TITLE: SMALL ENTIRE FUNCTIONS WITH EXTREMELY FAST GROWTH.

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SMALL ENTIRE FUNCTIONS WITH EXTREMELY FAST GROWTH

by

LUIS BERNAL–GONZÁLEZ*

Abstract. We prove in this note that, given $\alpha \in (0, 1/2)$, there exists a linear manifold $\mathcal{M}$ of entire functions satisfying that $\mathcal{M}$ is dense in the space of all entire functions such that $\lim_{z\to\infty} \exp(|z|^\alpha) f^{(j)}(z) = 0$ on any plane strip for every $f \in \mathcal{M}$ and for every derivation index j. Moreover, the growth index of each non-null function of $\mathcal{M}$ is infinite with respect to any prefixed sequence of nonconstant entire functions.

1. INTRODUCTION AND NOTATION

Liouville's theorem asserts that each bounded entire function is constant. Several authors have obtained nonconstant entire functions (nonbounded, of course) which are small on large plane subsets. For instance, Mittag-Leffler [**8**, pp. 290-294]

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had already constructed in 1920 a nonnull entire function tending to zero on every line through the origin (therefore, this function is bounded on every ray from the origin). Two easy explicit examples of this kind of functions can be found in [**9**] and [**12**, chap. 16]. In [**6**] it is shown the existence of a nonconstant entire function having limit zero on every line. This is achieved by using a result of A. Roth about approximation on closed sets.

But much more sophisticated "anti-Liouville" functions have been made. In 1982, L. Zalcman [**14**] constructed a nonconstant entire function tending to zero on every line and such that

$$\int_l f\,ds = 0$$

for every $l \in L$. We have denoted by L the set of all straight lines in the plane and by s the length measure. For this, Zalcman used a result (Theorem 1 below) due to Arakelian concerning tangential approximation. By an elementary pole-pushing technique, D. H. Armitage [**2**] has recently constructed a nonconstant entire function f such that each derivative $f^{(n)}$ is integrable on every line l with respect to s and

$$\int_l f^{(n)}\,ds = 0$$

for every $n \in \mathbf{N_0}$, where $\mathbf{N_0} = \mathbf{N} \cup \{0\}$ and $\mathbf{N}$ is the set of positive integers. Moreover, f satisfies

$$\lim_{\substack{z \to \infty \\ z \in l}} f^{(n)}(z) = 0$$

for every $n \in \mathbf{N_0}$ and every line l. In [**4**] the author improves all above assertions and provides a dense linear manifold of entire functions "violating" Liouville's theorem. By using Phragmen–Lindelof's theorem (see, for instance, [**7**, pp. 125-126]), it is proved in [**4**] that all nonnull functions in this manifold have infinite growth order. There it is even found an anti-Liouville function having extremely fast growth, that is, an entire function f with infinite growth index $i(f)$.

In this note we improve in turn the results in [**4**] and furnish a dense linear manifold $\mathcal{M}$ all of whose functions satisfy the properties stated in [**4**] and, in addition, the growth index *of every nunnull function of the manifold* is infinite. In fact, we can get for the growth of each $f \in \mathcal{M} \setminus \{0\}$ to be extremely fast with respect to any prefixed sequence of nonconstant entire functions. Moreover, the asymptotical behaviour when $z \to \infty$ for all functions $f \in \mathcal{M}$ is sharpened by considering a positive, continuous function φ which is integrable on $(1, +\infty)$ (compare Theorem

2 in **[4]** for $j = 0$ with part b) of Theorem 3 in this paper).

We will need a bit of notation. If $r > 0$, B_r denotes the closed ball $\{z : |z| \le r\}$. A *strip* is the plane region lying between two parallel straight lines. If $\beta \in (0, 2\pi)$, the sector s_β is the set $s_\beta = \{z : 0 \le \arg z \le \beta\}$. Σ will stand for the family consisting of all strips in $\mathbf{C}$ and all sectors s_β $(\beta \in (0, 2\pi))$. $H(\mathbf{C})$ is the space of all entire functions, endowed with the compact-open topology. $H(\mathbf{C})$ is a separable Fréchet space. The extended plane $\mathbf{C}_\infty$ is the one-point compactification of $\mathbf{C}$. If $F \subset \mathbf{C}$ is a closed set, then $A(F)$ is the space of all continuous functions on F which are holomorphic in the interior of F. A closed subset $F \subset \mathbf{C}$ is said to be an *Arakelian set* **[11]** whenever $\mathbf{C}_\infty \setminus F$ is both connected and locally connected at infinity.

