Compositionally universal entire functions on the plane and the punctured plane

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To Professor Tomás Domínguez Benavides on his 60th birthday

Abstract. Necessary/sufficient conditions for a sequence of automorphisms of the complex plane to generate a sequence of composition operators that is universal on the punctured plane are provided. As a consequence, it is derived that only for translations and rotation-dilations there can be entire functions whose orbits present universality. Boundedness of these functions on unbounded sets as well as frequent universality on the whole plane are also analyzed.

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1. Introduction, notation, known results and aim of this paper

In 1929 Birkhoff [9] published his celebrated universality theorem asserting the existence of an entire function f on the complex plane $\mathbb{C}$ whose translates $f(\cdot+n)$ ($n \in \mathbb{N} := \{1, 2, \dots\}$) approximate uniformly any prescribed entire function on every compact subset of $\mathbb{C}$. His result can be put into the more general setting of universality, whose starting notions are collected in the next paragraph.

Assume that X and Y are topological spaces and that $T_n : X \to Y$ ($n \geq 1$) is a sequence of continuous mappings. Then (T_n) is said to be *universal* whenever there exists a point $x_0 \in X$, called universal for (T_n), whose orbit $\{T_n x_0 : n \geq 1\}$ under (T_n) is dense in Y. The set of universal points for (T_n) will be denoted by

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$\mathcal{U}((T_n))$. It is easy to see that, if X is a Baire space and Y is second countable, then $\mathcal{U}((T_n))$ is dense if and only if it is residual, and if and only if (T_n) is *transitive* (i.e. given nonempty open sets $U \subset X$, $V \subset Y$, there is $n \in \mathbb{N}$ such that $T_n(U) \cap V \neq \emptyset$). If $X = Y$, then a single continuous selfmapping $T : X \to X$ is called universal (transitive) if the sequence (T^n) of its iterates is universal (transitive, resp.), and we denote $\mathcal{U}(T) = \{\text{universal points for } T\} := \mathcal{U}((T^n))$. Note that $\mathcal{U}(T) \neq \emptyset$ if and only if $\mathcal{U}(T)$ is dense. If now X and Y are (Hausdorff) topological vector spaces and $\{T_n : n \in \mathbb{N}\} \subset L(X,Y) =: \{\text{continuous linear mappings } X \to Y\}$ then the words *universal* and *hypercyclic* are synonymous, although "hypercyclic" is mainly used for single mappings $T \in L(X) = \{\text{operators on } X\} := L(X,X)$. The reader is referred to the surveys [4], [18] and [20] for history, concepts, results and references about universality and hypercyclicity.

For a domain (nonempty connected open subset) $G \subset \mathbb{C}$, we denote by $H(G)$ the vector space of all holomorphic functions $G \to \mathbb{C}$. Then $H(G)$ becomes an F-space (that is, a complete metrizable topological vector space) when it is endowed with the topology of convergence on compacta; in addition, $H(G)$ is separable, hence second countable. We will denote by $\mathcal{M}(G)$ the family of Mergelyan subsets of G, that is, $\mathcal{M}(G) = \{K \subset \mathbb{C} : K \text{ is compact and } \mathbb{C} \setminus K \text{ is connected}\}$. Recall that the sets $U(f,\varepsilon,K) := \{g \in H(G) : |g(z) - f(z)| < \varepsilon \ \forall z \in K\}$ ($f \in H(G)$, $\varepsilon > 0$, $K \subset G$ a compact set) form a basis of open subsets in $H(G)$. For the sake of brevity, we introduce the following two concepts. A subset $\mathcal{A} \subset H(G)$ will be called $\mathcal{M}$-*dense* provided that for every $\varepsilon > 0$ and every $K \in \mathcal{M}(G)$, one has $\mathcal{A} \cap U(f,\varepsilon,K) \neq \emptyset$. By using Runge's approximation theorem (see [16]), we have that the polynomials form an $\mathcal{M}$-dense subset of $H(G)$. Of course, denseness matches $\mathcal{M}$-denseness if G is simply connected. If G, Ω are domains in $\mathbb{C}$ and $\varphi \in H(G,\Omega)$ (i.e. $\varphi \in H(G)$ and $\varphi(G) \subset \Omega$), then the composition operator $C_\varphi : H(\Omega) \to H(G)$ is the continuous linear mapping defined as $C_\varphi f = f \circ \varphi$. If $(\varphi_n) \subset H(G,\Omega)$, then we say that the sequence of composition operators (C_{φ_n}) is $\mathcal{M}$-*universal* provided that there is $f \in H(G)$ –called $\mathcal{M}$-universal for (C_{φ_n})– such that $\{f \circ \varphi_n\}_{n \geq 1}$ is $\mathcal{M}$-dense in $H(G)$. A corresponding notion of $\mathcal{M}$-hypercyclicity arises when $G = \Omega$ and each φ_n is the nth-iterate $\varphi^{[n]} = \varphi \circ \cdots \circ \varphi$ (n-fold) of φ.

In the terminology of the preceding paragraphs, Birkhoff's theorem tells us that the translation operator $f \mapsto f(\cdot + 1)$ is hypercyclic on $H(\mathbb{C})$. With essentially the same proof as in [9] we have (see [22]) that if $(a_n) \subset \mathbb{C}$ is an unbounded sequence then the sequence of translations $\tau_{a_n} : f \in H(\mathbb{C}) \mapsto f(\cdot + a_n) \in H(\mathbb{C})$ ($n \in \mathbb{N}$) is universal (a function $f \in H(\mathbb{C})$ (τ_{a_n})-universal for some $(a_n) \subset \mathbb{C}$ will be called Birkhoff-universal). In 1988, Zappa [28] replaced the additive group $(\mathbb{C}, +)$ by the multiplicative group $(\mathbb{C}^* := \mathbb{C} \setminus \{0\}, \cdot)$ (see [1] and [2] for extensions to complex special or general linear groups) and demonstrated the existence of a "multiplicatively universal" function $f \in H(\mathbb{C}^*)$, that is, given $g \in H(\mathbb{C}^*)$, $\varepsilon > 0$ and $K \in \mathcal{M}(\mathbb{C}^*)$, there exists $a \in \mathbb{C}^*$ such that $|f(az) - g(z)| < \varepsilon$ for all $z \in K$ (by Mergelyan's approximation theorem [16, Chap. 3], one may in fact prescribe $g \in A(K)$, the space of continuous functions $f : K \to \mathbb{C}$ that are holomorphic in the interior K^0 of K). The same approach of [28] shows that if $(a_n) \subset \mathbb{C}^*$ is an unbounded sequence then $(C_{z \mapsto a_n z})$ is $\mathcal{M}$-universal on $H(\mathbb{C}^*)$, that is, there is an $f \in H(\mathbb{C}^*)$ whose sequence of dilation-rotations $\{f(a_n \cdot)\}_{n \geq 1}$ is $\mathcal{M}$-dense in $H(\mathbb{C}^*)$.

