



**TITLE:** A LOT OF “COUNTEREXAMPLES” TO LIOUVILLE’S THEOREM.

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# A LOT OF “COUNTEREXAMPLES” TO LIOUVILLE’S THEOREM

by

LUIS BERNAL–GONZÁLEZ\*

**Abstract.** We prove in this paper that, given  $\alpha \in (0, 1/2)$ , there exists a linear manifold  $M$  of entire functions satisfying that  $M$  is dense in the space of all entire functions and, in addition,  $\lim_{z \rightarrow \infty} \exp(|z|^\alpha) f^{(j)}(z) = 0$  on any plane strip for every  $f \in M$  and for every derivation index  $j$ . Moreover, it is shown the existence of an entire function with infinite growth index satisfying the latter property.

## 1. INTRODUCTION AND NOTATION

One of the most elementary, surprising and beautiful results in Complex Analysis is Liouville’s theorem: each bounded entire function is constant. Nevertheless, if boundedness condition is slightly weakened (for instance, by allowing boundedness

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on every line), then nonconstant entire functions can be obtained. For this, see for instance [7, pp. 9-10], where it is even shown a nonconstant entire function having limit zero on any line. This is achieved by using a result of A. Roth about approximation on closed sets.

But several sharper results have been obtained. Recently, D.H. Armitage [3] has constructed a nonconstant entire function  $f$  such that each derivative  $f^{(n)}$  is integrable on every line  $l$  with respect to length measure  $s$  and

$$\int_l f^{(n)} ds = 0$$

for every  $n \in \mathbf{N}_0$ , where  $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$  and  $\mathbf{N}$  is the set of positive integers.

Moreover,  $f$  satisfies

$$\lim_{\substack{z \rightarrow \infty \\ z \in l}} f^{(n)}(z) = 0$$

for every  $n \in \mathbf{N}_0$  and every line  $l$  (therefore each derivative is bounded on each line). This is obtained by an elementary pole-pushing technique. In the same paper it is noted that if  $f$  is a continuous function on the complex plane  $\mathbf{C}$ , integrable on

$\mathbf{C}$  with respect to plane Lebesgue measure, and if

$$\int_l f ds = 0 \tag{1}$$

for every line  $l$ , then  $f$  is identically zero (see [2] for instance). We remark here that the mapping  $\hat{f}(l) = \int_l f ds$  ( $l \in L$ ) is known as the *Radon transform* of  $f$ . We

have denoted by  $L$  the set of all straight lines in the plane. Thus, we are dealing with the injectivity of the linear transform  $f \mapsto \hat{f}$  in  $\mathbf{R}^2$ . The corresponding problem for hyperplanes in  $\mathbf{R}^n$  is extensively studied in [8] for certain function spaces.

L. Zalcman [16] in 1982 had indicated that it actually suffices for (1) to hold only for *almost* every line belonging to a *dense* set of directions in order that  $f \equiv 0$ . In the same paper, he constructed –by using a result (Theorem 1 below) due to Arakelian concerning tangential approximation (see [1, p. 1189] and [6, pp. 160-162])– a nonconstant entire function satisfying (1) on every line  $l$ . Nonnull entire functions tending to zero on every line had already been constructed by Mittag-Leffler [10, pp. 290-294], and D.J. Newman [11] provided an explicit nonconstant entire function which is bounded on every line *through the origin*. Zalcman also pointed out that, if nearly nothing is assumed for  $f$ , then there exist nonnull functions for

which (1) holds: take  $f = \chi_E$ , where  $E$  is Sierpinski's nonmeasurable set with the property that any line intersects  $E$  in at most two points (see [15]). All these results can be carried into the setting of harmonic functions on  $\mathbf{R}^n$  ( $n \geq 2$ ) (see [4] and [5]).

In this paper we improve all above assertions about entire functions and provide a dense linear manifold of entire functions “violating” Liouville's theorem. All nonnull functions in this manifold have infinite growth order. We can get even for at least one of these functions to have extremely fast growth.

