On the growth of zero-free MacLane-universal entire functions

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Abstract

We show that exponential growth is the critical discrete rate of growth for zero-free entire functions which are universal in the sense of MacLane. Specifically, it is proved that if the lower exponential growth order of a zero-free entire function f is finite, then f cannot be hypercyclic for the derivative operator; and, if a positive function φ having infinite exponential growth is fixed, then there exist zero-free hypercyclic functions which are controlled by φ along a sequence of radii tending to infinity.

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1 Introduction

In 1952 MacLane [23] proved the existence of an entire function whose sequence of derivatives is dense in the space of entire functions. A large number of related results have been published from his finding. Among them, one can find several statements dealing with combinations of the mentioned density with other special properties, such as rate of growth or absence of zeros, see below. Our aim in this paper is to contribute in this research by considering all three properties simultaneously.

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In the next paragraphs we will settle the pertinent terminology, notation and framework, and make a brief account of results.

If G is a domain in the complex plane $\mathbb{C}$, by $H(G)$ we denote, as usual, the space of holomorphic functions on G endowed with the topology of convergence in compacta. Then $H(G)$ is a Polish space, that is, it is separable and completely metrizable. In the special case $G = \mathbb{C}$ we obtain the space $H(\mathbb{C})$ of entire functions. The extended complex plane is $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. If $a \in \mathbb{C}$ and $r > 0$, then $D(a, r)$ and $\overline{D}(a, r)$ will stand, respectively, for the open disk and the closed disk with center a and radius r. If $f \in H(\mathbb{C})$, its maximum modulus function is defined as $M(f, r) = \max\{|f(z)| : z \in \overline{D}(0, r)\}$ $(r > 0)$. For a function $\varphi : (0, +\infty) \to (0, +\infty)$ with $\lim_{r\to\infty} \varphi(r) = \infty$, the exponential growth order and the lower exponential growth order are respectively defined by $\rho(\varphi) = \limsup_{r\to\infty} \dfrac{\log\log\varphi(r)}{\log r}$ and $\widetilde{\rho}(\varphi) = \liminf_{r\to\infty} \dfrac{\log\log\varphi(r)}{\log r}$. Observe that $\rho(\varphi) = \inf\{\mu > 0 : \text{there is } r_0 = r_0(\mu) > 0 \text{ such that } \varphi(r) < e^{r^\mu}$ for all $r > r_0\}$, while $\widetilde{\rho}(\varphi) = \sup\{\mu > 0 : \text{there is } r_0 = r_0(\mu) > 0$ such that $\varphi(r) > e^{r^\mu}$ for all $r > r_0\}$. Note specially that $\widetilde{\rho}(\varphi) = +\infty$ if and only if $\lim_{r\to\infty} \dfrac{\varphi(r)}{e^{r^k}} = +\infty$ for all $k \geq 1$. The same concepts for an entire function f are defined by $\widetilde{\rho}(f) = \widetilde{\rho}(M(f, \cdot))$ and $\rho(f) = \rho(M(f, \cdot))$ (with $\widetilde{\rho}(f) := 0 =: \rho(f)$ if f is constant), see [11].

As for universality, [18] and [5] are excellent surveys. Assume that X and Y are topological spaces. Then a sequence $T_n : X \to Y$ $(n \geq 1)$ is called *universal* whenever there is a point $x_0 \in X$ –called universal for (T_n)– such that the orbit $\{T_n x_0 : n \geq 1\}$ of x_0 under (T_n) is dense in Y. The set of universal elements for (T_n) is denoted by $\mathcal{U}((T_n))$. If $X = Y$ and X is a topological vector space, then an operator T on X (that is, $T : X \to X$ is a continuous linear selfmapping) is said to be *hypercyclic* provided that the sequence (T^n) of iterates ($T^1 := T$, $T^2 = T \circ T$ and so on) is universal; in this case the corresponding universal vectors are called hypercyclic for T. The symbol $HC(T)$ will stand for the set of such hypercyclic vectors.