Finally, let f be an entire function. If $r > 0$, we define $\exp_1 r = \exp r$, $\exp_{m+1} r = \exp(\exp_m r)$ $(m \in \mathbf{N})$, and $M_f(r) = \max\{|f(z)| : |z| = r\}$. For $r > 0$ large enough, we denote $\log_1 r = \log r$, $\log_{m+1} r = \log(\log_m r)$ $(m \in \mathbf{N})$. The *m-order growth* $\rho_m = \rho_m(f)$ of f (see **[10]** and **[13]**) is defined to be

$$\rho_m = \limsup_{r\to\infty} \frac{\log_{m+1} M_f(r)}{\log r}.$$

Note that ρ_1 is the ordinary growth order of f. The *growth index* of f is $i(f) = \min\{m \in \mathbf{N} : \rho_m(f) < \infty\}$, and we set $i(f) = \infty$ when $\rho_m(f) = \infty$ for all m. We can generalize these definitions (see [**3**]) as follows. Let h be a nonconstant entire function. The *relative growth order of f with respect to h* is

$$\rho_h(f) = \limsup_{r\to\infty} \frac{\log M_h^{-1}(M_f(r))}{\log r}.$$

Let $\mathcal{F} = \{h_m\}_1^\infty$ be any sequence of nonconstant entire functions. We define the *growth index of f with respect to $\mathcal{F}$* as $i_{\mathcal{F}}(f) = \min\{m \in \mathbf{N} : \rho_{h_m}(f) < \infty\}$, and we set $i_{\mathcal{F}}(f) = \infty$ if $\rho_{h_m}(f) = \infty$ for all m. Note that $\rho_m = \rho_{\exp_m}$ and $i(f) = i_{\mathcal{F}}(f)$ for $\mathcal{F} = \{\exp_m\}_1^\infty$.

1. PRELIMINARY RESULTS

We will use the following theorem due to Arakelian. For the proof and comments, see [**1**, p. 1189] and [**5**, pp. 160-162].

THEOREM 1. *Assume that $F \subset \mathbf{C}$ is an Arakelian set and that $\varepsilon(t)$ is con-*

tinuous and positive for $t \geq 0$. In addition, suppose that

$$\int_1^\infty t^{-3/2} \log \varepsilon(t)\, dt > -\infty. \tag{1}$$

Then for every $g \in A(F)$ there exists an entire function f such that

$$|f(z) - g(z)| < \varepsilon(|z|) \quad \forall z \in F.$$

The statement does not remain valid for every F if (1) is violated.

Note that, for instance, $\varepsilon(t) = \exp(-t^{1/2})$ does not satisfy (1), but $\varepsilon(t) = \exp(-t^\alpha)$ does for $\alpha < 1/2$.

Moreover, we will need an elementary lemma about the maximum modulus.

LEMMA 2. *If h is a nonconstant entire function and a, b are positive constants with $a > \max\{1, b\}$, then there is $r_0 > 0$ such that*

$$b\, M_h(r) < M_h(ar) \quad \forall r > r_0.$$

PROOF. We obtain $|g(z)| \leq |z| M_g(R)/R$ if $|z| \leq R$ by applying Schwarz's Lemma to $g(z) = h(z) - h(0)$. If $R = ar$, then $M_h(r) - |h(0)| \leq M_g(r) \leq (r/R)(M_h(R) + |h(0)|) = (M_h(ar) + |h(0)|)/a$. Choose $\varepsilon = \frac{a-b}{2a+2b}$. Then there is $r_0 > 0$ with $|h(0)| < \varepsilon M_h(r_0)$. Thus $M_h(ar) \geq \frac{(1-\varepsilon)a}{1+\varepsilon} M_h(r) > b\, M_h(r)$ for all $r > r_0$. ////

3. THE MAIN THEOREM

We are now ready to state our results. They are collected in the following theorem. Note that it generalizes Theorems 2,3,4 in [**4**].