We will need a bit of notation and a preliminary result. If  $r > 0$ ,  $B_r$  denotes the closed ball  $\{z : |z| \leq r\}$ . A *strip* is the plane region lying between two parallel straight lines. If  $\beta \in (0, 2\pi)$ , the sector  $s_\beta$  is the set  $s_\beta = \{z : 0 \leq \arg z \leq \beta\}$  (here  $\arg z$  is evidently being allowed to be outside the “principal interval”  $(-\pi, \pi]$  used by some authors) .  $\Sigma$  will stand for the family consisting of all strips in  $\mathbf{C}$  and all sectors  $s_\beta$  ( $\beta \in (0, 2\pi)$ ).  $H(\mathbf{C})$  is the space of all entire functions, endowed with the compact-open topology. The extended plane  $\mathbf{C}_\infty$  is the one-point compactification of  $\mathbf{C}$ . If  $F \subset \mathbf{C}$  is a closed set, then  $A(F)$  is the space of all continuous functions

on  $F$  which are holomorphic in the interior of  $F$ . A closed subset  $F \subset \mathbf{C}$  is said to be an *Arakelian set* [13] whenever  $\mathbf{C}_\infty \setminus F$  is both connected and locally connected at infinity.

We will use the following (above mentioned) theorem due to Arakelian.

**THEOREM 1.** *Assume that  $F \subset \mathbf{C}$  is an Arakelian set and that  $\varepsilon(t)$  is continuous and positive for  $t \geq 0$ . In addition, suppose that*

$$\int_1^\infty t^{-3/2} \log \varepsilon(t) dt > -\infty. \quad (2)$$

*Then for every  $g \in A(F)$  there exists an entire function  $f$  such that*

$$|f(z) - g(z)| < \varepsilon(|z|) \quad \forall z \in F.$$

*The statement does not remain valid for every  $F$  if (2) is violated.*

Note that, for instance,  $\varepsilon(t) = \exp(-t^{1/2})$  does not satisfy (2), but  $\varepsilon(t) = \exp(-t^\alpha)$  does for  $\alpha < 1/2$ .

Finally, let  $f$  be an entire function. If  $r > 0$ , we define  $\exp_1 r = \exp r$ ,  $\exp_{k+1} r = \exp(\exp_k r)$  ( $k \in \mathbf{N}$ ), and  $\mu(r) = \max\{|f(z)| : |z| = r\}$ . For  $r > 0$

large enough, we denote  $\log_1 r = \log r$ ,  $\log_{k+1} r = \log(\log_k r)$  ( $k \in \mathbf{N}$ ). The following definitions can be found in [12] and [14]. The *growth k-order*  $\rho_k = \rho_k(f)$  of  $f$  is

$$\rho_k = \limsup_{r \rightarrow \infty} \frac{\log_{k+1} \mu(r)}{\log r}.$$

If  $k = 1$ , then  $\rho_k$  is called the *order* of  $f$ , and we shall denote it by  $\rho(f)$ . The *growth index*  $i(f)$  of  $f$  is  $i(f) = \min\{k \in \mathbf{N} : \rho_k(f) < \infty\}$ , where we set  $i(f) = \infty$  when  $\rho_k(f) = \infty$  for all  $k$ .

## 2. RESULTS

Recall that  $H(\mathbf{C})$  is separable: the family  $\{P_n\}_1^\infty$  of holomorphic polynomials having coefficients with rational real and imaginary parts is an example of a countable dense subset of  $H(\mathbf{C})$ . With this in mind, we are now ready to state the following theorem.

**THEOREM 2.** *Assume that  $\alpha \in (0, 1/2)$ . Then there is a linear manifold  $M \subset H(\mathbf{C})$  which is dense in  $H(\mathbf{C})$  and satisfying*

$$\lim_{\substack{z \rightarrow \infty \\ z \in S}} \exp(|z|^\alpha) f^{(j)}(z) = 0 \tag{3}$$

$\forall S \in \Sigma, \forall f \in M$  and  $\forall j \in \mathbf{N}_0$ .