Denote by $\mathrm{Aut}(G)$ the group of holomorphic automorphisms of a domain G. Recall that $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b : a \in \mathbb{C}^*, b \in \mathbb{C}\}$ and $\mathrm{Aut}(\mathbb{C}^*) = \{z \mapsto az : a \in \mathbb{C}^*\} \cup \{z \mapsto a/z : a \in \mathbb{C}^*\}$. Birkhoff's and Zappa's results have been extended or improved in several directions. Restricting ourselves to $\mathbb{C}$ and $\mathbb{C}^*$, we remark the following:

(a) In 1995, Montes and the author [7] proved that, for a sequence $\{z \overset{\varphi_n}{\mapsto} a_n z + b_n\}_{n \geq 1} \subset \mathrm{Aut}(\mathbb{C})$, the sequence of composition operators $C_{\varphi_n} : f \mapsto f(a_n \cdot + b_n)$ $(n \geq 1)$ is universal on $H(\mathbb{C})$ if and only if the sequence

$$\{\min\{|b_n|, |b_n/a_n|\}\}_{n \geq 1} \tag{1}$$

is unbounded. If this is the case, then there is a residual set of universal entire functions. In particular, the composition operator $f \mapsto f(a \cdot + b)$ associated to the automorphism $z \mapsto az + b$ is hypercyclic if and only if $a = 1$ and $b \neq 0$. Moreover, given $(a_n) \subset \mathbb{C}^*$, the sequence of composition operators $f \in H(\mathbb{C}^*) \mapsto f(a_n \cdot) \in H(\mathbb{C}^*)$ is $\mathcal{M}$-universal if and only if the sequence

$$\{\max\{|a_n|, |1/a_n|\}_{n \geq 1}$$

is unbounded, in which case there is a residual subset in $H(\mathbb{C}^*)$ of $\mathcal{M}$-universal functions. In particular, the operator $f \mapsto f(a \cdot)$ is $\mathcal{M}$-hypercyclic on $H(\mathbb{C}^*)$ if and only if $|a| \neq 1$. By contrast, *no* sequence $(\varphi_n) \subset \mathrm{Aut}(\mathbb{C}^*)$ generates a universal sequence (C_{φ_n}) on $H(\mathbb{C}^*)$, see [7] and [19].

(b) In 1996, Luh [23] improved Zappa's theorem by constructing, for a given unbounded sequence $(a_n) \subset \mathbb{C}^*$, an *entire* function f such that $\{f(a_n \cdot)\}_{n \geq 1}$ is $\mathcal{M}$-dense in $H(\mathbb{C}^*)$. The set of such functions f is in fact residual in $H(\mathbb{C})$ [8]. Notice that the unboundedness of (a_n) is necessary for the existence of these multiplicatively universal entire functions, for if $|a_n| \leq M < +\infty$ $(n \geq 1)$ then the sequence $(f(a_n \cdot))$ cannot approximate the constant function $1 + \sup_{|z| \leq M} |f(z)|$ on the compact set $\{1\}$.

In view of (a)–(b), a natural question arises: which sequences $\{a_n z + b_n\}_{n \geq 1} \subset \mathrm{Aut}(\mathbb{C})$ support entire functions f satisfying that $\{f(a_n \cdot + b_n)\}_{n \geq 1}$ is $\mathcal{M}$-dense in $H(\mathbb{C}^*)$? We provide necessary as well as sufficient conditions in Section 2.

In the last few years, a number of authors have invested much effort in making universality compatible with some additional property, specially boundedness on large sets. In this respect, Tenthoff [26] proved in 2000 that there are Birkhoff-universal entire functions that are bounded on every (straight) line. M.C. Calderón [13] (see also [17]) showed in 2002 the existence of a dense linear manifold of Birkhoff-universal entire functions f (see [10] for similar results in the harmonic setting) which are bounded on any sector $\{z : 0 \leq \arg z \leq \alpha\}$ $(0 < \alpha < 2\pi)$ and any strip (i.e. any domain lying between two parallel lines); even $\exp(|z|^\alpha) f(z) \to 0$

$(z \to \infty)$ on these subsets for fixed $\alpha \in (0, 1/2)$, and many more additional properties hold. In 2003, Costakis and Sambarino [14] proved that there are τ_1-hypercyclic entire functions f tending to 0 as $z \to \infty$ on each sector $\{z : \varepsilon \leq \arg z \leq 2\pi - \varepsilon\}$ $(0 < \varepsilon < \pi)$. In 2006, Niess [25] gave the following necessary and sufficient condition for the existence of (τ_{a_n})-universal entire functions bounded on every line: there exists a subsequence (a_{n_k}) such that, for every $R > 0$ and every line L, there is $k_0 \in \mathbb{N}$ with $L \cap B(a_{n_k}, R) = \emptyset$ for all $k \geq k_0$ (as usual, $B(a, R) := \{z \in \mathbb{C} : |z - a| < r\}$ and $\overline{B}(a, R) := \{z \in \mathbb{C} : |z - a| < r\}$). Gharibyan, Luh and Niess [17, Theorem 1.2] established the existence of an entire function φ and of sequences $(a_n), (b_n) \subset \mathbb{C}$ with $a_n \to 0$, $b_n \to \infty$ such that φ is $(C_{a_n z + b_n})$-universal and bounded on every line. Bonilla and the author [5] proved that, for given $F \subset \mathbb{C}$, there exist a Birkhoff-universal entire function that is bounded on F if and only if there exists an *Arakelian subset* F_0 of $\mathbb{C}$ (i.e. F_0 is closed and $\mathbb{C}_\infty \setminus F_0$ is connected and locally connected at ∞, where $\mathbb{C}_\infty$ is the one-point compactification of $\mathbb{C}$) such that $\rho_i(\mathbb{C} \setminus F_0) = +\infty$ (here $\rho_i(A) = \sup\{r > 0 :$ there exists a closed ball B of radius r with $B \subset A\}$, the inscribed radius of a subset $A \subset \mathbb{C}$). In 2010, Calderón, Luh and the author [6] stated the following: if $A \subset \mathbb{C}$ is an unbounded Arakelian set with $\rho_i(\mathbb{C} \setminus A) = +\infty$, there is a dense linear manifold M of entire functions all of whose nonzero members are Birkhoff-universal and $\exp(|z|^\alpha) f(z) \to 0$ $(z \to \infty,\ z \in A)$ for all $\alpha < 1/2$ and $f \in M$. Passing to $\mathbb{C}^*$, A. Vogt [27] has recently constructed a multiplicative universal entire function φ with respect to a given unbounded sequence (a_n) such that φ is bounded on some curve Γ tending to ∞. On the contrary, he has proved that any such φ is necessarily *unbounded on every line*.

In Section 3, we investigate boundedness and unboundedness on large subsets of $\mathbb{C}$ for entire functions being compositionally universal ($\mathcal{M}$-universal) on $H(\mathbb{C})$ (on $H(\mathbb{C}^*)$, resp.) with respect to sequences in $\mathrm{Aut}(\mathbb{C})$.

In 2006, Bayart and Grivaux [3] presented the more stringent notion of frequent hypercyclicity, that is a kind of quantified hypercyclicity. If X is a topological vector space, then an operator $T \in L(X)$ is said to be *frequent hypercyclic* provided that there is a vector $x \in X$, called frequent hypercyclic for T, such that, for every nonempty open set $U \subset X$, the set $\{n \in \mathbb{N} : T^n x \in U\}$ has lower positive density, that is,

$$\liminf_{n \to \infty} \frac{\operatorname{card}\{k \in \{1, \dots, n\} : T^k x \in U\}}{n} > 0. \tag{2}$$

As in the mere hypercyclicity, the concept can be extended to a sequence of continuous mappings $T_n : X \to Y$ $(n \in \mathbb{N})$ between two topological spaces X, Y (simply replace $T^k x$ by $T_k x$ in (2)), in which case (T_n) is called *frequent universal*, and x a frequent universal element for (T_n), see [12]. Sufficient conditions for frequent hypercyclicity/universality have been given in [3] and [12]. In [3] it is proved the frequent universality of any translation operator τ_a $(a \neq 0)$ on $H(\mathbb{C})$. This is also shown by Bonilla and Grosse-Erdmann in three different ways (see [11], [12, Example 3.2] and [12, Theorem 4.2]) as a consequence of more general criteria.

Our third and final goal, which will be performed in Section 4, is to extend this result by analyzing the frequent universality of a sequence of composition operators on $H(\mathbb{C})$ generated by a sequence $(\varphi_n) \subset \mathrm{Aut}(\mathbb{C})$.

2. $\mathcal{M}$-universal entire functions

Our first theorem extends and unifies the assertions given by Luh, Prado and Bernal about existence of entire functions that are $\mathcal{M}$-universal on $H(\mathbb{C}^*)$ with respect to sequences of composition operators generated by automorphisms of $\mathbb{C}$, see Section 1. As usual, we adopt the convention $\alpha/0 = +\infty$ whenever $\alpha \in (0, +\infty)$.