In the latter terminology, MacLane's theorem tells us that the derivative operator $D : f \in H(\mathbb{C}) \to f' \in H(\mathbb{C})$ is hypercyclic. In fact, he proved in [23] that there are D-hypercyclic functions of exponential type $\tau(f) = 1$. Recall that $\tau(f) = \inf\{\mu > 0 : \text{there is } r_0 = r_0(\mu) > 0 \text{ such that } M(f, r) < e^{\mu r}$ for all $r > r_0\}$. Duyos-Ruiz [16] noted in 1984 that no D-hypercyclic function f with $\tau(f) < 1$ can exist. Herzog [20] showed in 1988 the existence of a D-hypercyclic function growing not faster than re^r. In 1990, Grosse-Erdmann proved the next sharp statement about the growth of MacLane-universal entire functions, see [17].

Theorem 1.1. *There is no D-hypercyclic entire function such that* $|f(z)| = O(e^r/\sqrt{r})$ *as* $r \to \infty$*. However, given any function* $\varphi : (0, +\infty) \to (0, +\infty)$ *with* $\varphi(r) \to \infty$ *as* $r \to \infty$*, the set of D-hypercyclic functions* f *with* $|f(z)| = O(\varphi(r)e^r/\sqrt{r})$ *as* $r \to \infty$ *is dense in* $H(\mathbb{C})$*.*

Corresponding results about harmonic functions on $\mathbb{R}^N$, shift operators on $H(\mathbb{C})$ and certain differential operators on $H(\mathbb{C})$ can be found in [1,2], [19, Section 6] and [9,15], respectively. The growth of entire functions presenting the stronger property of *frequent hypercyclicity* (for D) introduced by Bayart and Grivaux [4] is considered in [12], [10] and [13].

In 1994, Herzog [21] demonstrated the existence of zero-free entire functions which are universal for D (even with zero-free first derivative; see extensions to differential operators in [7] and [8]). The study of permissible rates of growth of D-hypercyclic zero-free entire functions seems to be new, and we initiate it in this note. We show that exponential growth is the critical discrete rate of growth for these functions. Specifically, it is proved that if the lower exponential growth order of a zero-free entire function f is finite, then f cannot be hypercyclic for the derivative operator; and, if a positive function φ having infinite exponential growth is fixed, then there exist zero-free hypercyclic functions which are controlled by φ along a sequence of radii tending to infinity, see Section 3. Section 2 will be devoted to give some instrumental assertions.

2 Preliminary results

We are going to state three statements that will be crucial in the proof of our main result. The first one (Theorem 2.1) comes from G. Pólya ([25], see also [26]) and concerns the dynamics of zeros of derivatives. The second one (Theorem 2.3) is due to G. Herzog [21] and provides a useful criterion for induced universality. Finally, the third statement is the so-called Borel-Carathéodory inequality (Theorem 2.4), see for instance [22, pp. 53–54].

Definition 2.1. If $f \in H(\mathbb{C})$, then the *final set* L_f of f is the set of all points $z_0 \in \mathbb{C}_\infty$ satisfying the following property: every neighborhood of z_0 contains zeros of infinitely many of the functions $f^{(n)}$ $(n \geq 1)$.

Theorem 2.1. *If* $f(z) = p(z)e^{q(z)}$*, where* p *and* q *are polynomials with* degree$(q) \geq 2$ *then* L_f *consists of* degree(q) *equally spaced rays emanating from one point.*

We insert here an elementary lemma which will be used later. Note that $\mathcal{F}$ might be, for instance, the family of translations $z \mapsto z + a$ $(a \in \mathbb{C})$.