PROOF. Consider the sequence  $\{P_n\}_1^\infty$  mentioned at the beginning of the paragraph and fix  $\alpha \in (0, 1/2)$ . Fix also a number  $\beta \in (\alpha, 1/2)$ . For every  $n \in \mathbf{N}$ , the function

$$\varepsilon(t) = \varepsilon_n(t) \equiv \min\{1/n, \exp(-t^\beta)\}$$

is positive and continuous for  $t \geq 0$  and satisfies (2). Let  $F_n = B_n \cup E_n$ , where

$$E_n = \{z \in \mathbf{C} : |z| \geq n + 1 \text{ and } \text{dist}(z, P) \geq 1\}$$

and  $P$  is the parabolic curve

$$P = \{x - ix^{1/2} : x \geq 0\}.$$

Then  $F_n$  is closed. In addition, a glance over its complement reveals that  $F_n$  is an

Arakelian set. Define the function  $g_n : F_n \rightarrow \mathbf{C}$  by

$$g_n(z) = \begin{cases} P_n(z) & \text{if } z \in B_n \\ 0 & \text{if } z \in E_n. \end{cases}$$

Since  $B_n \cap E_n = \emptyset$ ,  $g_n$  is well-defined. Trivially,  $g_n \in A(E_n)$ . By Theorem 1, there

exists an entire function  $f_n$  such that

$$|f_n(z) - g_n(z)| < \varepsilon(|z|) \quad \forall z \in F_n,$$

so

$$|f_n(z) - P_n(z)| < 1/n \quad \forall z \in B_n \quad (4)$$

and

$$|f_n(z)| < \exp(-|z|^\beta) \quad \forall z \in E_n. \quad (5)$$

Let  $G \in H(\mathbf{C})$  and  $K \subset \mathbf{C}$  be a compact set. There exists a sequence of natural numbers  $n_1 < n_2 < \dots < n_k < \dots$  such that  $P_{n_k} \rightarrow G$  ( $k \rightarrow \infty$ ) uniformly on  $K$ . Moreover, there is  $k_0 \in \mathbf{N}$  satisfying  $K \subset B_{n_k}$  whenever  $k > k_0$ . Then, by using (4), we have that  $f_{n_k} \rightarrow G$  ( $k \rightarrow \infty$ ) uniformly on  $K$ . Thus the sequence  $\{f_n\}_1^\infty$  is dense in  $H(\mathbf{C})$ .

Let us define  $M$  as the linear span of  $\{f_n\}_1^\infty$ . Evidently,  $M$  is a linear dense manifold of  $H(\mathbf{C})$ . In order to verify that (3) holds for every  $f \in M$ , it suffices to check that it is satisfied for every function  $f = f_n$ .

Define  $E_n^*$  as

$$E_n^* = \{z \in \mathbf{C} : |z| \geq n + 2 \text{ and } \text{dist}(z, P) \geq 2\}.$$

Observe that  $\{w : |w - z| \leq 1\} \subset E_n$  for all  $z \in E_n^*$ . By (5) and Cauchy's

inequalities,

$$|f_n^{(j)}(z)| \leq (j!/1^j) \max\{|f_n(w)| : |w - z| = 1\} \leq j! \exp(-(|z| - 1)^\beta)$$

for all  $z \in E_n^*$  and all  $j \in \mathbf{N}_0$ . Hence

$$|\exp(|z|^\alpha) f_n^{(j)}(z)| \leq j! \exp(|z|^\alpha - (|z| - 1)^\beta) \rightarrow 0 \quad (z \rightarrow \infty; z \in E_n^*).$$

