UNIVERSAL FUNCTIONS FOR TAYLOR SHIFTS*

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In this paper a new sort of operators, the *Taylor shifts*, is introduced. They appear as a generalization of weighted backward shifts on the spaces of entire functions and of holomorphic functions in the unit disk. Necessary and sufficient conditions for the existence of universal functions with respect to a sequence of such operators are investigated. Several earlier results are derived as consequences.

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1. INTRODUCTION AND NOTATION

Let G denote an open subset of the complex plane $\mathbf{C}$, $\mathbf{N}$ the set of positive integers, $\mathbf{N_o} = \mathbf{N} \cup \{0\}$ and $\mathbf{D}$ the open unit disk in $\mathbf{C}$. $H(G)$ denotes, as usual, the space of holomorphic functions on G, endowed with the topology of uniform convergence on compact subsets. $H(G)$ is a second-countable Fréchet space, so a Baire space. In a Baire space X, a subset is *residual* when it contains a dense G_δ subset of X. Such a subset is "very large" in X (see [**9**, pp.213-214 and 238] and [**12**, pp.40-41]). If the complement of G in the extended plane $\mathbf{C} \cup \{\infty\}$ is connected, then polynomials are dense in $H(G)$ by Runge's theorem (see [**14**, pp.288-291]). In

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particular, polynomials are dense in $H(\mathbf{C})$ and $H(\mathbf{D})$.

In [**5**] the following definition of universality is given: If X is a Fréchet space, T is a continuous linear operator on X and $x \in X$, then the vector x is called *T–universal* if its orbit $\{T^n(x) : n \in \mathbf{N}\}$ is dense in X (in [**6**] universal vectors are called *hypercyclic*). It is obvious that T has a non-zero non-universal vector if and only if X has a nontrivial T-invariant closed subset. We will also use a more general notion of universality, which can be found in [**7**], namely: Let X and Y be topological spaces and $T_n : X \to Y$ $(n \in \mathbf{N})$ a sequence of continuous mappings. Then an element $x \in X$ is called *$\{T_n\}_1^\infty$–universal* if the set $\{T_n(x) : n \in \mathbf{N}\}$ is dense in Y. Clearly, in order that universal vectors can exist in X, Y must be separable.

In [**5**] Gethner and Shapiro used topological categories to obtain quick proofs of existence of universal functions on some function spaces. For instance, they strengthen MacLane's theorem ([**11**]; see also [**2**]): There exist entire functions (even a residual set of them) for which the sequence of successive derivatives is dense in $H(\mathbf{C})$. In fact, this result had been found by Duyos Ruiz [**4**] in 1984. In [**8**] Große-Erdmann established the following sharp result on the growth of this kind of functions: Given any function $\varphi : (0, +\infty) \to (0, +\infty)$ with $\varphi(r) \to \infty$ as $r \to \infty$, there is an entire function f for which $\{f^{(n)} : n \in \mathbf{N_o}\}$ is dense in $H(\mathbf{C})$ such that $|f(z)| = O(\varphi(r) \cdot \frac{\exp r}{r^{1/2}})$ as $|z| = r \to \infty$, while there is no such a function with $|f(z)| = O(\frac{\exp r}{r^{1/2}})$.

In other order of ideas, the author and A. Montes-Rodríguez have recently studied [**1**] the universality of a sequence of composition operators on $H(G)$ defined by a sequence of automorphisms on a plane domain G.