Lemma 2.2. *Assume that f is an entire function for which there exists a family $\mathcal{F} \subset H(\mathbb{C})$ satisfying the following properties:*
(a) $\mathcal{F} \subset \overline{\{f^{(n)} : n \geq 1\}}$.
(b) *There is a dense subset $\mathcal{D}$ of $\mathbb{C}$ such that, for every $z_0 \in \mathcal{D}$, there exists $\varphi \in \mathcal{F} \setminus \{0\}$ with $\varphi(z_0) = 0$.*
Then L_f is maximal, that is, $L_f = \mathbb{C}_\infty$. In particular, L_f is maximal if f is D-hypercyclic.

Proof. Let $z_0 \in \mathcal{D}$. According to (b), we can select a function $\varphi \in \mathcal{F} \setminus \{0\}$ with $\varphi(z_0) = 0$. Consider an open neighborhood V of z_0. Without loss of generality we may assume that $V = D(z_0, R)$ for some $R > 0$ and that z_0 is the unique point of V where φ vanishes. Set $\varepsilon := \min_{|z-z_0|=R/2} |\varphi(z)| > 0$. By (a), there exists a strictly increasing sequence $\{k_n\}$ of positive integers such that

$$\sup_{z \in \overline{D}(z_0, \frac{R}{2})} |f^{(k_n)}(z) - \varphi(z)| < \varepsilon$$

for every $n = 1, 2, \ldots$. A straightforward application of Rouché's theorem [3] shows that V contains at least one zero of each function $f^{(k_n)}$ and therefore $z_0 \in L_f$. Hence $\mathcal{D} \subset L_f$. But L_f is clearly closed in $\mathbb{C}_\infty$, so $L_f = \mathbb{C}_\infty$. $\square$

Now, we are going to see that under appropriate conditions the universality of a sequence (T_n) can be transmited to the sequence $(T_n|_A)$ of its restrictions to a G_δ-subset $A \subset X$. Recall that, by Alexandroff's theorem (see [24]), a subset A of a completely metrizable topological space X is completely metrizable (so Baire) whenever A is G_δ.

Theorem 2.3. *Assume that X is a Polish space and Y is a separable metrizable space. Let d_X, d_Y be distances inducing the topologies of X, Y, respectively. Let $\{A_k\}_{k\geq 1}$ be a sequence of open subsets of X with $A = \bigcap_{k=1}^\infty A_k \neq \emptyset$. Suppose that $T_n : X \to Y$ $(n \geq 1)$ is a sequence of continuous mappings satisfying that $\mathcal{U}((T_n))$ is residual in X. If*

$$\lim_{k\to\infty} \sup_{n\geq 1} \inf_{z\in A} (d_X(a_k, z) + d_Y(T_n a_k, T_n z)) = 0$$

for every sequence $(a_k)_{k\geq 1}$ with $a_k \in A_k$ $(k = 1, 2, \ldots)$, then $\mathcal{U}((T_n|_A))$ is residual in A.

Theorem 2.4. *Let f be analytic on $\overline{D}(0, R)$. Then for $0 < r < R$ we have*

$$M(f, r) \leq \frac{R + r}{R - r}\left(\max_{|z|=R} \mathrm{Re} f(z) + |f(0)|\right).$$

3 Main result

We start with the following two lemmas. The first one prevents certain functions to be MacLane-universal, while the second one –which might be of some interest in itself– gives a sufficient condition for induced universality on the space of analytic functions.

Lemma 3.1. *If $f(z) = p(z)e^{q(z)}$, where p and q are polynomials, then $f \notin HC(D)$.*

Proof. If degree$(q) \geq 2$, simply combine Theorem 2.1 and Lemma 2.2. Suppose now that degree$(q) \leq 1$. Then $f(z) = p(z)e^{az+b}$ for some $a, b \in \mathbb{C}$. By induction, it can be seen that $f^{(n)}(z) = p_n(z)e^{az+b}$, where p_n is a polynomial with degree$(p_n) \leq$ degree(p). Therefore $\{f^{(n)} : n \geq 1\} \subset M$, where M is a finite-dimensional (so closed) subspace of $H(\mathbb{C})$. Hence $\{f^{(n)}\}_{n\geq 1}$ cannot be dense in $H(\mathbb{C})$. □