Given a sector or a strip  $S \in \Sigma$ , we have that  $S \setminus E_n^*$  is a bounded set (this would not be true if  $P$  is chosen to be the positive real semiaxis instead of a parabolic curve). Consequently,

$$\exp(|z|^\alpha) f_n^{(j)}(z) \rightarrow 0 \quad (z \rightarrow \infty; z \in S),$$

which is (3) for  $f = f_n$ . This finishes the proof. /////

**THEOREM 3.** *There exists a linear manifold  $M \subset H(\mathbf{C})$  which is dense in*

*$H(\mathbf{C})$  and satisfying the following five properties:*

a)  $f^{(j)}$  is bounded on  $S \quad \forall S \in \Sigma, \forall f \in M$  and  $\forall j \in \mathbf{N}_0$ .

b)  $f^{(j)}$  is integrable on  $S$  with respect to plane Lebesgue measure  $\forall S \in \Sigma, \forall f \in M$

and  $\forall j \in \mathbf{N}_0$ .

c)  $f^{(j)}$  is integrable on  $l$  with respect to length measure  $\forall l \in L, \forall f \in M$  and  $\forall j \in \mathbf{N}_0$ .

d)  $\int_l f^{(j)} ds = 0 \quad \forall l \in L, \forall f \in M$  and  $\forall j \in \mathbf{N}$ .

e)  $\rho(f) = \infty \quad \forall f \in M \setminus \{0\}$ .

PROOF. Choose, for instance,  $\alpha = 1/3 \in (0, 1/2)$ . We only must show that the linear manifold  $M$  constructed in Theorem 2 satisfies all five properties stated. Fix  $f \in M, S \in \Sigma, j \in \mathbf{N}_0$  and  $l \in L$ . Property a) is immediate from (3). b) and c) are straightforward because, by (3), there is  $R > 0$  such that

$$|f^{(j)}(z)| < \exp(-|z|^{1/3}) \quad \forall z \in (S \cup l) \setminus B_R$$

and  $\exp(-|z|^{1/3})$  is integrable on  $\mathbf{C}$  with respect to plane Lebesgue measure and integrable on any line with respect to length measure.

As for d), fix  $j \in \mathbf{N}$ . Then, from fundamental calculus theorem and (3), we have

$$\int_l f^{(j)} ds = \lim_{\substack{b \rightarrow \infty \\ b \in l}} f^{(j-1)}(b) - \lim_{\substack{a \rightarrow \infty \\ a \in l}} f^{(j-1)}(a) = 0 - 0 = 0,$$

as required.

Finally, assume, by way of contradiction, that  $f \in M \setminus \{0\}$  and  $\rho(f)$  is finite.

Then there is  $\delta \in (0, +\infty)$  such that

$$\mu(r) = O(\exp(r^\delta)) \quad (r \rightarrow \infty).$$

Take  $\gamma > \max\{1/2, \delta\}$ . Consider the rays from the origin  $r_1 = [0, +\infty)$  and  $r_2 = \{z : \arg z = -\pi/\gamma\}$ . Let  $A = \sup\{|f(z)| : z \in s_{\pi(2-(1/\gamma))}\}$ , which is finite because of a).

Then  $|f| \leq A$  on  $r_1 \cup r_2$ . By the Phragmén-Lindelöf theorem (see, for instance, [9, pp. 125-126]), we have that  $|f| \leq A$  on the sector  $\{z : -\pi/\gamma \leq \arg z \leq 0\}$ . Thus

$|f| \leq A$  on the whole plane  $\mathbf{C}$ . But Liouville's theorem implies that  $f$  is constant,

so  $f \equiv 0$  because  $f(z) \rightarrow 0$  ( $z \rightarrow \infty$ ) on every line. This contradiction proves e)

and finishes the proof.

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Now, we state the existence of entire functions having very fast growth but with rapidly decreasing modulus on every strip.