In [**5**] conditions are obtained for a backward shift *on a Hilbert space* to have universal vectors. For a survey of results on shift operators on Hilbert spaces, the reader is referred to Shields' survey article [**15**]. We point out that Bourdon [**3**] has recently proved that if an arbitrary operator on a separable *Banach space* has a universal vector, then there is a dense, invariant linear manifold consisting, except for zero, entirely of universal vectors. In [**6**] the existence of a dense, invariant linear submanifold each of whose non-zero elements is universal is proved by Godefroy and Shapiro for differential operators other than the scalar multiples of the identity on the Fréchet space of entire functions $H(\mathbf{C}^n)$ and for λS, where S is a generalized backward shift *on a Banach space* and $\lambda \in \mathbf{C}$ is a scalar of sufficiently large modulus. A *generalized backward shift* on a Fréchet space X is defined (see [**6**, pp.238 and 262]) as a bounded linear operator S on X obeying the following conditions: 1) the kernel of S is one dimensional; 2) the set $\cup\{\ker(S^n) : n = 0, 1, ...\}$ is dense in X. This is a generalization of Rolewicz's theorem [**13**] which asserts that if B is the ordinary backward shift on a separable Hilbert space relative to a fixed orthonormal basis $\{e_j\}_1^\infty$, i.e., $B(\sum_{j=1}^\infty a_j e_j) = \sum_{j=1}^\infty a_{j+1} e_j$, then λB has a universal vector whenever $|\lambda| > 1$. In the negative side, it is shown in [**6**, Section 5.4] that no scalar multiple λB of the ordinary backward shift B defined on $H(\mathbf{C})$ relative to the monomial basis $\{z^n\}_0^\infty$, i.e.,

$$f(z) = \sum_{j=0}^{\infty} a_j z^j \mapsto Bf(z) = \frac{f(z) - f(0)}{z} = \sum_{j=0}^{\infty} a_{j+1} z^j,$$

has universal functions.

Our aim in this paper is to present a new class of operators on $H(\mathbf{C})$ and $H(\mathbf{D})$, namely, the *Taylor shifts*, and show under what conditions they have universal functions. They appear as a generalization of weighted backward shifts relative to the latter basis. The differentiation operator $Tf = f'$ is a special case. Our theorems extend several earlier results. Several problems are proposed in the last section.

2. A SUFFICIENT CONDITION FOR UNIVERSALITY

We now present a slight improvement of a result about universality given in [**5**, **6**, **8** and **10**]. The hypotheses on the spaces X and Y are the same as in [**8**], but the conditions on the operators are weaker. Furthermore, we do not use any product spaces in our proof.

THEOREM 2.1 *Let X be a linear topological space that is a Baire space, Y a linear topological space that is second-countable, $D \subset X$ dense in X, $D' \subset Y$ dense in Y and $T_n : X \to Y$ $(n \in \mathbf{N})$ a countable family of continuous linear mappings satisfying the following condition:*

> *(C) For every $d \in D$ and every $d' \in D'$ there exist a sequence $\{x_p : p \in \mathbf{N}\} \subset X$ and positive integers $n_1 < n_2 < n_3 < \dots$ such that $x_p \to 0$, $T_{n_p}(d) \to 0$ and $T_{n_p}(x_p) \to d'$ $(p \to \infty)$.*

Then the set of $\{T_n\}_1^\infty$–universal vectors of X is residual.

Proof Let $\{V_m : m \in \mathbf{N}\}$ be a countable basis for the topology of Y. Denote by E the set

$$E = \{x \in X : \{T_n(x) : n \in \mathbf{N}\} \text{ is dense in } Y\}.$$

We have to prove that E is residual. But we can rewrite E as

$$E = \bigcap\{H_m : m \in \mathbf{N}\}$$

where

$$H_m = \bigcup\{T_n^{-1}(V_m) : n \in \mathbf{N}\}.$$

Clearly, each H_m is open and E is G_δ. Thus, it suffices to show that H_m is dense. For this, fix a nonempty open subset $S \subset X$ and choose $a \in X$, $b \in Y$, $A \subset X$, $B \subset Y$ such that $a + A + A \subset S$, $b + B + B + B \subset V_m$ and A (B, respectively) is a neighborhood of the origin of X (of Y, respectively). By hypothesis, we can find $d \in D$ and $d' \in D'$ with $d \in a + A$ and $d' \in b + B$. We apply (C) on these elements d, d':

There is $p \in \mathbf{N}$ such that $x_p \in A$, $T_{n_p}(d) \in B$ and $T_{n_p}(x_p) \in d' + B$. Define $x \in X$ by $x = x_p + d$. Then $x \in d + A \subset a + A + A \subset S$ and $T_{n_p}(x) = T_{n_p}(x_p) + T_{n_p}(d) \in d' + B + B \subset b + B + B + B \subset V_m$. Hence $x \in S \cap H_m$ and H_m is dense.