THEOREM 3. *Assume that* $\alpha \in (0,1/2)$ *and that* $\varphi : [0,+\infty) \to (0,+\infty)$ *is a continuous function which is integrable on* $(1,+\infty)$. *Assume, in addition, that* $\mathcal{F} = \{h_m\}_1^\infty$ *is a sequence of nonconstant entire functions. Then there is a linear manifold* $\mathcal{M} = \mathcal{M}(\alpha,\varphi,\mathcal{F}) \subset H(\mathbf{C})$ *satisfying the following seven properties:*

a) $\mathcal{M}$ *is dense in* $H(\mathbf{C})$.

b) $\lim_{\substack{z\to\infty \\ z\in S}} \exp(|z|^{3/2}\varphi(z)) f(z) = 0 \ \forall S \in \Sigma$ *and* $\forall f \in \mathcal{M}$.

c) $\lim_{\substack{z\to\infty \\ z\in S}} \exp(|z|^{\alpha}) f^{(j)}(z) = 0 \ \forall S \in \Sigma$, $\forall f \in \mathcal{M}$ *and* $\forall j \in \mathbf{N_0}$.

d) $f^{(j)}$ *is bounded on* S $\ \forall S \in \Sigma$, $\forall f \in \mathcal{M}$ *and* $\forall j \in \mathbf{N_0}$.

e) $f^{(j)}$ *is integrable on* S *with respect to plane Lebesgue measure* $\forall S \in \Sigma$, $\forall f \in \mathcal{M}$ *and* $\forall j \in \mathbf{N_0}$.

f) $f^{(j)}$ *is integrable on* l *with respect to length measure* $\forall l \in L$, $\forall f \in \mathcal{M}$ *and* $\forall j \in \mathbf{N_0}$.

g) $\int_l f^{(j)}\, ds = 0 \ \ \forall l \in L$, $\forall f \in \mathcal{M}$ *and* $\forall j \in \mathbf{N}$.

h) $i_{\mathcal{F}}(f) = \infty \ \ \forall f \in \mathcal{M} \setminus \{0\}$.

PROOF. Fix a dense sequence $\{P_n\}_1^\infty$ in $H(\mathbf{C})$, $\alpha \in (0, 1/2)$, $\beta \in (\alpha, 1/2)$ and a function φ and a family $\mathcal{F} = \{h_m\}_1^\infty$ as in the hypothesis. For every $n \in \mathbf{N}$, the function

$$\varepsilon(t) = \varepsilon_n(t) \equiv \min\{1/n, \exp(-t^{3/2}\varphi(t) - t^\beta)\}$$

is positive and continuous for $t \geq 0$ and satisfies (1). Let P be the parabolic curve

$$P = \{x - ix^{1/2} : \, x \geq 0\}.$$

Fix any denumerable set $H = \{z_k : \, k \in \mathbf{N}\} \subset \{z \in \mathbf{C} : \, \mathrm{dist}\,(z, P) < 1\}$ satisfying $\lim_{z \to \infty} z_k = \infty$. H does not depend upon n. Let $F_n = B_n \cup E_n \cup H$, where

$$E_n = \{z \in \mathbf{C} : \, |z| \geq n + 1 \text{ and } \mathrm{dist}(z, P) \geq 1\}.$$

Then F_n is closed and $\mathbf{C}_\infty \setminus F_n$ is connected and locally connected at infinity, so F_n is an Arakelian set. Define inductively a sequence $\{f_n\}_1^\infty$ of entire functions as follows. Denote $r_k = |z_k|$ for all k. Let $g_1 : F_1 \to \mathbf{C}$ denote the function

$$g_1(z) = \begin{cases} P_1(z) & \text{if } z \in B_1 \\ 0 & \text{if } z \in E_1 \\ 1 + \max_{1 \le j \le k} M_{h_j}(\exp r_k) & \text{if } z = z_k \text{ and } |z_k| > 1. \end{cases}$$

