in (0, +\infty)$ such that

$$|\Phi(z)| \leq K e^{\varepsilon|z|} \quad (z \in \mathbf{C}).$$

Each function of subexponential type is obviously of exponential type. With essentially the same methods of Valiron [9, p. 35] (see also [3, pp. 58-60] and [2, Theorem 5]), it can be proved that $\Phi(D)$ is a well-defined operator on $H(G)$ as soon as Φ

is of subexponential type (and, in fact, on $H(\mathbf{C})$ if Φ is just of exponential type). The reader is referred to [8] for a systematic study of this kind of operators.

From now on, if X is a linear metric space, then we denote $||x|| = d(x,0)$ for $x \in X$, where d is the (translation-invariant) metric of X. For instance, the translation-invariant metric

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\sup_{K_n} |f-g|}{1+\sup_{K_n} |f-g|} \quad (f,g \in H(G))$$

generates the natural topology of G. Here (K_n) is an exhaustive nondecreasing sequence of compact subsets of G. In 1987, R. M. Gethner and J. H. Shapiro [5], when studying the existence of universal vectors for certain kinds of operators on spaces of holomorphic functions, proved the following related result [5, Theorem 2.4].

THEOREM A. *Suppose T is a continuous linear operator on a separable complete linear metric space X and $T^n x \to 0$ $(n \to \infty)$ for every $x \in \mathcal{D}$, where $\mathcal{D}$ is a dense subset of X and $T^n = T \circ T \circ \cdots \circ T$ (n times). Let (x_n) be a sequence in X such that $x_n \to 0$ $(n \to \infty)$. Then the set of vectors $x \in X$ for which* $\liminf_{n\to\infty} ||T^n x - T^n x_n|| = 0$ *is a dense G_δ subset of X.*

As an application of Theorem A to function theory, it is shown in [5, Theorem 3.7] the next theorem, that establishes the existence of entire functions for which many derivatives have "large images" on prefixed arbitrarily small open subsets.

THEOREM B. *Suppose* (ρ_n) *is an unbounded, increasing sequence of positive numbers for which* $\lim_{n\to\infty} \frac{\rho_n^{1/n}}{n} = 0$. *Then there exists a dense* G_δ *subset* M *of* $H(G)$ *satisfying that, for every* $f \in M$ *and every nonempty open set* $V \subset G$, *there are infinitely many* $n \in \mathbf{N}$ *such that* $f^{(n)}(V) \supset D(0, \rho_n)$.

Our aim in this note is to extend the latter result to certain kinds of infinite order differential operators on simply connected domains. In passing, Theorem A can also be manifestly strengthened.

2. SETS OF POINTS CLOSE TO THE ORBIT OF A RELATIVELY COMPACT SEQUENCE

We start with an elementary lemma. Let X be a topological space, (Y, d) a metric space and denote, as usual, by $C(X, Y)$ the space of all continuous mappings from X into Y. If $\sigma = (s_n)$ and $\tau = (T_n)$ are sequences in X and $C(X, Y)$ respectively, then we put

$$M(\sigma, \tau) = \{x \in X : \liminf_{n\to\infty} d(T_n x, T_n s_n) = 0\}.$$

LEMMA 1. *If* X, (Y, d), $\sigma = (s_n)$ *and* $\tau = (T_n)$ *are as before, then* $M(\sigma, \tau)$ *is a* G_δ *subset of* X.

Proof. Fix a point $x \in X$. Observe that $x \in M(\sigma, \tau)$ if and only if for each pair $N, k \in \mathbf{N}$ there is $n > N$ such that $d(T_n x, T_n s_n) < 1/k$, that is,

$$M(\sigma, \tau) = \bigcap_{N \in \mathbf{N}} \bigcap_{k \in \mathbf{N}} \bigcup_{n > N} T_n^{-1}(G_{n,k}),$$

where $G_{n,k} = \{y \in Y : d(y, T_n s_n) < 1/k\}$, which is an open ball in X. Since T_n is continuous, $T_n^{-1}(G_{n,k})$ is open in X and so $M(\sigma, \tau)$ is a G_δ subset. ////

For every sequence $\sigma = (s_n)$ in X, denote

$$LP(\sigma) = \{\alpha \in X : \alpha \text{ is a limit point for } (s_n)\}.$$

$LP(\sigma)$ may well be empty. If X and Y are topological vector spaces, then $L(X, Y)$ will stand for the subspace of $C(X, Y)$ of all linear mappings from X into Y. Recall that a subset in a Baire space is residual if and only if it contains a dense G_δ subset. The next result generalizes Theorem A.