Theorem 2.1. *Assume that* $\{\varphi_n\}_{n\geq 1} \subset \mathrm{Aut}(\mathbb{C})$ *and* $\varphi_n(z) = a_n z + b_n$ $(n \geq 1)$.

(a) *If the sequence*

$$\{\max\{\min\{|a_n|, |a_n/b_n|\}, \min\{|b_n|, |b_n/a_n|\}\}\}_{n\geq 1} \tag{3}$$

is unbounded then the sequence $C_{\varphi_n} : H(\mathbb{C}) \to H(\mathbb{C}^*)$ $(n \in \mathbb{N})$ *is* $\mathcal{M}$-*universal. In fact, the set* $\mathcal{U}_{\mathcal{M}}((C_{\varphi_n})) := \{f \in H(\mathbb{C}) : (f \circ \varphi_n)$ *is* $\mathcal{M}$-*dense in* $H(\mathbb{C}^*)\}$ *is a residual subset of* $H(\mathbb{C})$.

(b) *If the sequence* $C_{\varphi_n} : H(\mathbb{C}) \to H(\mathbb{C}^*)$ $(n \in \mathbb{N})$ *is* $\mathcal{M}$-*universal then*
   (i) *at least one of the sequences* $(a_n), (b_n)$ *is unbounded, and*
   (ii) *there exists a sequence* $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ *such that the sequence of zeros* $\{\varphi_{n_k}^{-1}(0)\}_{k\geq 1}$ *is not contained in any set* $K \in \mathcal{M}(\mathbb{C}^*)$.

Proof. (a) Assume that (3) is unbounded. Then at least one of the sequences $\{\min\{|a_n|, |a_n/b_n|\}\}_{n\geq 1}$, $\{\min\{|b_n|, |b_n/a_n|\}\}_{n\geq 1}$ is unbounded. If the latter one is unbounded, then by [7, Proposition 2.3 and Theorem 3.1] the set $\mathcal{U}((C_{\varphi_n})) = \{f \in H(\mathbb{C}) :$ the set $\{f \circ \varphi_n\}_{n\geq 1}$ is dense in $H(\mathbb{C})\}$ is a residual subset of $H(\mathbb{C})$. But by Runge's theorem $H(\mathbb{C})$ is $\mathcal{M}$-dense in $H(\mathbb{C}^*)$. Therefore $\mathcal{U}_{\mathcal{M}}((C_{\varphi_n})) \supset \mathcal{U}((C_{\varphi_n}))$, so $\mathcal{U}_{\mathcal{M}}((C_{\varphi_n}))$ is residual in $H(\mathbb{C})$. Suppose now that $\{\min\{|a_n|, |a_n/b_n|\}\}_{n\geq 1}$ is unbounded. Fix a compact set $K \subset \mathbb{C}$, a number $\varepsilon > 0$, a set $L \in \mathcal{M}(\mathbb{C}^*)$ and functions $f \in H(\mathbb{C})$, $g \in H(\mathbb{C}^*)$. Consider the sets $U(f, \varepsilon, K) := \{h \in H(\mathbb{C}) : |h(z) - f(z)| < \varepsilon$ for all $z \in K\}$ and $V(g, \varepsilon, L) := \{h \in H(\mathbb{C}^*) : |h(z) - g(z)| < \varepsilon$ for all $z \in L\}$, which are respectively basic open sets in $H(\mathbb{C})$ and $H(\mathbb{C}^*)$. Choose $r, s > 0$ with $K \subset \overline{B}(0, r)$ and $L \cap \overline{B}(0, 2s) = \emptyset$, then select $n \in \mathbb{N}$ such that $|a_n| > r/s$ and $|a_n/b_n| > 1/s$. Then we obtain, for all $z \in L$,

$$|\varphi_n(z)| = |a_n z + b_n| = |a_n| \left| z + \frac{b_n}{a_n} \right| > \frac{r}{s}(2s - s) = r.$$

Hence $\overline{B}(0, r) \cap \varphi_n(L) = \emptyset$. Define the function $F : \overline{B}(0, r) \cup \varphi_n(L) \to \mathbb{C}$ by

$$F(z) = \begin{cases} f(z) & \text{if } z \in \overline{B}(0, r) \\ g(\frac{z - b_n}{a_n}) & \text{if } z \in \varphi_n(L). \end{cases}$$

Note that $S := \overline{B}(0, r) \cup \varphi_n(L) \in \mathcal{M}(\mathbb{C})$ because $\varphi_n \in \mathrm{Aut}(\mathbb{C})$. Moreover, F is holomorphic on a neighborhood of S. By Runge's theorem, there exists a polynomial h (so $h \in H(\mathbb{C})$) with $|h(z) - F(z)| < \varepsilon$ for all $z \in S$. In particular, $|h(z) - f(z)| < \varepsilon$

for all $z \in K$ and $|h(\varphi_n(z)) - g(z)| < \varepsilon$ for all $z \in L$. In other words, $h \in U(f, \varepsilon, K)$ and $C_{\varphi_n} h \in V(g, \varepsilon, L)$. Therefore $C_{\varphi_n}(U(f, \varepsilon, K)) \cap V(g, \varepsilon, L) \neq \emptyset$, whence each set $\bigcup_{n \in \mathbb{N}} C_{\varphi_n}^{-1}(V(g, \varepsilon, L))$ is (open and) dense in $H(\mathbb{C})$. Now, for every domain G there exists a sequence $\{K_j\}_{j \geq 1} \subset \mathcal{M}(G)$ that is exhaustive, that is, given $L \in \mathcal{M}(G)$ there is $j_0 \in \mathbb{N}$ with $L \subset K_{j_0}$ [7, Lemma 2.9]. For the special case $G = \mathbb{C}^*$, it follows that

$$\mathcal{U}_{\mathcal{M}}((C_{\varphi_n})) = \bigcap_{\substack{g \in H(\mathbb{C}^*) \\ L \in \mathcal{M}, \varepsilon > 0}} \bigcup_{n \in \mathbb{N}} C_{\varphi_n}^{-1}(V(g, \varepsilon, L)) = \bigcap_{k,l,j \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} C_{\varphi_n}^{-1}(V(g_k, 1/l, K_j)),$$

where (g_k) is any fixed dense sequence in $H(\mathbb{C}^*)$. Since $H(\mathbb{C})$ is a Baire space, we obtain that $\mathcal{U}_{\mathcal{M}}((C_{\varphi_n}))$ is a dense G_δ (so residual) subset of $H(\mathbb{C})$.

(b) Suppose that there exists $f \in H(\mathbb{C})$ that is $\mathcal{M}$-universal for the sequence $C_{\varphi_n} : H(\mathbb{C}) \to H(\mathbb{C}^*)$ $(n \in \mathbb{N})$. By way of contradiction, assume also that (a_n) and (b_n) are bounded. Then $|a_n| + |b_n| \leq M < +\infty$ $(n \in \mathbb{N})$, say. Therefore no sequence $(f(a_{n_k} z + b_{n_k}))$ would be able to approximate the constant function $1 + \max_{|w| \leq M} |f(w)|$ on the compact set $\{1\} \in \mathcal{M}(\mathbb{C}^*)$, so contradicting the hypothesis. Thus, (i) is satisfied. Suppose now, again by way of contradiction, that (ii) does not hold. By hypothesis, there is $f \in H(\mathbb{C})$ as well as a sequence $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ such that $f(a_{n_k} z + b_{n_k}) \to g(z)$ $(k \to \infty)$ uniformly on each member of $\mathcal{M}(\mathbb{C}^*)$, where $g(z) := 1 + |f(0)|$. But one can find a set $K \in \mathcal{M}(\mathbb{C}^*)$ with $\{\varphi_{n_k}^{-1}(0) = -b_{n_k}/a_{n_k}\}_{n \geq 1} \subset K$. Then

$$\sup_{z \in K} |f(a_{n_k} z + b_{n_k}) - g(z)| \geq |f(\varphi_{n_k}(\varphi_{n_k}^{-1}(0))) - g(z)| = |f(0) - (1 + |f(0)|)| \geq 1$$

for all $k \in \mathbb{N}$, which is absurd. This proves the theorem. □

Corollary 2.2. *Let $\varphi \in \mathrm{Aut}(\mathbb{C})$, with $\varphi(z) = az + b$. Then the following properties are equivalent:*