Since $B_1 \cap E_1 = H \cap E_1 = \emptyset$, g_1 is well-defined. Trivially, $g_1 \in A(F_1)$. By Theorem 1, there exists an entire function f_1 such that

$$|f_1(z) - g_1(z)| < \varepsilon_1(|z|) \quad \forall z \in F_1.$$

Assume that $n \in \{2, 3, ...\}$ and that we have constructed $2n - 2$ functions $g_1, f_1, g_2, f_2, ..., g_{n-1}, f_{n-1}$ in such a way that $g_i \in A(F_i)$, $f_i \in H(\mathbf{C})$ and $|f_i(z) - g_i(z)| < \varepsilon_i(|z|)$ $\forall z \in F_i$, $\forall i \in \{1, 2, ..., n-1\}$. Now, define the function $g_n : F_n \to \mathbf{C}$ by

$$g_n(z) = \begin{cases} P_n(z) & \text{if } z \in B_n \\ 0 & \text{if } z \in E_n \\ 1 + \max_{1 \le j \le k} M_{h_j}(\exp r_k) + k \sum_{i=1}^{n-1} M_{f_i}(r_k) & \text{if } z = z_k \text{ and } |z_k| > n. \end{cases}$$

It is trivial that $g_n \in A(F_n)$. By Theorem 1, there exists an entire function f_n such that

$$|f_n(z) - g_n(z)| < \varepsilon_n(|z|) \quad \forall z \in F_n.$$

Thus, for all $n \in \mathbf{N}$,

$$|f_n(z) - P_n(z)| < 1/n \quad \forall z \in B_n, \tag{2}$$

$$|f_n(z)| < \exp(-|z|^{3/2}\varphi(|z|) - |z|^{\beta}) \quad \forall z \in E_n \tag{3}$$

and

$$|f_n(z_k) - (1 + \max_{1\leq j\leq k} M_{h_j}(\exp r_k) + S(n,k))| < 1/n \leq 1 \tag{4}$$

for all k such that $|z_k| > n$, where $S(n,k) = 0$ if $n = 1$ and $S(n,k) = k\sum_{i=1}^{n-1} M_{f_i}(r_k)$ if $n \geq 2$.

Let $G \in H(\mathbf{C})$ and $K \subset \mathbf{C}$ be a compact set. There exists a sequence of natural numbers $n_1 < n_2 < ... < n_k < ...$ such that $P_{n_k} \to G$ $(k \to \infty)$ uniformly on K. Moreover, there is $k_0 \in \mathbf{N}$ satisfying $K \subset B_{n_k}$ whenever $k > k_0$. Then, by using (2), we have that $f_{n_k} \to G$ $(k \to \infty)$ uniformly on K. Thus the sequence $\{f_n\}_1^{\infty}$ is dense in $H(\mathbf{C})$.

Let us define $\mathcal{M}$ as the linear span of $\{f_n\}_1^{\infty}$. Evidently, $\mathcal{M}$ is a linear dense manifold of $H(\mathbf{C})$: this is a). In order to verify that b), c) hold for every $f \in \mathcal{M}$, it suffices to check that these two properties are satisfied for every function $f = f_n$. We have from (3) that

$$\exp(|z|^{3/2} \cdot \varphi(z))|f_n(z)| < \exp(-|z|^{\beta}) \to 0 \quad (z \to \infty;\ z \in E_n).$$

Given a sector or a strip $S \in \Sigma$, we have that $S \setminus E_n$ is a bounded set. Consequently,

$$\exp(|z|^{3/2} \cdot \varphi(z))|f_n(z)| \to 0 \quad (z \to \infty;\ z \in S).$$

This proves b).

Define E_n^* as

$$E_n^* = \{z \in \mathbf{C} : |z| \geq n + 2 \text{ and } \operatorname{dist}(z, P) \geq 2\}.$$

Observe that $\{w : |w - z| \leq 1\} \subset E_n$ for all $z \in E_n^*$. By (3) and Cauchy's inequalities,

$$|f_n^{(j)}(z)| \leq (j!/1^j) \max\{|f_n(w)| : |w - z| = 1\}$$

$$\leq j! \max\{\exp(-|w|^{-\beta}) : |w| \geq |z| - 1\} \leq j! \exp(-(|z| - 1)^{\beta})$$

for all $z \in E_n^*$ and all $j \in \mathbf{N_0}$ (we have used that φ is positive). Hence

$$|\exp(|z|^{\alpha}) f_n^{(j)}(z)| \leq j! \exp(|z|^{\alpha} - (|z| - 1)^{\beta}) \to 0 \quad (z \to \infty;\ z \in E_n^*).$$

If $S \in \Sigma$, we have again that $S \setminus E_n^*$ is bounded, so $\lim\limits_{\substack{z \to \infty \\ z \in S}} \exp(|z|^{\alpha}) f_n^{(j)}(z) = 0$, which proves c).