THEOREM 1. *Assume that X and Y are linear metric spaces, in such a way that X is a Baire space. Let $\sigma = (s_n)$ and $\tau = (T_n)$ be sequences in X and $L(X, Y)$, respectively. Suppose that the following three conditions are satisfied:*

(a) σ is relatively compact.

(b) $\lim_{n \to \infty} T_n \alpha = 0$ for every $\alpha \in LP(\sigma)$.

(c) There exists a dense subset $\mathcal{D} \subset X$ such that $\liminf_{n \to \infty} ||T_n x|| = 0$ for all $x \in \mathcal{D}$.

Then $M(\sigma, \tau)$ is residual in X.

Proof. Note that, by (a), $LP(\sigma)$ is not empty. Recall that $||x|| = d_1(x,0)$ for every $x \in X$ and $||y|| = d(y,0)$ for every $y \in Y$, where d_1, d are the metrics on X, Y (resp.), which are translation-invariant. By Lemma 1, $M(\sigma,\tau)$ is a G_δ subset of X. Let us keep in mind the notation of the proof of that lemma. Since X is Baire and the sets $S(N,k) := \bigcup_{n>N} T_n^{-1}(G_{n,k})$ $(N,k \in \mathbf{N})$ are open, it suffices to show that every $S(N,k)$ is dense in X. For this, fix $N,k \in \mathbf{N}$, a point $x_0 \in \mathcal{D}$ and $\varepsilon > 0$. By (c), there is a sequence $n_1 < n_2 < ... < n_j < ...$ of positive integers such that

$$\lim_{j\to\infty} ||T_{n_j} x_0|| = 0.$$

But σ is relatively compact, so there is a point $\alpha \in X$ and a subsequence $m_1 < m_2 < ... < m_j < ...$ of (n_j) such that

$$\lim_{j\to\infty} ||s_{m_j} - \alpha|| = 0.$$

From (b), we have that

$$\lim_{j\to\infty} ||T_{m_j} \alpha|| = 0.$$

In particular, there exists $n > N$ such that $||T_n x_0|| < \frac{1}{2k}$, $||T_n \alpha|| < \frac{1}{2k}$ and $||s_n - \alpha|| < \varepsilon$. Define the point

$$x = x_0 + s_n - \alpha.$$

Then $||x - x_0|| = ||s_n - \alpha|| < \varepsilon$ and, by linearity, $d(T_n x, T_n s_n) = ||T_n x - T_n s_n|| = ||T_n x_0 - T_n \alpha + T_n s_n - T_n s_n|| \leq ||T_n x_0|| + ||T_n \alpha|| < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}$.

Thus, $x \in S(N,k) \cap \{z \in X : d_1(z,x_0) < \varepsilon\}$, that is, every point $x_0 \in \mathcal{D}$ is in the closure of $S(N,k)$. But $\mathcal{D}$ is dense in X. Consequently, $S(N,k)$ is also dense in X, as required. ////

3. DIFFERENTIAL OPERATORS AND LARGE IMAGES

In this section we extend Theorem B to differential operators. Recall that if $\Phi(z) = \sum_{j=0}^{\infty} c_j z^j$ is a nonconstant entire function, then its multiplicity for the zero at the origin is $m = \min\{j \in \{0,1,2,...\} : c_j \neq 0\}$. We start with the following easy lemma, whose proof is omited since it is a simple calculation.

LEMMA 2. *If a, b are complex numbers with $a \neq 0$, and $m \in \mathbf{N}$, then*

$$(aD^m + bD^{m+1})(\frac{1}{a}\frac{z^{m+1}}{(m+1)!} - \frac{b}{a^2}\frac{z^m}{m!}) = z \quad (z \in \mathbf{C}).$$

THEOREM 2. *Let G be a simply connected domain of $\mathbf{C}$, $\Phi_n(z) = \sum_{j=0}^{\infty} c_j^{(n)} z^j$ $(n \in \mathbf{N})$ nonconstant entire functions of subexponential type and (ρ_n) an unbounded sequence of positive numbers. Denote by $m(n)$ the multiplicity of Φ_n for the zero at the origin. Suppose that the following conditions are fulfilled:*

(1) The sequence $(m(n))$ is unbounded.