(a) *There is an entire function f whose orbit $\{f \circ \varphi^{[n]}\}_{n \geq 1}$ is $\mathcal{M}$-dense in $H(\mathbb{C}^*)$.*
(b) *The set $\mathcal{U}_{\mathcal{M}}((C_{\varphi^{[n]}}))$ is residual in $H(\mathbb{C})$.*
(c) *Either $|a| > 1$ and $b = 0$ or $a = 1$ and $b \neq 0$.*

Proof. It is evident that (b) implies (a). Suppose that (c) holds. If $a = 1$ and $b \neq 0$ (i.e. φ is a translation) then we have in fact that C_φ is hypercyclic on $H(\mathbb{C})$, so the sequence $(C_\varphi^n) = (C_{\varphi^{[n]}})$ is transitive from $H(\mathbb{C})$ into itself. Therefore the set $\mathcal{U}((C_\varphi^n))$ of its hypercyclic functions is residual in $H(\mathbb{C})$. But, again by Runge's theorem, $\mathcal{U}((C_\varphi^n)) \subset \mathcal{U}_{\mathcal{M}}((C_{\varphi^{[n]}}))$, from which the residuality of $\mathcal{U}_{\mathcal{M}}((C_{\varphi^{[n]}}))$ follows. If $a \neq 1$ then $\varphi^{[n]}(z) = a^n z + a^{n-1} b + a^{n-2} b + \cdots + ab = a_n z + b_n$, where $a_n = a^n$ and $b_n = b \cdot \frac{a^n - 1}{a - 1}$. Therefore, if $|a| > 1$ and $b = 0$, the sequence $\{\min\{|a_n|, |a_n/b_n|\}_{n \geq 1} = \{|a|^n\}_{n \geq 1}$ is unbounded. By Theorem 2.1, (b) is satisfied. Finally, if (a) is true and $a \neq 1$ then by Theorem 2.1 at least one of the sequences (a^n), $(b\frac{a^n - 1}{a - 1})$ is unbounded, hence $|a| > 1$. If $a \neq 1$ and $b \neq 0$ then $(\varphi^{[n]})^{-1}(0) = \frac{b(1 - a^{-n})}{1 - a} \underset{n}{\longrightarrow} \frac{b}{1 - a} \in \mathbb{C}^*$, so the full sequence $\{(\varphi^{[n]})^{-1}(0)\}_{n \geq 1}$ is contained in the set $K := \{\frac{b}{1 - a}\} \cup \{(\varphi^{[n]})^{-1}(0) : n \in \mathbb{N}\} \in \mathcal{M}(\mathbb{C}^*)$. According to

Theorem 2.1, there is no entire function f such that $\{f \circ \varphi^{[n]}\}_{n\geq 1}$ is $\mathcal{M}$-dense. If $a = 1$ and $b = 0$ then φ is the identity, whose iterates clearly do not generate compositional universality. Hence (a) implies (c). □

Remark 2.3. 1. As an example, there is no entire function f such that the set $A(f) := \{z \mapsto f(n^2 z + n) : n \in \mathbb{N}\}$ is dense in $H(\mathbb{C})$, because the sequence (1) (see Section 1) for $a_n = n^2$, $b_n = n$ is $\{1/n\}_{n\in\mathbb{N}}$, which is bounded. Nevertheless, entire functions f such that $A(f)$ is $\mathcal{M}$-dense in $H(\mathbb{C}^*)$ do exist: indeed, $\min\{|a_n|, |a_n/b_n|\} = n$ ($n \in \mathbb{N}$) and Theorem 2.1 applies.
2. Let (q_n) be an enumeration of the set $\{e^{it} : t \in \mathbb{Q}\}$ ($\mathbb{Q}$ = the set of rational numbers), which is dense in the unit circle $\mathbb{T}$. Then $\varphi_n(z) = a_n z + b_n := nz + nq_n \in \mathrm{Aut}(\mathbb{C})$ ($n \geq 1$) satisfies that the sequences (1) and (3) are bounded, (a_n) and (b_n) are unbounded and the full sequence $(\varphi_n^{-1}(0)) = (-q_n)$ is not contained in any member of $\mathcal{M}(\mathbb{C}^*)$. Hence (C_{φ_n}) is not $H(\mathbb{C})$-universal but we can extract from Theorem 2.1 neither the $\mathcal{M}$-universality nor the non-$\mathcal{M}$-universality of (C_{φ_n}) on $H(\mathbb{C}^*)$. This raises the question of finding an intermediate property between the unboundedness of (3) and (i)+(ii) of Theorem 2.1 characterizing the $\mathcal{M}$-universality of $\{H(\mathbb{C}) \xrightarrow{C_{\varphi_n}} H(\mathbb{C}^*)\}_{n\geq 1}$.

3. Bounded universal entire functions

In this section we show that, under appropriate conditions, no $\mathcal{M}$-universal entire function on $H(\mathbb{C}^*)$ is bounded on every member of a large family of unbounded sets (including lines), see Theorem 3.1. In the positive direction, it will be proved (Theorem 3.3) the existence of an $\mathcal{M}$-universal entire function f for which there is an unbounded set Γ such that f tends to 0 as $z \to \infty$ (in particular, f is bounded) on it. We may even prescribe the set Γ within a large family, including again all lines, in the case of $H(\mathbb{C})$-universality, see Theorem 3.5. This improves recent results by A. Vogt [27] and completes a number of findings by several authors, see Section 1.

We introduce the following notion. By $\mathcal{S}_0$ we denote the collection of all closed sectors S with vertex at the origin and amplitude $\mathrm{ampl}(S) \in (0, 2\pi)$. We say that a set Γ is *totally non-spiral-like* provided that there are two unbounded connected sets Γ_0, L and a sector $S \in \mathcal{S}_0$ such that $\Gamma_0 \subset \Gamma$, $L \subset S$ and $\Gamma_0 \cap L = \emptyset$. For instance, every set Γ with non-total oscillation near ∞ (i.e. such that there are $R > 0$, $S \in \mathcal{S}_0$ and an unbounded connected set $\Gamma_0 \subset \Gamma$ with $\Gamma_0 \cap S \cap \{|z| > R\} = \emptyset$) is totally non-spiral-like. In particular, every line, every parabola, and in general every unbounded algebraic curve $P(x, y) = 0$ (P = a nonconstant polynomial of two real variables with real coefficients) are totally non-spiral-like sets.

Theorem 3.1. *Assume that* $\{\varphi_n(z) = a_n z + b_n\}_{n\geq 1} \subset \mathrm{Aut}(\mathbb{C})$ *is a sequence such that* $a_n \to \infty$ *and* $b_n/a_n \to 0$ *as* $n \to \infty$. *Let* Γ *be a totally non-spiral-like subset of* $\mathbb{C}$. *Then every* $\mathcal{M}$*-universal entire function for the sequence* $C_{\varphi_n} : H(\mathbb{C}) \to H(\mathbb{C}^*)$ $(n \geq 1)$ *is unbounded on* Γ.