If $G \subset \mathbb{C}$ is a domain, the topology of $H(G)$ is generated by the distance

$$d(f, g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|f - g\|_j}{1 + \|f - g\|_j},$$

where $\|h\|_j = \sup_{z\in K_j} |h(z)|$ ($j \geq 1$, $h \in H(G)$) and (K_j) is a fixed exhaustive sequence of compact subsets of G, that is, $K_j \subset K_{j+1}^0$ (S^0 denotes the interior of S) and $G = \bigcup_{j=1}^{\infty} K_j$.

Lemma 3.2. *Let (A_k) be a sequence of subsets of G whose intersection $A := \bigcap_{k=1}^{\infty} A_k$ is nonempty. Suppose that $T_n : H(G) \to H(G)$ $(n \geq 1)$ is a sequence of continuous mappings such that $\mathcal{U}((T_n))$ is residual in $H(G)$. Assume that the following conditions are fulfilled:*

- (i) *For every $k \geq 1$ and every $f \in A_k$ there exist a domain $\Omega = \Omega(k, f) \supset K_k$ and a sequence $(h_m) \subset A$ such that $h_m \to f$ $(m \to \infty)$ uniformly on compacta in Ω.*
- (ii) *For every $n \geq 1$, every domain $\Omega \subset G$, every sequence $(f_m) \subset H(G)$ and every function $f \in H(G)$, we have:*
  $$f_m \to f \text{ in } H(\Omega) \implies T_n f_m \to T_n f \text{ in } H(\Omega).$$

Then $\mathcal{U}((T_n|_A))$ is residual in A.

Proof. Let us apply Theorem 2.3. Choose $X := H(G) =: Y$ and $d_X := d =: d_Y$. Let $(f_k) \subset H(G)$ be a sequence with $f_k \in A_k$ $(k \geq 1)$. Fix $k \geq 1$ and consider $f := f_k$. By (i), there are a domain $\Omega \supset K_k$ and a sequence $(h_m) \subset$

A such that $h_m \to f$ compactly in Ω. From (ii), $T_n h_m \to T_n f$ compactly in Ω for each n. In particular, we obtain that $\lim_{m\to\infty} ||T_n h_m - T_n f||_k = 0$.

Hence $\lim_{m\to\infty}(||h_m - f||_j + ||T_n h_m - T_n f||_j) = 0$ for every $n \geq 1$ and every $j \in \{1, 2, ..., k\}$. Given $\delta > 0$, a positive integer $m = m(\delta, n, k)$ can be found in such a way that $||h_m - f||_j + ||T_n h_m - T_n f||_j < \delta$ or all $j \in \{1, \ldots, k\}$, so

$$d(f, h_m) + d(T_n f, T_n h_m) < \sum_{j=1}^{k} \frac{\delta}{2^j} + \sum_{j=k+1}^{\infty} \frac{1}{2^j} + \sum_{j=k+1}^{\infty} \frac{1}{2^j} = \delta + 2^{1-k}.$$

Then $\inf_{h\in A}(d(f, h) + d(L^n f, L^n h)) < \delta + 2^{1-k}$ for all $\delta > 0$ and all $n \geq 1$. Therefore we get

$$\sup_{n\geq 1} \inf_{h\in A} (d(f_k, h) + d(T_n f_k, T_n h)) \leq 2^{1-k} \to 0 \quad (k \to \infty).$$

Consequently, the conditions in Theorem 2.3 are fulfilled, so obtaining the residuality of $\mathcal{U}((T_n|_A))$ in A. □