(2) $\max\{\limsup_{n\to\infty} \frac{1}{1+m(n)}(\frac{\rho_n}{|c_{m(n)}^{(n)}|})^{\frac{1}{1+m(n)}}, \limsup_{n\to\infty} \frac{1}{1+m(n)}(\frac{\rho_n |c_{m(n)+1}^{(n)}|}{|c_{m(n)}^{(n)}|^2})^{\frac{1}{1+m(n)}}\}$ $\leq \frac{1}{e \cdot \mathrm{diam}\,(G)}$.

Then there exists a residual subset M of $H(G)$ satisfying that, for every $f \in M$

and every nonempty open subset $V \subset G$, there are infinitely many $n \in \mathbf{N}$ for which $(\Phi_n(D)f)(V) \supset D(0, \rho_n)$.

Proof. Let us try to apply Theorem 1. Firstly, observe that every $\Phi_n(D)$ is a well-defined linear operator on $H(G)$. We start in a way that is very similar to the proof of [5, Theorem 3.7]. By the definition of multiplicity, $c_j^{(n)} = 0$ whenever $j < m(n)$ $(n \in \mathbf{N})$. Since $(m(n))$ is unbounded, we can assume without loss of generality that $1 \leq m(n) \uparrow \infty$. Indeed, there exists a subsequence $(m(n_k))$ such that $1 \leq m(n_k) \uparrow \infty$. Then (2) obviously holds if we change $(m(n))$ to $(m(n_k))$. If the statement of the theorem is true when $(m(n))$ is nondecreasing and unbounded (as we will prove inmediately), then one would get the existence of a residual subset $M \subset H(G)$ with the property that, for every $f \in M$ and every nonempty open subset $V \subset G$, $(\Phi_{n_k}(D)f)(V) \supset D(0, \rho_{n_k})$ for infinitely many $k \in \mathbf{N}$. But then, trivially, the same conclusion holds by changing (n_k) to the entire sequence of positive integers.

Consequently, we can start with the following hypotheses: $1 \leq m(n) \uparrow \infty$ and $\max\{\limsup_{n\to\infty} \frac{1}{m(n)} (\frac{\rho_n}{|c_{m(n)}^{(n)}|})^{\frac{1}{m(n)}}, \limsup_{n\to\infty} \frac{1}{m(n)} (\frac{\rho_n |c_{m(n)+1}^{(n)}|}{|c_{m(n)}^{(n)}|^2})^{\frac{1}{m(n)}}\} \leq \frac{1/e}{\mathrm{diam}\,(G)}$, because of (2) and the fact $\frac{m(n)}{m(n)+1} \to 1$ $(n \to \infty)$. By Stirling's formula, $\frac{1}{m(n)} \sim \frac{1}{e \cdot m(n)!^{1/m(n)}}$ $(n \to \infty)$, so the two numbers $\limsup_{n\to\infty} (\frac{\rho_n}{(m(n)+1)! \cdot |c_{m(n)}^{(n)}|})^{\frac{1}{m(n)+1}}$ and $\limsup_{n\to\infty} (\frac{\rho_n |c_{m(n)+1}^{(n)}|}{m(n)! \cdot |c_{m(n)}^{(n)}|^2})^{\frac{1}{m(n)}}$ are $\leq \frac{1}{\mathrm{diam}\,(G)}$.

Fix a disk $D(w, \varepsilon) \subset G$ and associate to it the sequence of degree one monomials

(f_n) defined as

$$f_n(z) = \frac{2\rho_n}{\varepsilon} \cdot (z - w) \quad (z \in \mathbf{C}).$$

Consider the sequence of functions (g_n) given by

$$g_n(z) = \frac{2\rho_n}{\varepsilon} \cdot \left[\frac{1}{c^{(n)}_{m(n)}} \cdot \frac{(z-w)^{m(n)+1}}{(m(n)+1)!} - \frac{c^{(n)}_{m(n)+1}}{(c^{(n)}_{m(n)})^2} \cdot \frac{(z-w)^{m(n)}}{m(n)!} \right] \quad (z \in \mathbf{C}).$$