Proof. Consider the compact set given by the "snail" curve

$$K := \{z(\theta) = \left(1 + \frac{\theta}{2\pi}\right) e^{i\theta} \in \mathbb{C} : \theta \in [0, 6\pi]\}.$$

Observe that $K \in \mathcal{M}(\mathbb{C}^*)$ and note that any ray σ starting from any point in $\mathbb{D}$ intersects K at least at two different points $z(\theta_{1,\sigma}), z(\theta_{2,\sigma})$ with $\theta_{2,\sigma} - \theta_{1,\sigma} > \pi$ and $|z(\theta_{1,\sigma})| > 1$, $|z(\theta_{2,\sigma})| > 2$. It follows that if M is an unbounded, connected set lying in a non-degenerated closed sector Σ whose vertex is at $\mathbb{D}$ with $M \cap \mathbb{D} \neq \emptyset$, then M must intersect both curvilinear arcs of the closed annular "sector" determined by K and the segments joining the pair of points $(z(\theta_{1,\sigma}), z(\theta_{2,\sigma}))$ and $(z(\theta_{1,\lambda}), z(\theta_{2,\lambda}))$, where σ and λ are the rays forming the boundary of Σ (there may be more than one of such annular sectors; then simply select one of them). As a consequence, if A is an unbounded connected subset of $\mathbb{C}$ such that $A \cap \mathbb{D} \neq \emptyset = A \cap M$, then A should intersect the snail, i.e. $A \cap K \neq \emptyset$. This property will be used later.

Notice that by Theorem 2.1, there are in fact $\mathcal{M}$-universal entire functions for (C_{φ_n}). Let f be one of them and assume, by way of contradiction, that f is bounded on Γ, say $|f(z)| \leq \alpha$ $(z \in \Gamma)$. By universality, there is a sequence $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ such that $C_{\varphi_{n_k}} f$ tends to the constant function $1 + \alpha$ uniformly on K as $k \to \infty$. Without loss of generality, we may assume that (n_k) is the full sequence $\mathbb{N}$. In particular, there is $n_0 \in \mathbb{N}$ such that

$$\sup_{z \in K} |f(a_n z + b_n) - (1 + \alpha)| < 1 \text{ for all } n \geq n_0. \tag{4}$$

Let Γ_0, L, S the sets supplied by the definition of non-total spiral-likeness for our set Γ. Fix any $z_0 \in \Gamma_0$ and any $w_0 \in L$. Since $a_n \to \infty$ and $b_n/a_n \to 0$, there is $m \in \mathbb{N}$ with $m \geq n_0$ such that $|z_0/a_m| < 1/2$, $|w_0/a_m| < 1/2$ and $|b_m/a_m| < 1/2$. Then $\frac{z_0 - b_m}{a_m}, \frac{w_0 - b_m}{a_m} \in \mathbb{D}$. We have:

- The sets $A := \{\frac{z - b_m}{a_m} : z \in \Gamma_0\}$ and $M := \{\frac{z - b_m}{a_m} : z \in L\}$ are unbounded connected sets, and $\Sigma := \{\frac{z - b_m}{a_m} : z \in S\}$ is a closed sector with ampl(Σ) $\in (0, 2\pi)$ and vertex at the point $-b_m/a_m \in \mathbb{D}$.
- $A \cap \mathbb{D} \neq \emptyset \neq M \cap \mathbb{D}$, $A \cap M = \emptyset$ and $M \subset \Sigma$.

From the first paragraph, $A \cap K \neq \emptyset$. Therefore there exist $z_1 \in K$ and $z_2 \in \Gamma_0$ (so $z_2 \in \Gamma$) such that $z_1 = \frac{z_2 - b_m}{a_m}$. It follows from (4) that $|f(a_m z_1 + b_m) - (1 + \alpha)| < 1$. Hence $|f(z_2) - (1 + \alpha)| < 1$, so $|f(z_2)| > \alpha$ due to the triangular inequality. This contradicts the fact that $|f| \leq \alpha$ on Γ. □

Remark 3.2. If the snail K of the last proof had only two turns (i.e. $\theta \in [0, 4\pi]$) then not every ray σ with vertex at a point of $\mathbb{D}$ would intersect at least twice the set K: take for instance $\sigma = \{z = t + i(-1 + (t/2)) : t \geq 0\}$.

In the next result, notice that similarly to Theorem 3.1 the condition imposed on (a_n), (b_n) guarantees $\mathcal{M}$-universality on $H(\mathbb{C}^*)$, but discards $H(\mathbb{C})$-universality. By contrast, the restriction imposed in Theorem 3.4 entails $H(\mathbb{C})$-universality for the sequence (C_{φ_n}), with $\varphi_n(z) = a_n z + b_n$.

Theorem 3.3. *Let* $\{\varphi_n(z) = a_n z + b_n\}_{n\ge 1} \subset \mathrm{Aut}(\mathbb{C})$. *Suppose that the sequence* $\{\min\{|a_n|, |a_n/b_n|\}\}_{n\ge 1}$ *is unbounded. Let* $\omega : \mathbb{C} \to (0, +\infty)$ *be a continuous function. Then there exists an* $\mathcal{M}$*-universal entire function* f *with respect to* $\{C_{\varphi_n} : H(\mathbb{C}) \to H(\mathbb{C}^*)\}_{n\ge 1}$ *that admits an unbounded connected set* $\Gamma \subset \mathbb{C}$ *such that* $\lim\limits_{\substack{z\to\infty \\ z\in\Gamma}} \omega(z) f(z) = 0$.

Proof. We will follow the approach of [27] in the first steps. Choose an exhaustive sequence $\{K_n\}_{n\ge 1}$ in $\mathcal{M}(\mathbb{C}^*)$ as in the proof of Theorem 2.1 (see [7, Lemma 2.9] or [23, Lemma 3]). Let $\{Q_n\}_{n\ge 1}$ be an enumeration of all polynomials whose coefficients are in $\mathbb{Q} + i\mathbb{Q}$. In addition, let $\{(Q_n^*, K_n^*)\}_{n\ge 1}$ be a sequence with the property that every pair (Q_i, K_j) $(i, j \in \mathbb{N})$ occurs infinitely many often.

By hypothesis, and without loss of generality, one can assume that $\min\{|a_n|, |a_n/b_n|\} \to +\infty$ as $n \to \infty$. Then $a_n \to \infty$ and $b_n/a_n \to 0$. Therefore, given $r, s \in (0, +\infty)$, one has

$$|a_n|(r - |b_n/a_n|) > s \ \text{ for } \ n \ \text{ large enough.} \tag{5}$$

Set $n_1 = 1$. Proceeding by induction, and assuming that $n_1 < n_2 < \cdots < n_k$ have been already determined, we can choose due to (5) an integer $n_{k+1} > n_k$ such that

$$|a_{n_{k+1}}|(\inf\{|z| : z \in K_k^*\} - |b_{n_{k+1}}/a_{n_{k+1}}|) > 1 + \sup\{|z| : z \in a_{n_k} K_k^* + b_{n_k}\}. \tag{6}$$

With the help of the triangular inequality, (6) shows that the sets $A_k := a_{n_k} K_k^* + b_{n_k}$ $(k \in \mathbb{N})$ are mutually disjoint. Of course, each A_k is compact and has connected complement. In addition, due to (6), one gets $\mathrm{dist}(A_j, A_k) \ge 1$ $(j, k \in \mathbb{N};\ j \ne k)$. Then $\bigcup_{k\in\mathbb{N}} A_k$ is closed and $G := \mathbb{C} \setminus \bigcup_{k\in\mathbb{N}} A_k$ is an unbounded connected open subset of $\mathbb{C}$. Hence we can find an unbounded regular curve Γ in G, that is, Γ is the image of some differentiable function $\gamma : [0, +\infty) \to G$ with $\lim_{t\to\infty} \gamma(t) = \infty$. In particular, Γ is an unbounded connected closed subset of $\mathbb{C}$ with empty interior Γ^0.

Consider the set $A := G \cup \bigcup_{k\in\mathbb{N}} A_k$. Then A is a *Carleman set*, that is, it satisfies the following properties:

- A is closed.
- $\mathbb{C}_\infty \setminus A$ is connected and locally connected at ∞.
- For every compact set $K \subset \mathbb{C}$ there is a neighborhood V of ∞ in $\mathbb{C}_\infty$ such that no component of A^0 intersects both K and V.