Remark 3.3. Condition (ii) in the preceding lemma is a kind of "supercontinuity" for each T_n. This property is satisfied, for instance, by the differential operators $\Phi(D)$, with Φ an entire function of subexponential type (see for instance [6] or [7, Theorem 4]). In particular, (ii) is satisfied by the sequence $T_n := D^n$ $(n \geq 1)$. Recall that an entire function is said to be of subexponential type $\tau(f) = 0$. Operators $\Phi(D)$ with Φ nonconstant and of subexponential type are hypercyclic on $H(G)$ for any domain $G \subset \mathbb{C}$ (see [8]). Godefroy and Shapiro had demonstrated in 1991 the hypercyclicity of $\Phi(D)$ on $H(\mathbb{C})$ for every nonconstant entire function Φ of exponential type (i.e. with $\tau(\Phi) < +\infty$).

We are now ready to state our theorem.

Theorem 3.4. (a) *There is no D-hypercyclic entire function f with finitely many zeros satisfying $\widetilde{\rho}(f) < +\infty$.*
(b) *If $\varphi : (0, +\infty) \to (0, +\infty)$ is a function with $\rho(\varphi) = +\infty$ (so if $\widetilde{\rho}(\varphi) = +\infty$) then there is a D-hypercyclic zero-free entire function f such that* $\liminf_{r\to\infty} \dfrac{M(f, r)}{\varphi(r)} < +\infty$.

Proof. Let us proof the negative part (a). Suppose, by way of contradiction, that there is an entire function $f \in HC(D)$ with finitely many zeros such that $\widetilde{\rho}(f) < +\infty$. We set $p(z) := 1$ if f has no zeros, and $p(z) := (z - z_1) \cdots (z - z_N)$ if $z_1, \ldots, z_N$ are the zeros of f (counting multiplicities). Since

$M(p,r) = 1$ or $M(p,r) \sim r^N$ $(r \to \infty)$, we also have $\widetilde{\rho}(g) < +\infty$, where $g := f/p$. Observe that g is an entire function without zeros. Hence $g = e^h$ for some $h \in H(\mathbb{C})$. From $\widetilde{\rho}(g) < +\infty$ and $|g| = e^{\mathrm{Re} h}$ we derive the existence of a natural number k and of a sequence $0 < R_1 < R_2 < \cdots < R_n \to \infty$ such that

$$\mathrm{Re}\, h(z) \le R_n^k \quad (|z| = R_n;\ n = 1, 2, \dots).$$

Let $r_n := R_n/2$. By Theorem 2.4, we get

$$\begin{aligned} M(h, r_n) &\le \frac{2r_n + r_n}{2r_n - r_n}\big(\max_{|z|=R_n} \mathrm{Re}\, h(z) + |h(0)|\big) \\ &\le 3(2^k r_n^k + |h(0)|) \le C r_n^k \ \ (n \ge 1) \end{aligned} \tag{1}$$

for an appropriate constant $C \in (0, +\infty)$. Now, by Cauchy's inequalities and (1) we have for all $n \ge 1$ that

$$|a_j| \le \frac{M(h, r_n)}{r_n^j} \le C r_n^{k-j},$$

where a_j $(j = 0, 1, 2, ...)$ are the MacLaurin coefficients of h. Letting $n \to \infty$ we get $a_j = 0$ for all $j > k$. Thus h is a polynomial. Since $f = pe^h$, Lemma 3.1 tells us that $f \notin HC(D)$. This contradiction proves (a).