Observe that $D^j((z-w)^k) = 0$ whenever $j > k$. This and Lemma 2 yield

$$(\Phi_n(D)g_n)(z) = (c^{(n)}_{m(n)} D^{m(n)} + c^{(n)}_{m(n)+1} D^{m(n)+1}) g_n(z)$$

$$= \frac{2\rho_n}{\varepsilon}(z-w) = f_n(z) \quad (z \in \mathbf{C}).$$

Now, fix a compact subset $K \subset G$. Then there are positive numbers r, R such that $\sup_{z\in K} |z - w| < r < R < \operatorname{diam}(G)$, so $\frac{1}{R} > \frac{1}{\operatorname{diam}(G)}$. Hence there exists $n_0 \in \mathbf{N}$ such that $(\frac{\rho_n}{(m(n)+1)! \cdot |c^{(n)}_{m(n)}|})^{\frac{1}{m(n)+1}}$ and $(\frac{\rho_n |c^{(n)}_{m(n)+1}|}{m(n)! \cdot |c^{(n)}_{m(n)}|^2})^{\frac{1}{m(n)}}$ are less that $1/R$ for all $n \geq n_0$. If $z \in K$ and $n \geq n_0$, we obtain

$$|g_n(z)| \leq \frac{2}{\varepsilon} \cdot \frac{\rho_n}{|c^{(n)}_{m(n)}|} \cdot \frac{|z-w|^{m(n)+1}}{(m(n)+1)!} + \frac{2}{\varepsilon} \cdot \frac{\rho_n |c^{(n)}_{m(n)+1}|}{|c^{(n)}_{m(n)}|^2} \cdot \frac{|z-w|^{m(n)}}{m(n)!}$$

$$< \frac{2}{\varepsilon} \cdot (\frac{r}{R})^{1+m(n)} + \frac{2}{\varepsilon} \cdot (\frac{r}{R})^{m(n)} \to 0 \quad (n \to \infty).$$

Thus, $g_n \to 0$ $(n \to \infty)$ uniformly on compact subsets of G. In particular, the null function is the unique limit point of (g_n), i.e., $LP((g_n)) = \{0\}$. If P is a fixed polynomial, then $D^j P = 0$ for all $j > \operatorname{degree}(P)$. But there is $n_1 \in \mathbf{N}$ with $m(n) > \operatorname{degree}(P)$ for all $n > n_1$, so $\Phi_n(D)P = \sum_{j=m(n)}^{\infty} c^{(n)}_j D^j P = 0$

for all $n > n_1$, which implies, trivially, that $\Phi_n(D)P \to 0$ $(n \to \infty)$ in $H(G)$. Then conditions (a), (b), (c) of Theorem 1 are fulfilled if we take $X = H(G) = Y$, $T_n = \Phi_n(D)$ $(n \in \mathbf{N})$, $s_n = g_n$ $(n \in \mathbf{N})$ and $\mathcal{D} = \{\text{polynomials}\}$ (note that, trivially, $T_n\alpha \to 0$ as $n \to \infty$ if $\alpha = 0$). Therefore the set $M(\sigma, \tau)$ is residual in $H(G)$ for $\sigma = (s_n)$, $\tau = (T_n)$. We now relabel $M(\sigma, \tau) = H(w, \varepsilon)$ because it depends upon the disk $D(w, \varepsilon)$.

Next, let us observe that $H(w, \varepsilon)$ is precisely the set of holomorphic functions f on G for which some subsequence of $(\Phi_n(D)f - f_n)$ tends to zero uniformly on compact subsets of G. Therefore, for every $f \in H(w, \varepsilon)$, there are infinitely many positive integers n satisfying $|(\Phi_n(D)f)(z) - f_n(z)| < 1$ for all $z \in \partial D(w, \varepsilon)$ and $\rho_n > 1$, since (ρ_n) is unbounded. Fix one of these n, a point $a \in D(0, \rho_n)$ and a point $z \in \partial D(w, \varepsilon)$. We have that

$$|((\Phi_n(D)f)(z) - a) - (f_n(z) - a)| < 1 < \rho_n = 2\rho_n - \rho_n$$

$$< \frac{2}{\varepsilon}\rho_n \cdot \varepsilon - |a| = |f_n(z)| - |a| \leq |f_n(z) - a|,$$

so by Rouché's Theorem [1, p. 153], $\Phi_n(D)f$ takes the value a in $D(w, \varepsilon)$. Then $D(0, \rho_n) \subset (\Phi_n(D)f)(D(w, \varepsilon))$.