3. TAYLOR SHIFTS

Let E be either of the spaces $H(\mathbf{C})$ or $H(\mathbf{D})$. We have to distinguish these two cases in the definition. If $f \in E$ and $R > 0$ (with $R < 1$ if $E = H(\mathbf{D})$), we denote $||f||_R = \sup\{|f(z)| : |z| \leq R\}$.

DEFINITION 3.1 *An operator $T : E \to E$ is said to be a Taylor shift if there are two functions $m : \mathbf{N_o} \to \mathbf{N_o}$ and $\alpha : \mathbf{N_o} \to \mathbf{C}$ satisfying:*

1) m is injective.

2) There exists a constant $\beta \in (0, \infty)$ such that $m(j) \geq \beta j \ \forall j \in \mathbf{N}$ if $E = H(\mathbf{C})$ (such that $m(j) \leq \beta j \ \forall j \in \mathbf{N}$ if $E = H(\mathbf{D})$).

3) The sequence $\{|\alpha(j)|^{1/j} : j \in \mathbf{N}\}$ is bounded if $E = H(\mathbf{C})$ (has a lim sup not greater than 1 if $E = H(\mathbf{D})$).

4) If $f \in E$ and $f(z) := \sum_{j=0}^{\infty} a_j z^j$, then

$$Tf(z) := \sum_{j=0}^{\infty} \alpha(j) a_{m(j)} z^j. \tag{1}$$

If T is as in the definition, we denote $T = \tau(m, \alpha)$. The ordinary backward shift is $\tau(m, \alpha)$ with $m(j) = j + 1$, $\alpha(j) = 1$. The ordinary derivative operator $Tf = f'$ is the Taylor shift $\tau(m, \alpha)$ with $m(j) = j + 1$, $\alpha(j) = j + 1$.

THEOREM 3.2 *Let $T = \tau(m, \alpha)$ be a Taylor shift. Then:*

a) T is well defined, that is, the series in (1) converges at every $z \in \mathbf{C}$ or $\mathbf{D}$ for each $f \in E$.

b) T is linear and continuous.

Proof a) Let $f(z) := \sum_{j=0}^{\infty} a_j z^j \in E$. Assume that $E = H(\mathbf{C})$. We must prove that the convergence radius of the series in (1) is infinite or, equivalently, $\lim_{j \to \infty} |\alpha(j) a_{m(j)}|^{1/j} = 0$. There exists a constant $A \in (0, \infty)$ with $|\alpha(j)| \le A^j$ for all $j \in \mathbf{N}$. Since $|a_{m(j)}|^{1/m(j)} \to 0$ $(j \to \infty)$ we have, for j large enough,

$$|\alpha(j) a_{m(j)}|^{1/j} = |\alpha(j)|^{1/j} (|a_{m(j)}|^{1/m(j)})^{m(j)/j} \le A(|a_{m(j)}|^{1/m(j)})^{\beta},$$

and the right hand side tends to zero when $j \to \infty$.

Assume that $E = H(\mathbf{D})$. We must prove that the convergence radius of the series in (1) is not less than one or, equivalently: Given $\mu > 1$, $|\alpha(j) a_{m(j)}| < \mu^{(1+\beta)j}$ for j large enough. But $|\alpha(j)| < \mu^j$ and $|a_{m(j)}| < \mu^{m(j)}$ for j large enough. Consequently, $|\alpha(j) a_{m(j)}| < \mu^{j+m(j)} \le \mu^{(1+\beta)j}$ for j large enough.