Define the function $g : A \to \mathbb{C}$ by

$$g(z) = \begin{cases} 0 & \text{if } z \in \Gamma \\ Q_k^*\left(\frac{z - b_{n_k}}{a_{n_k}}\right) & \text{if } z \in A_k \ (k \in \mathbb{N}). \end{cases}$$

This function is evidently holomorphic in A^0 and continuous on A. Moreover, the function $\varepsilon(z) = \begin{cases} \frac{1}{(1+|z|)\omega(z)} & \text{if } z \in \Gamma \\ \frac{1}{k} & \text{if } z \in A_k \ (k \in \mathbb{N}) \end{cases}$ is continuous on A and positive.

According to the Nersesjan tangential approximation theorem (see [16, pp. 155–159] or [24]), there exists an entire function f satisfying

$$|f(z) - g(z)| < \varepsilon(z) \quad (z \in A). \tag{7}$$

In particular, $|f(z)\omega(z)| < \frac{1}{1+|z|}$ for all $z \in \Gamma$. Hence $f(z)\omega(z) \to 0$ ($z \to \infty$; $z \in \Gamma$).

Finally, we show the $\mathcal{M}$-universality of f for (C_{φ_n}). From (7) we get $|f(z) - Q_k^*(\frac{z-b_{n_k}}{a_{n_k}})| < \frac{1}{k}$ for all $z \in A_k$ or, that is the same, $|f(a_{n_k}z+b_{n_k}) - Q_k^*(z)| < \frac{1}{k}$ for all $z \in K_k^*$. Fix $h \in H(\mathbb{C}^*)$, $\varepsilon > 0$ and $K \in \mathcal{M}(\mathbb{C}^*)$. Select $n_0 \in \mathbb{N}$ with $K \subset K_{n_0}$. By Runge's approximation theorem there exists $m_0 \in \mathbb{N}$ with $|Q_{m_0}(z) - h(z)| < \varepsilon/2$ for all $z \in K$. Now choose a sequence $\{k(1) < k(2) < \cdots\} \subset \mathbb{N}$ such that $(Q_{k(j)}^*, K_{k(j)}^*) = (Q_{m_0}, K_{n_0})$ for all $j \geq 1$. Pick $j \in \mathbb{N}$ such that $k(j) > 2/\varepsilon$ and let $\nu := n_{k(j)}$. Since $K \subset K_{n_0} = K_{k(j)}^*$ we obtain for all $z \in K$ that

$$|(C_{\varphi_\nu} f)(z) - h(z)| \leq |f(a_\nu z + b_\nu) - Q_{k(j)}^*(z)| + |Q_{m_0}(z) - h(z)| < \frac{1}{k(j)} + \frac{\varepsilon}{2} < \varepsilon.$$

To summarize, $\{C_{\varphi_n} f\}_{n\geq 1}$ is $\mathcal{M}$-dense in $H(\mathbb{C}^*)$, as required. □

Remark 3.4. It is also possible to obtain an $\mathcal{M}$-universal entire function f as in the last theorem and an unbounded connected *open* set $\Gamma \subset \mathbb{C}$ such that $\lim_{\substack{z\to\infty \\ z\in\Gamma}} \omega(z)f(z) = 0$ for prescribed functions $\omega(z)$, but within a more restrictive class of weight functions, see the approach of Theorem 3.5.

The following theorem extends the results on bounded universal functions by Niess [25], Bernal and Bonilla [5] and Gharibyan, Luh and Niess [17, Theorem 1.2] given in Section 1.

Theorem 3.5. *Let $\Gamma \subset \mathbb{C}$ be an unbounded subset. Assume that $\{\varphi_n(z) = a_n z + b_n\}_{n\geq 1} \subset \mathrm{Aut}(\mathbb{C})$ and that $b_n \to \infty$ and $b_n/a_n \to \infty$ $(n \to \infty)$. Then the following conditions are equivalent:*

(a) *There is a universal entire function for the sequence $C_{\varphi_n} : H(\mathbb{C}) \to H(\mathbb{C})$ $(n \geq 1)$ that is bounded on Γ.*

(b) *For each continuous function $\varphi : [0, +\infty) \to (0, +\infty)$ that is integrable on $[1, +\infty)$, there is a (C_{φ_n})-universal entire function f such that*

$$\lim_{\substack{z\to\infty \\ z\in\Gamma}} \exp(\varphi(|z|)|z|^{3/2}) f(z) = 0.$$

(c) *There is an Arakelian subset $\Gamma_0 \subset \mathbb{C}$ with $\Gamma_0 \supset \Gamma$ satisfying the following property: for every $R > 0$ there exists $n \in \mathbb{N}$ such that $\Gamma_0 \cap B(b_n, R|a_n|) = \emptyset$.*

Proof. To prove that (b) implies (a), take $\varphi(t) := \frac{1}{1+t^{4/3}}$. Then there is a C_{φ_n}-universal entire function f with $\lim_{\substack{z\to\infty \\ z\in\Gamma}} \exp(\frac{|z|^{3/2}}{1+|z|^{4/3}}) f(z) = 0$. Hence $\lim_{\substack{z\to\infty \\ z\in\Gamma}} f(z) = 0$, showing the boundedness of f on Γ.

Assume that (a) holds for some f. Since f is bounded on Γ, a theorem by Danielyan and Schmieder [15] tells us that the set $\Gamma_0 := \{z \in \mathbb{C} : |f(z)| \leq$

$\sup_\Gamma |f|\}$ is an Arakelian set. Obviously, $\Gamma_0 \supset \Gamma$. Suppose that there is $R > 0$ such that $\Gamma_0 \cap B(b_n, R|a_n|) \neq \emptyset$ for all $n \in \mathbb{N}$. Then there exists a sequence $\{z_n\}_{n\geq 1} \subset B(0,R)$ such that $\{\varphi_n(z_n)\}_{n\geq 1} \subset \Gamma_0$. Let $g(z) := 1+M \in H(\mathbb{C})$, where $M := \sup_\Gamma |f| = \sup_{\Gamma_0} |f|$. Then no subsequence of $(C_{\varphi_n} f)$ can approximate g on the compact set $K = \overline{B}(0,R)$, because $\sup_K |C_{\varphi_n} f - g| \geq |f(\varphi_n(z_n)) - (1+M)| \geq 1 + M - |f(\varphi_n(z_n))| \geq 1$ ($n \in \mathbb{N}$). This contradiction shows that (c) is true.

Finally, we start from (c). To prove (b), we can assume without loss of generality that $\Gamma = \Gamma_0$, so Γ is itself an Arakelian set. Fix a continuous function $\varphi : [0,+\infty) \to (0,+\infty)$ such that $\int_1^{+\infty} \varphi(t)\,dt < +\infty$. Then the function $\varepsilon(t) := \exp(-\varphi(t)t^{3/2} - t^{1/3})$ ($t \geq 0$) is continuous, positive and satisfies

$$\int_1^{+\infty} t^{-3/2} \log \frac{1}{\varepsilon(t)}\,dt < +\infty. \tag{8}$$

Let $\{P_n : n \in \mathbb{N}\}$ be a dense countable subset of $H(\mathbb{C})$, for instance, the set of all polynomial whose coefficients have rational real and imaginary parts. We can construct a sequence (Q_n) whose members are in $\{P_n : n \in \mathbb{N}\}$ such that every function P_m occurs infinitely many times in (Q_n). We denote $B_n := \overline{B}(0,n)$ ($n \geq 1$).

By hypothesis, given $R > 0$ there is $n \in \mathbb{N}$ with $\Gamma \cap \overline{B}(b_n, R|a_n|) = \emptyset$. Let us show that, in fact, there are infinitely many of such n's. Indeed, if it were not true, there would be some $R > 0$ and some $n_0 \in \mathbb{N}$ with $\Gamma \cap \overline{B}(b_n, R|a_n|) \neq \emptyset$ for all $n > n_0$. Set $\widetilde{R} := R + \max\{\frac{\operatorname{dist}(b_k,\Gamma)}{|a_k|} : 1 \leq k \leq k_0\}$. Then it is plain that $\Gamma \cap B(b_n, \widetilde{R}|a_n|) \neq \emptyset$ for all $n \geq 1$, which is absurd.