Denote $M = \{f \in H(G) :$ given any nonempty open subset $V \subset G$ there are infinitely many $n \in \mathbf{N}$ with $D(0, \rho_n) \subset (\Phi_n(D)f)(V)\}$. If $\{D(w_j, \varepsilon_j) : j \in \mathbf{N}\}$ is the set of all open disks contained in G having rational radii and centers with rational coordinates, then it is an open basis for G and

$$M = \bigcap_{j \in \mathbf{N}} H(w_j, \varepsilon_j).$$

Since $H(G)$ is a Baire space and each $H(w_j, \varepsilon_j)$ is residual, we conclude that M is residual, as required. ////

Remarks. 1. Observe that the "high order" coefficients of the functions Φ_n do not appear in condition (2) of Theorem 2.

2. We have put $m(n)+1$ instead of $m(n)$ on some denominators in condition (2) in order to avoid that such denominators can be zero.

3. The statement of Theorem 2 holds for $G = \mathbf{C}$ if the word "subexponential" is replaced to "exponential" (see Section 1). The same is true for Theorems 3,4 below.

4. Theorem B is a special case of Theorem 2: Take $G = \mathbf{C}$, $\Phi_n(z) = z^n$ $(n \in \mathbf{N})$. Note that here $\operatorname{diam}(G) = \infty$, $m(n) = n$, $c_{m(n)}^{(n)} = 1$ and $c_{m(n)+1}^{(n)} = 0$ $(n \in \mathbf{N})$. Note also that, trivially, the hypothesis $\lim_{n\to\infty} \frac{\rho_n^{1/n}}{n} = 0$ in Theorem B is equivalent to $\max\{\limsup_{n\to\infty} \frac{\rho_n^{1/(n+1)}}{n+1}, 0\} \leq 0$.

Next, we state an additional result in order to have large images under differential operators. This time, the result does not contain Theorem B as a special case. However, the sequence of multiplicities $(m(n))$ need not be bounded. Since the proof is parallel to that of Theorem 2, we will abridge it. Just one observation before giving the promise assertion: If $a \in \mathbf{C}$, then $\Phi(D)e_a = \Phi(a)e_a$, because

$D^j e_a = a^j e_a \ (j \geq 0)$.

THEOREM 3. *Let G be a simply connected domain of $\mathbf{C}$, $\Phi_n(z) = \sum_{j=0}^{\infty} c_j^{(n)} z^j$ ($n \in \mathbf{N}$) nonconstant entire functions of subexponential type and (ρ_n) an unbounded sequence of positive numbers. Denote by $m(n)$ the multiplicity of Φ_n for the zero at the origin. Suppose that the following conditions are fulfilled:*

(1) There exists a point $c \in \mathbf{C} \setminus \{0\}$ with $\Phi_n(c) \neq 0$ for every $n \in \mathbf{N}$ and

$\lim_{n \to \infty} \frac{\rho_n}{\Phi_n(c)} = 0.$

(2) $\lim_{n \to \infty} \frac{\rho_n R^{m(n)}}{|c_{m(n)}^{(n)}| \cdot m(n)!} = 0$ *for every $R \in (0, \operatorname{diam}(G))$.*

(3) There exists a subset $A \subset \mathbf{C}$ with at least one finite accumulation point such that $\lim_{n \to \infty} \Phi_n(a) = 0$ for all $a \in A$.

Then the same conclusion of Theorem 2 holds.

Proof. Fix $\varepsilon_0 > 0$ with the property that $|\frac{e^{ct}-1}{t}| > |c|/2$ for all $t \in D(0, \varepsilon_0)$, where c is the point provided by condition (1). This time we associate to every disk $D(w, \varepsilon) \subset G$ (with $0 < \varepsilon < \varepsilon_0$) the sequence of functions (f_n) given by

$$f_n(z) = \frac{4}{c\varepsilon} \rho_n \cdot (e^{c(z-w)} - 1) \quad (z \in \mathbf{C}).$$

Then $\Phi_n(D) g_n = f_n$, where we have denoted

$$g_n(z) = \frac{4\rho_n}{c\varepsilon \cdot \Phi_n(c)} \cdot e^{c(z-w)} - \frac{4\rho_n}{c\varepsilon \cdot c_{m(n)}^{(n)} \cdot m(n)!} \cdot (z-w)^{m(n)} \quad (z \in \mathbf{C}).$$