b) The linearity of T is trivial. Let us show the continuity. Fix $R \in (0, \infty)$ (with $R < 1$ if $E = H(\mathbf{D})$). We must find $M, S \in (0, \infty)$ (with $S < 1$ if $E = H(\mathbf{D})$) such that

$$||Tf||_R \le M ||f||_S \quad (\forall f \in E).$$

If $f(z) = \sum_{j=0}^{\infty} a_j z^j$ and $|z| \le R$, then $|Tf(z)| \le \sum_{j=0}^{\infty} |\alpha(j)||a_{m(j)}| R^j$. Assume that $E = H(\mathbf{C})$. If A is a constant as in the proof of a), then

$$|Tf(z)| \leq \sum_{j=0}^{\infty} |a_{m(j)}|(AR)^j \leq \sum_{j=0}^{\infty} |a_{m(j)}|(1+AR)^j$$

$$\leq \sum_{j=0}^{\infty} |a_{m(j)}|(1+AR)^{m(j)/\beta} \leq \sum_{j=0}^{\infty} |a_j|(1+AR)^{j/\beta},$$

because m is injective. If $S = (2+AR)^{1/\beta}$ then, by Cauchy's inequalities,

$$|a_j| \leq \frac{||f||_S}{S^j} \ (\forall j \in \mathbf{N_o}).$$

Thus

$$||Tf||_R \leq \sum_{j=0}^{\infty} ||f||_S \left(\frac{1+AR}{2+AR}\right)^{j/\beta} = M||f||_S,$$

as required, where

$$M = \left(1 - \left(\frac{1+AR}{2+AR}\right)^{1/\beta}\right)^{-1}.$$

Assume that $E = H(\mathbf{D})$. Fix $A \in (1, 1/R)$. There is $j_o \in \mathbf{N}$ such that $|\alpha(j)| < A^j$ $(\forall j > j_o)$. Then

$$|Tf(z)| \leq \sum_{j=0}^{j_o} |\alpha(j)a_{m(j)}|R^j + \sum_{j=j_o+1}^{\infty} |a_{m(j)}|(AR)^j \leq \max_{0\leq j\leq j_o} |\alpha(j)| \sum_{j=0}^{j_o} |a_{m(j)}| +$$

$$\sum_{j=j_o+1}^{\infty} |a_{m(j)}|(AR)^{m(j)/\beta} \leq \max_{0\leq j\leq j_o} |\alpha(j)| \sum_{j=0}^{j_o} |a_{m(j)}| + \sum_{j=0}^{\infty} |a_j|(AR)^{j/\beta},$$

because $AR < 1$ and m is injective. We apply again Cauchy's inequalities to obtain

$$||Tf||_R \leq M||f||_S,$$

as required, where

$$M = j_o \max_{0\leq j\leq j_o} |\alpha(j)| \left(\frac{2}{1+AR}\right)^{\max_{0\leq j\leq j_0} m(j)/\beta} + \left(1 - \left(\frac{2AR}{1+AR}\right)^{1/\beta}\right)^{-1}$$

and

$$S = \left(\frac{1+AR}{2}\right)^{1/\beta}.$$

The next theorem shows that, under a smooth hypothesis just in the case $E = H(\mathbf{C})$, compositions of Taylor shifts are Taylor shifts too. It also shows how to generate Taylor shifts linearly from two ones. Its proof is straightforward and left to the reader.

THEOREM 3.3 *Let $T = \tau(m, \alpha)$ and $S = \tau(m^*, \alpha^*)$ be two Taylor shifts. Assume that $\lambda, \mu \in \mathbf{C}$. Then:*
a) $\lambda\tau(m, \alpha) + \mu\tau(m, \alpha^*)$ is the Taylor shift $\tau(m, \lambda\alpha + \mu\alpha^*)$.
b) If $E = H(\mathbf{D})$, $S \circ T$ is the Taylor shift $\tau(m \circ m^*, \alpha^* \cdot (\alpha \circ m^*))$.
c) If $E = H(\mathbf{C})$ and the sequence $\{|\alpha(m^*(j))|^{1/j} : \ j \in \mathbf{N}\}$ is bounded, $S \circ T$ is the Taylor shift $\tau(m \circ m^*, \alpha^* \cdot (\alpha \circ m^*))$.