As for (b), assume that φ is as in the hypothesis. Then one can find a sequence $0 < s_1 < s_2 < \cdots < s_n < \cdots \to \infty$ satisfying

$$\varphi(s_n) > e^{s_n^n} \ \ (n = 1, 2, \dots). \tag{2}$$

Consider the set

$$B := \{\text{zero-free entire functions}\} \cap \{f \in H(\mathbb{C}) : \liminf_{r\to\infty} \frac{M(f,r)}{\varphi(r)} < +\infty\}.$$

Our aim is to show that $B \cap HC(D) \neq \emptyset$. For this, it is enough to show that $A \cap HC(D) \neq \emptyset$, where $A := \bigcap_{k=1}^{\infty} A_k$ and $A_k := \{f \in H(\mathbb{C}) : f(z) \neq 0$ for all $z \in \overline{D}(0, k)$ and there exists $n = n(k, f) > k$ such that $|f(z)| < e^{s_n^n}$ for all $z \in \overline{D}(0, s_n)\}$. Indeed, by (2) any function $f \in A$ satisfies $\liminf_{n\to\infty} \dfrac{M(f, s_n)}{\varphi(s_n)} \le 1$, so $A \subset B$.

Observe that each A_k is an open set in $H(\mathbb{C})$, so A is a G_δ-subset. In order to apply Lemma 3.2, we choose $G := \mathbb{C}$, $K_n := \overline{D}(0, n)$ and $T_n := D^n$ $(n \ge 1)$. Since D is hypercyclic on a completely metrizable space, we obtain that $\mathcal{U}((T_n))$ $(= HC(D))$ is residual in X (see [18]). Moreover, $A \neq \emptyset$; in fact, every function e^P with P polynomial belongs to A, because $M(P, r) < r^{1+\mathrm{degree}(P)}$ asymptotically. Note also that $A \cap HC(D) = \mathcal{U}((T_n|_A))$.

As said in Remark 3.3, condition (ii) in Lemma 3.2 is satisfied. Hence our task is to demonstrate that (i) is also satisfied. To do this, fix $k \geq 1$ and $f \in A_k$. Since $\overline{D}(0,k)$ is compact, there is $\varepsilon > 0$ such that $f(z) \neq 0$ for all $z \in D(0, k+2\varepsilon)$. By simple connectedness, we can find a function $g \in H(D(0, k+2\varepsilon))$ with $f(z) = e^{g(z)}$ on such disk. Select a sequence of polynomials (P_m) tending uniformly to g on $\overline{D}(0, k+\varepsilon)$. From the inequality $|e^z - e^w| \leq e^{\max\{\mathrm{Re} z, \mathrm{Re} w\}}|z-w|$ $(z, w \in \mathbb{C})$ we get

$$\lim_{m\to\infty} \sup_{z\in\overline{D}(0,k+\varepsilon)} |e^{P_m(z)} - e^{g(z)}| = 0.$$

In particular, we have that $e^{P_m} \to f$ compactly in $D(0, k+\varepsilon)$. Finally, choose $\Omega := D(0, k+\varepsilon)$ and $h_m := e^{P_m}$ $(m \geq 1)$. Since $h_m \in A$, we obtain (i) and the theorem is proved. □

Final remarks. 1. The special case of zero-free functions is included in part (a) of Theorem 3.4. Trivially, (a) also holds if we replace "$\widetilde{\rho}(f) < +\infty$" by "$\rho(f) < +\infty$". Furthermore, according to Remark 3.3 and the preceding proof, part (b) of the theorem holds for $\Phi(D)$-hypercyclic zero-free functions, where Φ is a nonconstant entire function of subexponential type.

2. We want to pose here the following question: can the conclusion of (b) be improved to $\limsup_{r\to\infty} \dfrac{M(f,r)}{\varphi(r)} < +\infty$, that is, $|f(z)| = O(\varphi(r))$ as $|z| = r \to \infty$? Our conjecture is *yes*, at least if one reinforce the condition on φ to $\widetilde{\rho}(\varphi) = +\infty$.

3. According to the results of Bourdon [14] on every hypercyclic operator and of Shkarin [28] on the operator D, we obtain thanks to Lemma 2.2 that the family of entire functions having maximal final set is *dense-lineable* (it contains, except for zero, a dense vector subspace) and *spaceable* (it contains, except for zero, a closed infinite dimensional vector subspace) in $H(\mathbb{C})$.

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