Now, select $n_1 \in \mathbb{N}$ with $\Gamma \cap \overline{B}(b_{n_1}, |a_{n_1}|) = \emptyset$. Since b_n and b_n/a_n tend to ∞, one derives $|b_n| - 2|a_n| = |b_n|(1 - 2|a_n/b_n|) \to \infty$, so $|b_n| - 2|a_n| > |b_{n_1}| + |a_{n_1}|$ for $n \geq m$, say. Take $n_2 > \max\{n_1, m\}$ such that $\Gamma \cap \overline{B}(b_{n_2}, 2|a_{n_2}|) = \emptyset$. Observe that we also obtain $|z - b_{n_1}| \geq |z| - |b_{n_1}| \geq |b_{n_2}| - |b_{n_2} - z| - |b_{n_1}| \geq |b_{n_2}| - 2|a_{n_2}| - |b_{n_1}| > |a_{n_1}|$ for all $z \in \overline{B}(b_{n_2}, 2|a_{n_2}|)$. Hence $\overline{B}(b_{n_1}, |a_{n_1}|) \cap \overline{B}(b_{n_2}, 2|a_{n_2}|) = \emptyset$. With this procedure, we may determine by induction a sequence $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ satisfying $\Gamma \cap J_k = \emptyset = J_j \cap J_k = \emptyset$ ($k, j \geq 1$, $k \neq j$), where $J_k := \overline{B}(b_{n_k}, k|a_{n_k}|)$. Note also that the balls J_k tend to ∞ as $k \to \infty$ in the sense that $\min\{|z| : z \in J_k\} \to +\infty$ (indeed, these minima are $|b_{n_k}| - k|a_{n_k}|$ and the n_k's were chosen by using $|b_n| - R|a_n| \to +\infty$ for all $R > 0$). Hence the closed sets $\Gamma, J_1, J_2, \ldots$ are mutually disjoint, the J_k's are compact with connected complement, Γ is an Arakelian set and the J_k's do not accumulate at any finite point. From this it is easy to see that $A := \Gamma \cup \bigcup_{n=1}^\infty J_n$ is also an Arakelian set.

Consider the function $g : A \to \mathbb{C}$ given by

$$g(z) = \begin{cases} 0 & \text{if } z \in \Gamma \\ Q_k\big(\frac{z - b_{n_k}}{a_{n_k}}\big) & \text{if } z \in J_k \ (k \in \mathbb{N}), \end{cases}$$

which is clearly continuous on A and holomorphic in A^0. By (8) and a tangential approximation theorem due to Arakelian [16, pp. 160–162], there exists an entire function f such that

$$|f(z) - g(z)| < \varepsilon(|z|) \quad (z \in A). \tag{9}$$

From (9) one obtains

$$|\exp(\varphi(|z|)|z|^{3/2})f(z)| < \exp(-|z|^{1/3}) \longrightarrow 0 \quad (z \to \infty;\ z \in \Gamma).$$

It remains to prove that $f \in \mathcal{U}((C_{\varphi_n}))$. Fix a compact set $K \subset \mathbb{C}$, an $\varepsilon \in (0,1)$ and a polynomial P_m. Since the sets J_k escape towards ∞, we can find $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, $K \subset B_k$ and $|z| > (\log(1/\varepsilon))^{1/3}$ for all $z \in J_k$. Select $k \geq k_0$ with $Q_k = P_m$. From (9), we get $|f(z) - Q_k(\frac{z-b_{n_k}}{a_{n_k}})| < \varepsilon$ $(z \in J_k)$, or equivalently, $|(C_{\varphi_{n_k}} f)(z) - P_m(z)| < \varepsilon$ $(z \in B_k)$. Since $K \subset B_k$, the function $C_{\varphi_{n_k}} f$ is in the basic neighborhood $U(P_m, \varepsilon, K)$. Thus, each P_m belongs to the closure of $\{C_{\varphi_n} f\}_{n \geq 1}$. Finally, the density of (P_m) in $H(\mathbb{C})$ shows the universality of f. □

Remark 3.6. 1. By using the approach of [6] it can be demonstrated the equivalence of (a), (b), (c) of Theorem 3.5 to the following property:

Given a continuous function $\varphi : [0, +\infty) \to (0, +\infty)$ that is integrable on $[1, +\infty)$, there is a *dense linear subspace* $M \subset H(\mathbb{C})$ such that any function $f \in M \setminus \{0\}$ is (C_{φ_n})-universal and satisfies $\lim_{z \to \infty, z \in \Gamma} \exp(\varphi(|z|)|z|^{3/2}) f(z) = 0$.

2. If Γ is a Carleman set (see definition in the proof of Theorem 3.3), (φ_n) is as in Theorem 3.5 and, for every $R > 0$, there is $n \in \mathbb{N}$ such that $\Gamma \cap \overline{B}(b_n, R|a_n|) = \emptyset$, then with essentially the same proof of Theorem 3.5 we obtain the following: given any continuous function $\omega : \mathbb{C} \to (0, +\infty)$, there is $f \in \mathcal{U}((C_{\varphi_n}))$ such that $\lim_{z \to \infty, z \in \Gamma} \omega(z) f(z) = 0$. Just use Nersesjan's instead of Arakelian's theorem and observe that the set $A = \Gamma \cup \bigcup_{k=1}^{\infty} J_k$ constructed in the last proof is a Carleman set.

4. Frequent universality

As said in Section 1, Bayart and Grivaux [3] established the frequent hypercyclicity of every translation operator τ_a $(a \in \mathbb{C}^*)$. By refining the approach of [3] we are going to prove that, under certain conditions, a more general sequence (C_{φ_n}) with $(\varphi_n) \subset \mathrm{Aut}(\mathbb{C})$ can be frequent universal.

Theorem 4.1. *Let* $\{\varphi_n(z) = a_n z + b_n\}_{n \geq 1} \subset \mathrm{Aut}(\mathbb{C})$. *Assume that there exists an unbounded nondecreasing sequence* $\{\omega_n\}_{n \geq 1} \subset (0, +\infty)$ *satisfying*

(a) $|b_n| - \omega_n |a_n| \to +\infty$ *as* $n \to \infty$, *and*
(b) $|b_m - b_n| \geq \omega_{m-n}(|a_m| + |a_n|)$ *for all* $m, n \in \mathbb{N}$ *with* $m > n$.

Then (C_{φ_n}) *is frequently universal on* $H(\mathbb{C})$.

Proof. According to [3, Lemma 2.2], there exist pairwise disjoint subsets $A(l,\nu)$ $(l,\nu \ge 1)$ of $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ of positive lower density such that, for any $n \in A(l,\nu)$ and $m \in A(k,\mu)$, we have

$$n \ge \nu \quad \text{and} \tag{10}$$

$$|n-m| \ge \nu + \mu \ \text{ if } n \ne m. \tag{11}$$

Let $\{P_l\}_{l\ge 1}$ be a dense sequence in $H(\mathbb{C})$, and let $\{n_k\}_{k\ge 1}$ be the increasing sequence of elements of $\bigcup_{l,\nu\ge 1} A(l,\nu)$. If $n_k \in A(l,\nu)$ then we define B_k as

$$B_k = \varphi_{n_k}(\overline{B}(0,\omega_\nu/2)) = \overline{B}(b_{n_k}, \omega_\nu |a_{n_k}|/2).$$