COROLLARY 3.4 *Let $T = \tau(m, \alpha)$ be a Taylor shift on E and $n \in \mathbf{N}$. Denote m^0 = the identity on $\mathbf{N_o}$, $m^n = m \circ m \circ ... \circ m$, $T^n = T \circ T \circ ... \circ T$ (n times). Then:*
a) If $E = H(\mathbf{D})$, T^n is the Taylor shift $\tau(m^n, \prod_{k=0}^{n-1} \alpha \circ m^k)$.
b) If $E = H(\mathbf{C})$ and each sequence $\{|\alpha(m^k(j))|^{1/j} : \ j \in \mathbf{N}\}$ $(k = 0, 1, ..., n-1)$ is bounded, T^n is the Taylor shift $\tau(m^n, \prod_{k=0}^{n-1} \alpha \circ m^k)$.

For instance, every power B^n of the ordinary backward shift B and every n-derivative operator $f \mapsto f^{(n)}$ are Taylor shifts $\tau(m, \alpha)$, with $m(j) = j + n$, $\alpha(j) = 1$ and $m(j) = j + n$, $\alpha(j) = (j+1)(j+2) \cdot ... \cdot (j+n)$, respectively.

4. EXISTENCE OF UNIVERSAL FUNCTIONS

We study in this paragraph necessary and sufficient conditions for a Taylor shift to have universal functions. A necessary one is provided by the following theorem.

THEOREM 4.1 *Let* $\{T_n = \tau(m_n, \alpha_n) : n \in \mathbf{N}\}$ *be a sequence of Taylor shifts satisfying:*

1) $\liminf_{n\to\infty}(\inf\{m_n(j) - j : j \in \mathbf{N_o}\}) > 0$.

2) The set $\{|\alpha_n(j)|^{1/(1+m_n(j))} : j \in \mathbf{N_o}, n \in \mathbf{N}\}$ *is bounded if* $E = H(\mathbf{C})$ *(bounded by a constant less than one if* $E = H(\mathbf{D})$*).*

Then $\{T_n\}_1^\infty$ *is an equicontinuous family of operators. In particular, there is no* $\{T_n\}_1^\infty$*–universal function in* E*.*

Proof Firstly, assume that $E = H(\mathbf{C})$. If we let $V(R,\varepsilon) = \{f \in E : ||f||_R < \varepsilon\}$, then $\{V(R,\varepsilon) : R, \varepsilon > 0\}$ is a fundamental system of neighborhoods of the origin for E. Fix $\varepsilon > 0$ and $R > 0$. Let $f(z) := \sum_{j=0}^\infty a_j z^j \in V(S,\delta)$ with $S, \delta > 0$. If $|z| \le R$, then

$$|T_n f(z)| \le \sum_{j=0}^\infty |\alpha_n(j) a_{m_n(j)}| R^j.$$

By hypothesis, there exist $\gamma, \rho \in (0, +\infty)$ and $N \in \mathbf{N}$ such that $|\alpha_n(j)| \le \gamma^{1+m_n(j)}$ $(\forall j \in \mathbf{N_o}, \forall n \in \mathbf{N})$ and $m_n(j) \ge j + \rho$ $(\forall j \in \mathbf{N_o}, \forall n \ge N)$. Now, Cauchy's inequalities say us that $|a_k| \le ||f||_S / S^k$ $(\forall k \in \mathbf{N_o})$, so

$$|T_n f(z)| \le \sum_{j=0}^\infty \gamma^{1+m_n(j)} \frac{||f||_S}{S^{m_n(j)}} R^j$$

and, if $S > \gamma \max\{1, R\}$ and $n \ge N$,

$$|T_n f(z)| \le \gamma ||f||_S \sum_{j=0}^\infty \frac{R^j}{(S/\gamma)^{m_n(j)}} \le \gamma ||f||_S \sum_{j=0}^\infty \frac{R^j}{(S/\gamma)^{j+\rho}}$$

$$= \frac{\gamma ||f||_S}{(S/\gamma)^\rho} \sum_{j=0}^\infty (\frac{\gamma R}{S})^j \le \frac{\gamma^{1+\rho}}{S^{\rho-1}(S-\gamma R)} ||f||_S.$$