**THEOREM 4.** *Assume that  $\alpha \in (0, 1/2)$ . Then there exists  $f \in H(\mathbf{C})$  such that  $i(f) = \infty$  and such that (3) holds for all  $S \in \Sigma$  and for all  $j \in \mathbf{N}_0$ .*

**PROOF.** The proof is similar to that of Theorem 2. Fix  $\alpha \in (0, 1/2)$  and

$\beta \in (\alpha, 1/2)$ . The function  $\varepsilon(t) = \exp(-t^\beta)$  ( $t \geq 0$ ) is positive and continuous and satisfies (2). Let  $P$  be the parabolic curve  $P = \{x - ix^{1/2} : x \geq 0\}$ . Define  $F$  by  $F = E \cup H$ , where

$$E = \{z \in \mathbf{C} : \text{dist}(z, P) \geq 1\}$$

and  $H = \{z_n : n \in \mathbf{N}\} \subset \mathbf{C} \setminus E$  is any fixed denumerable set satisfying  $\lim_{n \rightarrow \infty} z_n = \infty$ . Then  $F$  is closed. It is also an Arakelian set. Define the function  $g : F \rightarrow \mathbf{C}$  by

$$g(z) = \begin{cases} 0 & \text{if } z \in E \\ 1 + \exp_n(|z_n|) & \text{if } z = z_n \ (n \in \mathbf{N}). \end{cases}$$

By Theorem 1, there exists an entire function  $f$  such that

$$|f(z) - g(z)| < \varepsilon(|z|) \quad \forall z \in F.$$

Hence

$$|f(z)| < \exp(-|z|^\beta) \quad \forall z \in E \tag{6}$$

and

$$|f(z_n) - (1 + \exp_n(|z_n|))| < \exp(-|z_n|^\beta) < 1 \quad (\forall n \in \mathbf{N}),$$

so

$$\mu(r_n) \geq |f(z_n)| > \exp_n r_n > \exp_{k+1} r_n \quad (\forall n > k + 1)$$

where  $r_n = |z_n|$  and  $k \in \mathbf{N}$  is fixed. Then

$$\rho_k(f) = \limsup_{r \rightarrow \infty} \frac{\log_{k+1} \mu(r)}{\log r} \geq \limsup_{n \rightarrow \infty} \frac{r_n}{\log r_n} = \infty.$$

Thus  $\rho_k(f) = \infty$  for all  $k \in \mathbf{N}$  and  $i(f) = \infty$ . That  $f$  satisfies (3) can be proved

similarly to Theorem 2, by considering the set

$$E^* = \{z \in \mathbf{C} : \text{dist}(z, P) \geq 2\}$$

and using (6), Cauchy's inequalities and the fact that  $\{w : |w - z| \leq 1\} \subset E$  for all

$z \in E^*$ .

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As a final remark, let us denote by  $AL$  the “anti-Liouville” set, that is,

$$AL = \{f \in H(\mathbf{C}) : \exists a \in \mathbf{C} \text{ such that } \lim_{\substack{z \rightarrow \infty \\ z \in r}} f(z) = a \text{ for every ray}$$

$$r \text{ from the origin}\} \setminus \{\text{constants}\}.$$

Then  $AL$  is not very small, because  $M \setminus \{0\} \subset AL$  and, by Theorem 2,  $M \setminus \{0\}$

is dense in  $H(\mathbf{C})$ . But, simultaneously,  $AL$  is not very large, because it is of the

first category in the Baire space  $H(\mathbf{C})$ , i. e.,  $AL$  is a countable union of sets whose

closures have empty interiors. Indeed, it is easy to prove that if  $V \subset \mathbf{C}$  is unbounded,

then the set

$$T(V) = \{f \in H(\mathbf{C}) : f \text{ is bounded on } V\}$$

is of the first category, because  $T(V) = \bigcup_1^\infty C_n$  with  $C_n = \{f \in H(\mathbf{C}) : |f(z)| \leq n \text{ on } V\}$  and each  $C_n$  is closed and has empty interior (for this, note that  $C_n \cap \{\text{nonconstant polynomials}\} = \emptyset$  and that the second set in the intersection is dense in  $H(\mathbf{C})$ ). Finally, observe that

$$AL \subset \bigcap \{T(r) : r \text{ is a ray from the origin}\}.$$

So  $AL$  is, in this sense, very small.

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