The proof of d), e), f), g) is very easy. Indeed, fix $f \in \mathcal{M}$, $S \in \Sigma$, $j \in \mathbf{N_0}$ and $l \in L$. Property d) is inmediate from c). e) and f) are straightforward because, by c), there is $R > 0$ such that

$$|f^{(j)}(z)| < \exp(-|z|^\alpha) \quad \forall z \in (S \cup l) \setminus B_R$$

and $\exp(-|z|^\alpha)$ is integrable on $\mathbf{C}$ with respect to plane Lebesgue measure and integrable on any line with respect to length measure. In order to prove g), fix $j \in \mathbf{N}$. Then, from the fundamental calculus theorem and c), we have

$$\int_l f^{(j)}\, ds = \lim_{\substack{b \to \infty \\ b \in l}} f^{(j-1)}(b) - \lim_{\substack{a \to \infty \\ a \in l}} f^{(j-1)}(a) = 0 - 0 = 0,$$

as required.

As for h), fix $m \in \mathbf{N}$ and $f \in \mathcal{M} \setminus \{0\}$. Then there exists $p \in \mathbf{N}$ and complex constants $a_1, a_2, \ldots, a_p$ such that $f = a_1 f_1 + ... + a_p f_p$ and $a_p \neq 0$. Take $k_1 \in \mathbf{N}$ with $k_1 > \max\{m, |a_1/a_p|, ..., |a_{p-1}/a_p|\}$ such that $r_k > p \ \ \forall k > k_1$. Inequality (4) implies that, for all $k > k_1$, $M_{f_p}(r_k) \geq |f_p(z_k)| > \max_{1 \leq j \leq k} M_{h_k}(\exp r_k) + k(M_{f_1}(r_k) + ... + M_{f_{p-1}}(r_k)) > M_{h_m}(\exp r_k) + |a_1/a_p| M_{f_1}(r_k) + ... + |a_{p-1}/a_p| M_{f_{p-1}}(r_k)$. By using the triangle inequality, we get

$$M_f(r) \geq M_{a_p f_p}(r) - M_{a_1 f_1}(r) - M_{a_2 f_2}(r) - ... - M_{a_{p-1} f_{p-1}}(r) \quad \forall r > 0.$$

But $M_{a_jf_j}(r) = |a_j|M_{f_j}(r)$ for all $r > 0$, so $M_f(r_k) \geq |a_p|(M_{f_p}(r_k) - |a_1/a_p|M_{f_1}(r_k) - ... - |a_{p-1}/a_p|M_{f_{p-1}}(r_k)) > |a_p| \cdot M_{h_m}(\exp r_k)$ for all $k > k_1$. We now apply Lemma 2 with $a = 1 + \frac{1}{|a_p|}$, $b = \frac{1}{|a_p|}$ and $r = \frac{|a_p| \exp r_k}{|a_p|+1}$. Then there is $k_2 \in \mathbf{N}$ such that $M_{h_m}(\frac{|a_p| \exp r_k}{|a_p|+1}) < |a_p|M_{h_m}(\exp r_k)$ for all $k > k_2$. It is obvious that there exists $k_3 \in \mathbf{N}$ satisfying $\frac{|a_p| \exp r_k}{|a_p|+1} > \exp(r_k/2)$ for all $k > k_3$. If we set $k_0 = \max\{k_1, k_2, k_3\}$, then $M_f(r_k) > M_{h_m}(\exp(r_k/2))$ for all $k > k_0$, because the maximum modulus is increasing. Then

$$\rho_{h_m}(f) = \limsup_{r\to\infty} \frac{\log M_{h_m}^{-1}(M_f(r))}{\log r} \geq \limsup_{k\to\infty} \frac{\log M_{h_m}^{-1}(M_f(r_k))}{\log r_k}$$

$$\geq \limsup_{k\to\infty} \frac{\log \exp(r_k/2)}{\log r_k} = \limsup_{k\to\infty} \frac{r_k}{2\log r_k} = \infty.$$

Thus $\rho_{h_m}(f) = \infty$. But this holds for all $m \in \mathbf{N}$. Thus $i_{\mathcal{F}}(f) = \infty \; \forall f \in \mathcal{M} \setminus \{0\}$, as required. ////

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