Hence we get $\bigcup_{n\geq N} T_n(V(S,\delta)) \subset V(R,\varepsilon)$ just by taking $S \in (\gamma \max\{1,R\},+\infty)$ and $\delta \in (0, \frac{\varepsilon(S-\gamma R)S^{\rho-1}}{\gamma^{1+\rho}})$. Consequently, $\{T_n\}_N^\infty$ is equicontinuous. So, $\{T_n\}_1^\infty$ is too.

The proof for the case $E = H(\mathbf{D})$ runs over the same steps excepting that here R and γ are less than one and S should be chosen in the interval $(\gamma, 1)$, which in turn guarantees $S > \gamma R$ too.

Remark We have put $1 + m_n(j)$ instead of $m_n(j)$ in condition 2) just in order to avoid the denominator of the exponent of $|\alpha_n(j)|$ can be zero.

Theorem 4.1 enables us to obtain as an easy corollary the result on a scalar multiple λB of the ordinary backward shift B stated in the introduction [**6**, Section 5.4]. It suffices to take $\alpha_n(j) = \lambda^n$, $m_n(j) = j + n$ ($\forall j \in \mathbf{N_o}$, $\forall n \in \mathbf{N}$).

The next theorem gives us a sufficient condition for universality. It, indeed, provides a residual set of universal functions. Alternatively, it may be regarded as a "$H(\mathbf{C})-$ or $H(\mathbf{D})-$version" of Rolewicz's theorem [**13**]. The new conditions 1) and 2) are formally similar to those in Theorem 4.1 but, of course, they point at the opposite direction.

THEOREM 4.2 *Let $\{T_n = \tau(m_n, \alpha_n) : n \in \mathbf{N}\}$ be a sequence of Taylor shifts satisfying:*

1) $\lim_{n\to\infty} (\inf\{m_n(j) : j \in \mathbf{N_o}\}) = +\infty$.

2) For each $q \in \mathbf{N_o}$, the sequence $\{\min_{0\leq j\leq q} |\alpha_n(j)|^{1/(1+m_n(j))} : n \in \mathbf{N}\}$ is not bounded.

Then the set of all functions of E which are $\{T_n\}_1^\infty$–universal is residual.

Proof We apply Theorem 2.1 to $X = Y = E$, $D = D' = \{\text{polynomials}\}$. Let P be a polynomial. From 1) there exists $n_o \in \mathbf{N}$ such that $m_n(j) > \text{degree}(P)$ for all $j \in \mathbf{N_o}$ and all $n > n_o$. Then $T_n(P) = 0$ for all $n > n_o$. Now fix a polynomial

Q with degree$(Q) = q$, say

$$Q(z) = \sum_{j=0}^{q} b_j z^j.$$

By 2), there exists a sequence of positive integers $n_1 < n_2 < n_3 < ...$ with the following property: Given $\varepsilon \in (0,1)$ and $R \in (0,\infty)$ (with $R < 1$ if $E = H(\mathbf{D})$), there is $p_o \in \mathbf{N}$ such that

$$|\beta_p(j)|^{1/(1+q_p(j))} > \frac{R}{\varepsilon}$$

for all $p > p_o$ and all $j \in \{0, 1, ..., q\}$, where we have set

$$\beta_p = \alpha_{n_p} \quad \text{and} \quad q_p = m_{n_p}.$$

Then

$$|\beta_p(j)| > \left(\frac{R}{\varepsilon}\right)^{1+q_p(j)} \geq \frac{R^{q_p(j)}}{\varepsilon} R \quad (j = 0, 1, ..., q;\ p > p_o).$$

Hence

$$\max_{0 \leq j \leq q} \frac{R^{q_p(j)}}{|\beta_p(j)|} \longrightarrow 0 \quad (p \to \infty) \tag{2}$$

for all $R > 0$ (with $R < 1$ if $E = H(\mathbf{D})$).