Observe that the balls B_k are pairwise disjoint. Indeed, if $n = n_k \in A(l,\nu)$ and $m = n_{k'} \in A(j,\mu)$ with $k' > k$, then $m > n$ and the distance between the centers of B_k, $B_{k'}$ is $|b_m - b_n| \ge (|a_m| + |a_n|)\omega_{m-n} \ge (|a_m| + |a_n|)\omega_{\mu+\nu} \ge (|a_m| + |a_n|)(1/2)(\omega_\mu + \omega_\nu) > (1/2)\omega_\mu |a_m| + (1/2)\omega_\nu |a_n| = \text{radius}(B_k) + \text{radius}(B_{k'})$, where (b), (11) and the fact that (ω_n) is nondecreasing have been used. Now, from (a), (10) and the fact that (ω_n) is nondecreasing we obtain

$$\min_{z\in B_k} |z| = |b_{n_k}| - \frac{1}{2}\omega_\nu |a_{n_k}| > |b_{n_k}| - \omega_\nu |a_{n_k}| \ge |b_{n_k}| - \omega_{n_k} |a_{n_k}| \to +\infty \ (k \to \infty).$$

From the closedness and disjointness of the B_k's and the fact that they escape to ∞, it is easy to realize that the set $F := \bigcup_{k=1}^\infty B_k$ is a Carleman set (see the proof of Theorem 3.3). Define the functions $\alpha : F \to \mathbb{C}$ and $g : F \to \mathbb{C}$ as $\alpha(z) = \frac{1}{1+|z|}$ $(z \in F)$ and $g(z) = P_l(\frac{z-a_{n_k}}{b_{n_k}})$ $(z \in B_k,\ n_k \in A(l,\nu),\ k \in \mathbb{N})$. Then α and g are continuous on F, α is positive and $\alpha \in H(F^0)$. According to Nersesjan's tangential approximation theorem, there is $f \in H(\mathbb{C})$ such that

$$|f(z) - g(z)| < \alpha(z) \quad (z \in F). \tag{12}$$

We claim that f is frequently universal for (C_{φ_n}). To see this, fix $l,\nu \in \mathbb{N}$ and $\varepsilon > 0$. By (a), we can choose $n_0 \in \mathbb{N}$ such that $|b_n| - \omega_n |a_n| < 1/\varepsilon$ for every $n \ge n_0$. From (10) and since (ω_n) is nondecreasing we get for all $z \in \overline{B}(0,\omega_\nu/2)$ that

$$|\varphi_n(z)| = |b_n + a_n z| \ge |b_n| - |z||a_n| \ge |b_n| - \frac{\omega_\nu}{2}|a_n| \ge |b_n| - \omega_n |a_n| > \frac{1}{\varepsilon}. \tag{13}$$

Pick $k_0 \in \mathbb{N}$ with $n_{k_0} > n_0$. Then for all $z \in \overline{B}(0,\omega_\nu/2)$ and all $k > k_0$ with $n = n_k \in A(l,\nu)$ we obtain from (12) and (13) that

$$|(C_{\varphi_n} f)(z) - P_l(z)| = |f(\varphi_n(z)) - g(\varphi_n(z))| < \frac{1}{1 + |\varphi_n(z)|} < \varepsilon.$$

Since the lower density of $A(l,\nu)\setminus\{n_1,\dots,n_{k_0}\}$ is positive and the sets $\{g \in H(\mathbb{C}) : \sup_{|z|\le \omega_\nu/2} |g(z) - P_l(z)| < \varepsilon\}$ $(\varepsilon > 0;\ l,\nu \in \mathbb{N})$ form a basis of the topology of $H(\mathbb{C})$ (notice that the property $\omega_\nu \to \infty$ is crucial here) it follows the frequent universality of our f. □

Corollary 4.2. *If $(b_n) \subset \mathbb{C}$ is a sequence such that*

$$\lim_{k\to\infty} \inf_{n\in\mathbb{N}} |b_{n+k} - b_n| = +\infty$$

then the sequence of translations (τ_{b_n}) is frequently universal on $H(\mathbb{C})$.

Proof. Given $M > 0$ there is $k_0 \in \mathbb{N}$ such that $|b_{n+k} - b_n| > M + |b_1|$ for all $n \in \mathbb{N}$ and all $k \geq k_0$. Letting $n = 1$ and using the triangle inequality we get $|b_k| > M$ for all $k \geq k_0 + 1$. Setting $b_0 := 0$ one obtains $|b_{n+k} - b_n| > M$ for all $n \in \mathbb{N}_0$ and all $k \geq k_0 + 1$, hence $\delta_k := \inf_{\substack{n\in\mathbb{N}_0 \\ j\geq k}} |b_{n+j} - b_n| \to +\infty$ as $k \to \infty$. In particular $\delta_k > 0$ for $k \geq N$, say. Consider the sequence $c_n := b_{Nn}$. Then the sequence $\omega_k = (1/2) \inf_{\substack{n\in\mathbb{N}_0 \\ j\geq k}} |c_{n+j} - c_n|$ is positive, nondecreasing and unbounded, and fulfills (a) and (b) of Theorem 4.1 (with c_n instead of b_n and $a_n = 1$). Therefore $(\tau_{b_{Nn}})$ is frequently universal, so (τ_{b_n}) also is because (Nn) has positive density $1/N$ in the sequence $\mathbb{N}$. □

Remark 4.3. 1. By selecting $a_n = 1$, $b_n = an$ and $\omega_n = |a|n/2$, we get the already known frequent hypercyclicity of τ_a $(a \neq 0)$. As a different instance of Theorem 4.1, the sequence (C_{φ_n}) is frequently universal if $\varphi_n(z) = a_n z + b_n := n^\alpha z + n^\beta$, where $\beta > 0$ and $\beta \geq 1 + \alpha$. Indeed, let $\gamma := \beta - \alpha$ and consider the continuous function $\varphi : (1, +\infty) \to (0, +\infty)$ defined by $\varphi(t) := \frac{t^\beta - 1}{t^\alpha (t-1)^\gamma}$. Then $\lim_{t\to+\infty} \varphi(t) = 1 > 0$ and $\lim_{t\to 1^+} \varphi(t) = \begin{cases} \beta & \text{if } \gamma = 1 \\ +\infty & \text{if } \gamma > 1, \end{cases}$ so $d := \inf_{t>1} \varphi(t) > 0$. Set $\delta := \min\{1/2, d/2\}$ and take $\omega_n := \delta \cdot n^\gamma$. Then (ω_n) is positive, nondecreasing and unbounded. Moreover, $|b_n| - \omega_n |a_n| \geq n^\beta - (1/2) n^\alpha n^\gamma = (1/2) n^\beta \to +\infty$ $(n \to \infty)$, which yields the condition (a) of Theorem 4.1. Finally, if $m > n$, one has

$$\frac{|b_m - b_n|}{\omega_{m-n}(|a_m| + |a_n|)} = \frac{m^\beta - n^\beta}{\delta(m^\alpha + n^\alpha)(m-n)^\gamma} \geq \frac{m^\beta - n^\beta}{2\delta m^\alpha (m-n)^\gamma} = \frac{1}{2\delta} \varphi\left(\frac{m}{n}\right) \geq \frac{d}{2\delta} \geq 1,$$

that gives (b).

2. It would be interesting to characterize the sequences of automorphisms of $\mathbb{C}$ generating frequent universality, as well as to characterize the (or, at least, to give some criterion yielding) sequences $(\varphi_n) \subset \mathrm{Aut}(\mathbb{C})$ generating frequent $\mathcal{M}$-universality, that is, such that there exists $f \in H(\mathbb{C})$ with the following property: for every $g \in H(\mathbb{C}^*)$, every $\varepsilon > 0$ and every $K \in \mathcal{M}(\mathbb{C}^*)$, the set $\{n \in \mathbb{N} : |f(\varphi_n(z)) - g(z)| < \varepsilon \ \forall z \in K\}$ has positive lower density. In particular, we ask the questions:

(i) If $0 < \alpha < 1$, is $(C_{z\mapsto z+n^\alpha})$ frequently universal?

(ii) If $|a| > 1$, is $(C_{z\mapsto a^n z})$ frequently $\mathcal{M}$-universal?

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