Again the three conditions (a), (b), (c) in Theorem 1 are satisfied if we take $X = H(G) = Y$, $T_n = \Phi_n(D)$, $s_n = g_n$ ($n \in \mathbf{N}$) and $\mathcal{D} = \operatorname{span}\{e_a : a \in A\}$. Indeed,

$g_n \to 0$ $(n \to \infty)$ in $H(G)$ by (1) and (2). On the other hand, if $\varphi \in \mathcal{D}$, then there are finitely many complex constants $c_1, ..., c_p$ and points $a_1, ..., a_p$ in A such that $\varphi = \sum_{j=1}^p c_j e_{a_j}$. Therefore $T_n \varphi = \sum_{j=1}^p c_j T_n e_{a_j} = \sum_{j=1}^p c_j \Phi_n(a_j) e_{a_j} \to 0$ $(n \to \infty)$ by (3).

The remainder of the proof is similar to that of Theorem 2 as soon as one realizes that

$$|f_n(z)| = \frac{4}{|c|\varepsilon} \rho_n \, |e^{z-w} - 1| \geq \frac{4}{\varepsilon} \rho_n \frac{|z-w|}{2} = 2\rho_n$$

for every $n \in \mathbf{N}$ and every $z \in \partial D(w, \varepsilon)$. ////

We finish with a consequence of Theorems 2,3 for the iterates of a single differential operator $\Phi(D)$. Observe that the operator generated by a punctual product $\Phi(z) \cdot \ldots \cdot \Phi(z)$ (n times) is the compositional product $\Phi(D) \circ \ldots \circ \Phi(D)$ (n times), and that every Φ^n is of subexponential type whenever Φ is. The proof, which is left to the reader, is based upon the following three elementary facts about a non-constant entire function $\Phi(z) = \sum_{j=0}^{\infty} c_j z^j$ with multiplicity m for the zero at the origin:

1) For every $n \in \mathbf{N}$, mn is the multiplicity of Φ^n.

2) If $\{c_j^{(n)} : j \geq 0\}$ is the sequence of Taylor coefficients of Φ^n, then $c_{mn}^{(n)} = c_m^n$ and $c_{mn+1}^{(n)} = n c_m^{n-1} c_{m+1}$.

3) For each $r > 0$, the sets $A(r) = \{z \in \mathbf{C} : |\Phi(z)| < r\}$ and $B(r) = \{z \in \mathbf{C} : |\Phi(z)| > r\}$ are nonempty and open. In particular, $A(1)$ has at least one finite

acumulation point and $B(1+|c_0|) \setminus \{0\}$ is not empty.

Observe again that c_m is the only coefficient relevant to the conclusion of the next result. Theorem 4 also contains Theorem B as a special case.

THEOREM 4. *Let G be a simply connected domain of the complex plane, $\Phi(z) = \sum_{j=0}^{\infty} c_j z^j$ a nonconstant entire function of subexponential type with multiplicity m and (ρ_n) an unbounded sequence of positive numbers. Suppose that one of the following properties is satisfied:*

(1) $c_0 \neq 0$ and $\limsup_{n\to\infty} \frac{\rho_n}{|c_0|^n} = 0$.

(2) $c_0 = 0$ and $\limsup_{n\to\infty} \frac{\rho_n^{1/n}}{n^m} \leq (\frac{m}{e \cdot \mathrm{diam}\,(G)})^m \cdot |c_m|$.

Then the same conclusion of Theorem 2 holds.

Only a remark before the end. Fix $N \in \mathbf{N}$. By considering sequences of functions of the form $f_n(z) = \alpha(n,\varepsilon)(z-w)^N$ or of the form $f_n(z) = \alpha(n,\varepsilon)[e^{c(z-w)} - \sum_{j=0}^{N-1} \frac{c^j (z-w)^j}{j!}]$ $(n \in \mathbf{N})$, where c and $\alpha(n,\varepsilon)$ are appropriate constants, the interested reader (if any) could try to show that, under suitable conditions on the Taylor coefficients $c_j^{(n)}$ of the entire functions Φ_n (or on the coefficients c_j of a single Φ), there exists a residual subset $M \subset H(G)$ with the following property: for each member $f \in M$ and each nonempty open subset $V \subset G$, there are infinitely many $n \in \mathbf{N}$ for which the equation $f^{(n)}(z) = w$ has at least N solutions in V for every $w \in D(0,\rho_n)$.

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