Define f_p $(p \in \mathbf{N})$ by

$$f_p(z) = \sum_{j=0}^{q} \frac{b_j}{\beta_p(j)} z^{q_p(j)}.$$

Then $f_p \in E$ (in fact, it is a polynomial) for all $p \in \mathbf{N}$ and (2) shows that $||f_p||_R \to 0$ $(p \to \infty)$ for all $R > 0$ (with $R < 1$ if $E = H(\mathbf{D})$). Finally,

$$T_{n_p} f_p(z) = \sum_{j=0}^{\infty} \beta_p(j) \frac{b_j}{\beta_p(j)} z^j = Q(z)$$

for all $p \in \mathbf{N}$ and all $z \in \mathbf{C}$ or $\mathbf{D}$.

We summarize: Given $P \in D$ and $Q \in D'$, we have found sequences $n_1 < n_2 < \ldots$ and $\{f_p : p \in \mathbf{N}\} \subset E$ with $f_p \to 0$, $T_{n_p}(f_p) \to Q$ and, obviously, $T_{n_p}(P) \to 0$

$(p \to \infty)$. Consequently, condition (C) in Theorem 2.1 is satisfied and the proof is complete.

We conclude with two concrete examples.

Examples a) The n-derivative operators $T_n = \tau(m_n, \alpha_n)$ are Taylor shifts and satisfy hypotheses 1) and 2) in Theorem 4.2, because $\lim_{n\to\infty} (\inf\{m_n(j) : j \in \mathbf{N_o}\}) = \lim_{n\to\infty} (\inf\{j + n : j \in \mathbf{N_o}\}) = \lim_{n\to\infty} n = +\infty$ and, for each $q \in \mathbf{N_o}$, $\lim_{n\to\infty} (\min_{0\le j\le q} |\alpha_n(j)|^{1/(1+m_n(j))}) = \lim_{n\to\infty} (\min_{0\le j\le q} |(j+1)(j+2)\cdot \ldots \cdot (j+n)|^{1/(1+j+n)})$ $\geq \lim_{n\to\infty} (n!)^{1/(1+q+n)} \geq \lim_{n\to\infty} (\frac{n}{e})^{n/(n+q+n)} = +\infty$. Hence, we recover the extension of MacLane's theorem given in [**4**] and [**5**].

b) Theorem 4.2 allows much play. For instance, there exists a residual set of functions $f \in E$ whose orbits $\{T_n(f) : n \in \mathbf{N}\}$ are dense in E, where $T_n : E \to E$ is the operator defined as

$$f(z) = \sum_{j=0}^{\infty} a_j z^j \mapsto T_n f(z) = \sum_{j=0}^{\infty} \frac{n^{n^2 \log n}}{n+j} a_{2j+n^2} z^j .$$

5. FINAL REMARKS AND OPEN PROBLEMS

1) We propose the obvious problem of filling in the lack between theorems 4.1 and 4.2: Which are the *sharp* conditions on sequences $\{m_n\}_1^\infty$and $\{\alpha_n\}_1^\infty$ for the sequence $\{\tau(m_n, \alpha_n)\}_1^\infty$ to be equicontinuous (to have universal functions, respectively)?

2) We point out here that a Taylor shift need not be a generalized backward shift in the sense of Godefroy and Shapiro [**6**] (see Introduction), so their method cannot be used to furnish a linear manifold of universal vectors for $\{T_n\}_1^\infty$, even in the case that $\{T_n\}_1^\infty$ consists of the iterates of a Taylor shift.

3) In relation to Große-Erdmann's result **[8]** (see Introduction), it would be interesting to know which is the lowest growth allowed for an entire function f in order to be universal for a sequence of Taylor shifts on $H(\mathbf{C})$ (the sequence of derivative operators is a special case).

4) A weaker property than universality, namely, cyclicity, might also be studied: Is there a $\{T_n\}_1^\infty$*−cyclic* function in E, that is, a function $f \in E$ such that the linear span of $\{T_n(f) : n \in \mathbf{N}\}$ is dense in E?

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